



An Age-Structured Diffusive Model for Epidemic Modelling: Lie Symmetries and Exact Solutions

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Abstract

A new age-structured diffusive model for the mathematical modelling of epidemics is suggested. The model can be considered as a generalization of two models suggested earlier for similar purposes. The Lie symmetry classification of the model is derived. It is shown that the model admits an infinite-dimensional Lie algebra of invariance. Using the Lie symmetries, exact solutions, in particular those of the travelling wave types and in terms of special functions, are constructed. Examples of application of exact solutions with the correctly-specified parameters for calculation of the total number of infected individuals during an epidemic are presented.

1 Introduction

A mathematical model describing the spread of the COVID-19 pandemic was suggested in [5] and studied in detail in [6, 7]. The model is a generalization of a simpler model based on ordinary differential equations (see [5]) that was successfully used for modelling the first wave of the COVID-19 in some countries. The generalization takes into account spacial heterogeneity by introducing diffusion terms and reads as

$$\begin{aligned}u_t &= d_1 \Delta u + u(a - bu^\gamma), \\v_t &= d_2 \Delta u + k(t)u.\end{aligned}\tag{1}$$

In (1), the function $u(t, x, y)$ describes the density (rate) of the infected persons (the number of the COVID-19 cases) in a vicinity of the point (x, y) , while $v(t, x, y)$ means the density of the deaths from COVID-19. The diffusivity coefficients d_1 and d_2 describe the random movement of the infected persons, which lead to increasing the pandemic spread. Each coefficient in the reactions terms, a , b , γ and $k(t)$, has the clear

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meaning described in [5, 6]. Recently [7], it was demonstrated that the mathematical model with the governing equations (1) and relevant boundary and initial conditions adequately describes the second COVID-19 pandemic wave in Ukraine. A rigorous comparison of the numerical results obtained by numerical simulations with the official data proved that the model produces very plausible total numbers of the Covid-19 cases and deaths. An extensive analysis of impact of the parameters arising in the model is also presented in [7].

On the other hand, it is well-known that age of infective individuals has an essential impact on recovering rate. Generally speaking, old individuals have less chances to recover comparing with young those, therefore the death rate is growing with the age. The simplest epidemic model taking into account the age reads as

$$u_t + \mu_0 u_\tau = -\mu(\tau)u, \quad (2)$$

where $\mu(\tau)$ is a death rate and $\mu_0 > 0$ is a parameter (typically $\mu_0 = 1$ is taken but this assumption is questionable). Following the pioneering study [34], equation (2) is also called the Von Foerster equation. This linear equation was extensively studied many years ago [29, 31, 32]. We also note that there are recent studies, in which generalizations of (2) involving integral terms are suggested (see [4, 27] and papers cited therein).

Later, the epidemic model (2) was extended by taking into account the diffusion effect. As a result, the reaction-diffusion equation [15–17, 28]

$$u_t + \mu_0 u_\tau = d_1 \Delta u - \mu(\tau)u \quad (3)$$

was suggested as a natural generalization of the Von Foerster equation.

At the present time, the existence of solutions, including traveling waves and periodic solutions, of model (3) and its generalizations was studied in many works (see, e.g., [12, 19, 23, 25]). Obviously, the reaction-diffusion equation (3) is linear, therefore many classical techniques can directly be applied for its solving and rigorous analysis.

On the other hand, almost all known space distributed models used for modelling in epidemiology are based on nonlinear reaction-diffusion equations (see the recent review [11] and citations therein). In our opinion, there is some contradiction that the mathematical models used for the epidemic spread, but neglecting age of a population, are more complicated than those taking into account age structure of the population in question. Motivated by this observation, we suggest here a nonlinear generalization of the linear PDE (3), study its Lie symmetry, construct exact solutions and suggest their interpretation.

The reminder of this paper is organized as follows. In Section 2, a nonlinear generalization of equation (3) is presented and analyzed. In Section 3, the Lie symmetry classification of the generalized equation is derived. In Section 4, a wide range of exact solutions, including travelling waves and those involving special functions, are constructed and their properties are briefly discussed. An illustrative example of application of the correctly-specified exact solution for calculation of total numbers of infected individuals during an epidemic is presented as well. Finally, we discuss the results obtained and present some conclusions in the last section.

2 An Age-Structured Model with Diffusion

First of all, let us consider equation (3). One may easily observe that the assumptions $u_\tau = 0$ and $\mu(\tau) = \mu_0 = \text{const}$, i.e. the epidemic spread does not depend on age of the given population, immediately leads to the equation

$$u_t = d_1 \Delta u - \mu_0 u. \tag{4}$$

The latter is the classical diffusion equation with the sink $\mu_0 u$. To the best of our knowledge, equation (4) does not occur for the mathematical modelling of epidemics. However, if we combine the first equation from (1) with (2) then the following nonlinear PDE is obtained:

$$u_t + \mu_0 u_\tau = d_1 \Delta u + (a - \mu(\tau))u - bu^{\gamma+1}, \tag{5}$$

where $u(t, \tau, x, y)$ describes the density (rate) of the infected persons of age τ in a vicinity of the point (x, y) . We remind the reader that $a > 0$ is the coefficient for the virus transmission mechanism, $b > 0$ is the coefficient for the effectiveness of government restrictions (quarantine rules), $\gamma > 0$ is the exponent, which guarantees that the total number of the infected persons is bounded in time, and its value depends mostly on the country (large region) in question [5].

Thus, the total number of infected individuals of the age τ up to time t is calculated using the formula

$$U(t, \tau) = \int_{\Omega} u(t, \tau, x, y) dx dy,$$

where $\Omega \subset \mathbf{R}^2$ is a domain (continent/country/region) in which a pandemic is spread. The total number of infected population is obtained by the double integral

$$U_{\text{total}}(t) = \int_I \left(\int_{\Omega} u(t, \tau, x, y) dx dy \right) d\tau, \tag{6}$$

where $I = [0, \tau_{\text{max}}]$, τ_{max} is a maximum life span in population.

Equation can be rewritten in the form

$$u_t + \mu_0 u_\tau = d_1 \Delta u - \mu^*(\tau)u - bu^{\gamma+1}$$

by introducing the notation $\mu^*(\tau) = \mu(\tau) - a$. Moreover, as it was shown in [7], one should take nonconstant (in space) coefficients in the governing equations of (1) in order to ideally adopt the model to the official data of the COVID-19 pandemic. Thus, a natural generalization of (5) reads as

$$u_t + \mu_0 u_\tau = d_1 \Delta u + A(\tau, x, y)u - B(x, y)u^{\gamma+1}, \tag{7}$$

where $A(\tau, x, y)$ and $B(x, y)$ are given functions.

It is well known that epidemics spread much faster in agglomerations. It means that the coefficient b reflecting the effectiveness of the government restrictions (quarantine rules) takes the smallest value in the center of the agglomeration, say, a city with the coordinates $(x, y) = (0, 0)$. Notably, we have shown this for the Kyiv city agglomeration in the case of the COVID-19 pandemic in Ukraine [7]. Thus, one may expect that $B(x, y) = b_0 + b_1(x, y)$, where b_0 is a nonnegative parameter and $b_1(x, y)$ is

a nonnegative function satisfying the condition $b_1(0, 0) = 0$ that increases with growing $x^2 + y^2$. A simplest example could be

$$b_1(x, y) = b_1(x^2 + y^2)^\kappa, \quad \kappa > 0, \quad b_1 > 0.$$

As it is shown in [7], the coefficient a reflecting the virus transmission mechanism does not depend so much on the population density. So, one may take $A(\tau, x, y) = a_0 + \varepsilon(x^2 + y^2)^v - \mu(\tau)$ with $v > 0$, $a_0 > 0$ and $|\varepsilon| \ll 1$. As a result, a plausible example of the general model (7) has the form

$$u_t + \mu_0 u_\tau = d_1 \Delta u + \left(a_0 + \varepsilon(x^2 + y^2)^v - \mu(\tau) \right) u - \left(b_0 + b_1(x^2 + y^2)^\kappa \right) u^{\gamma+1}. \quad (8)$$

Obviously, the latter reduces to the form (5) provided $\varepsilon = b_1 = 0$. Another interesting case for possible applications occurs when $\varepsilon = b_0 = 0$.

In what follows, we consider (5) as a basic equation for age-structured models with diffusion. The nonlinear reaction-diffusion-convection equation (5) can be simplified provided natural assumptions about positivity of parameters μ_0 , a , b and d_1 take place (here we do not consider limiting case when some of them are zero). Let us apply the scale transformations

$$t^* = at, \quad \tau^* = \frac{a}{\mu_0} \tau, \quad \mu^* = \frac{1}{a} \mu, \quad x^* = \sqrt{\frac{a}{d_1}} x, \quad y^* = \sqrt{\frac{a}{d_1}} y, \quad u^* = \left(\frac{b}{a} \right)^{1/\gamma} u \quad (9)$$

in order to simplify equation (5). Thus, omitting stars $*$, one obtains the following nonlinear PDE involving only the parameter γ and the function $\mu(\tau)$:

$$u_t + u_\tau = \Delta u + (1 - \mu(\tau))u - u^{\gamma+1}. \quad (10)$$

The nonlinear PDE (10) is the main object of this study.

3 Lie Symmetry Classification of the Model

In this section, we study Lie symmetries of equation (10) with an arbitrary number of space variables:

$$u_t + u_\tau = \Delta u + (1 - \mu(\tau))u - u^{\gamma+1}, \tag{11}$$

where $u(t, \tau, x)$ is an unknown function of $n + 2$ variables $t, \tau, x = (x_1, \dots, x_n)$; $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian, and $\mu(\tau)$ is an arbitrary smooth function. Obviously, the cases $n = 1, 2, 3$ are the most important for prospective real-world applications. In what follows we assume that γ is an arbitrary constant satisfying the restriction $\gamma(\gamma + 1) \neq 0$, i.e. the equation in question is nonlinear.

According to the classical Lie method (see, e.g., the recent monographs [2, 9]), the most general form of Lie symmetry operators for (11) is

$$X = \xi^t(t, \tau, x, u)\partial_t + \xi^\tau(t, \tau, x, u)\partial_\tau + \xi^i(t, \tau, x, u)\partial_{x_i} + \eta(t, \tau, x, u)\partial_u,$$

where ξ^t, ξ^τ, ξ^i , and η are to-be-determined functions. In the case of the specific function $\mu(\tau)$, Lie symmetries of (11) with the fixed number of the space variables can be easily calculated using the computer algebra packages employed in Maple, Mathematica etc. However, we want to solve the *Lie symmetry classification (LSC) problem*, i.e. to identify all possible forms of the function $\mu(\tau)$ and γ leading to extensions of a so-called principal algebra. According to the definition, the principal algebra of equation (11) is calculated under assumption that $\mu(\tau)$ and γ are arbitrary. The detailed algorithm for solving LSC problem for a given PDE involving arbitrary function(s) as parameter(s) is described in [9, Chapter 2]. Here we apply this algorithm in the case of the nonlinear PDE (11).

Now we present a statement about the group of equivalence transformations (ETs) of equation (11) with $\gamma(\gamma + 1) \neq 0$. First of all, it can easily be noted that there is no any local transformation which transforms (11) to that of the same form but with another parameter $\gamma^* \neq \gamma$. Thus, ETs can change only the form of the function $\mu(\tau)$.

Theorem 1 *The group of the continuous ETs transforming equation (11) with $\gamma(\gamma + 1) \neq 0$ to that with the same structure:*

$$u_t^* + u_\tau^* = \Delta u^* + (1 - \mu^*(\tau^*))u^* - (u^*)^{\gamma+1}, \tag{12}$$

is an infinite-parameter Lie group generated by the transformations

$$\begin{aligned} t^* &= \alpha^2(t + T(t - \tau)), \quad \tau^* = \alpha^2(\tau + \tau_0), \\ x^* &= \alpha \mathbf{R}(t - \tau) x + X(t - \tau), \\ u^* &= \alpha^{-\frac{2}{\gamma}} u, \quad \mu^* = \frac{\mu - 1}{\alpha^2} + 1. \end{aligned} \tag{13}$$

Here $\alpha > 0$ and τ_0 are the real group parameters, $x^* = (x_1^*, x_2^*, \dots, x_n^*)^\top$, $x = (x_1, x_2, \dots, x_n)^\top$, $\mathbf{R}(t - \tau)$ is the $n \times n$ rotation matrix, $X(t - \tau) =$

$(X^1(t - \tau), X^2(t - \tau), \dots, X^n(t - \tau))^T$, and X^i ($i = 1, \dots, n$) are arbitrary smooth function. The function T is an arbitrary smooth function with the restriction $T \neq -(t - \tau)$.

Proof of Theorem 1 is based on the known technique for constructing the group of continuous ETs. Because this technique is rather cumbersome, there are not many papers, in which one was described in detail and successfully employed for nontrivial PDEs (good examples can be found in [18, 24]).

Thus, we should start from the infinitesimal operator

$$E = \xi^t(t, \tau, x, u)\partial_t + \xi^\tau(t, \tau, x, u)\partial_\tau + \xi^i(t, \tau, x, u)\partial_{x_i} + \eta(t, \tau, x, u)\partial_u + \zeta(t, \tau, x, u, \mu)\partial_\mu, \tag{14}$$

where ξ^t , ξ^τ , ξ^i , η and ζ are to-be-determined functions. We note that formula (14) without the last term gives the most general form of Lie symmetries of equation (11).

In order to find the operator E , we should apply the invariance criteria

$$E_2 \left(\Delta u - u_t - u_\tau + (1 - \mu)u - u^{\gamma+1} \right) \Big|_{\mathcal{M}} = 0, \quad E_1 \left(S^k \right) \Big|_{\mathcal{M}} = 0, \quad k = 1, 2, 3, \tag{15}$$

where E_1 and E_2 represent the first- and second-order prolongations of the operator (14), the manifold

$$\mathcal{M} = \{ \Delta u - u_t - u_\tau + (1 - \mu)u - u^{\gamma+1} = 0, \quad S_1 \equiv \mu_t = 0, \quad S_2 \equiv \mu_x = 0, \quad S_3 \equiv \mu_u = 0 \}.$$

After the straightforward calculations using formulae (15), we arrive at the infinitesimal operator

$$E = \xi^t(t, \tau)\partial_t + \xi^\tau(\tau)\partial_\tau + \xi^i(t, \tau, x)\partial_{x_i} + \left(\eta^1(t, \tau, x)u + \eta^0(t, \tau, x) \right) \partial_u + \zeta(\tau, \mu)\partial_\mu, \tag{16}$$

where the coefficients are to-be-determined functions that satisfy the system of linear PDEs

$$\xi_t^t + \xi_\tau^t - 2\xi_{x_i}^i = 0, \tag{17}$$

$$\xi_{x_j}^i + \xi_{x_i}^j = 0, \quad i \neq j, \tag{18}$$

$$\xi_t^t + \xi_\tau^t = \xi_\tau^\tau, \tag{19}$$

$$2\eta_{x_i}^1 + \xi_t^i + \xi_\tau^i = 0, \tag{20}$$

$$\left(\xi_t^t + \xi_\tau^t + \gamma\eta^1 \right) u^{\gamma+1} + (\gamma + 1)\eta^0 u^\gamma + \left(\eta_t^1 + \eta_\tau^1 - \Delta\eta^1 + \zeta - (1 - \mu)(\xi_t^t + \xi_\tau^t) \right) u + \eta_t^0 + \eta_\tau^0 - \Delta\eta^0 - (1 - \mu)\eta^0 = 0. \tag{21}$$

Table 1 Lie symmetries of the nonlinear equation (11)

	Restrictions	Additional Lie symmetries
1	$\mu(\tau) = \text{const}, \mu(\tau) \neq 1$	$F^2(t - \tau)\partial_\tau$
2	$\mu(\tau) = 1 + \frac{\alpha}{\tau}$	$F^3(t - \tau) \left(2t\partial_t + 2\tau\partial_\tau + x_i\partial_{x_i} - \frac{2}{\gamma}u\partial_u \right)$
3	$\mu(\tau) = 1$	$F^2(t - \tau)\partial_\tau, F^3(t - \tau) \left(2t\partial_t + 2\tau\partial_\tau + x_i\partial_{x_i} - \frac{2}{\gamma}u\partial_u \right)$
4	$\gamma = 1, \mu(\tau) = 1 - 2 \tan \tau$	$F^4(t - \tau) \left(\partial_\tau + \sec^2 \tau \partial_u \right)$
5	$\gamma = 1, \mu(\tau) = 1 + 2 \tanh \tau$	$F^5(t - \tau) \left(\partial_\tau - \text{sech}^2 \tau \partial_u \right)$
6	$\gamma = 1, \mu(\tau) = 1 + 2 \coth \tau$	$F^6(t - \tau) \left(\partial_\tau + \text{csch}^2 \tau \partial_u \right)$
7	$\gamma = 1, \mu(\tau) = 1 + \frac{2}{\tau}$	$F^3(t - \tau) \left(2t\partial_t + 2\tau\partial_\tau + x_i\partial_{x_i} - 2u\partial_u \right),$ $F^7(t - \tau) \left(\partial_\tau + \tau^{-2}\partial_u \right)$
8	$\gamma = 1, \mu(\tau) = 1 + \frac{1-2\alpha \tan(\alpha \ln \tau)}{\tau}$	$F^8(t - \tau) \left(2t\partial_t + 2\tau\partial_\tau + x_i\partial_{x_i} + \left(\frac{2\alpha^2}{\tau} \sec^2(\alpha \ln \tau) - 2u \right) \partial_u \right)$
9	$\gamma = 1, \mu(\tau) = 1 + \frac{1+2\alpha \tanh(\alpha \ln \tau)}{\tau}$ $\alpha \neq \pm \frac{1}{2}$	$F^9(t - \tau) \left(2t\partial_t + 2\tau\partial_\tau + x_i\partial_{x_i} - \left(\frac{2\alpha^2}{\tau} \text{sech}^2(\alpha \ln \tau) + 2u \right) \partial_u \right)$
10	$\gamma = 1, \mu(\tau) = 1 + \frac{1+2\alpha \coth(\alpha \ln \tau)}{\tau}$ $\alpha \neq \pm \frac{1}{2}$	$F^{10}(t - \tau) \left(2t\partial_t + 2\tau\partial_\tau + x_i\partial_{x_i} + \left(\frac{2\alpha^2}{\tau} \text{csch}^2(\alpha \ln \tau) - 2u \right) \partial_u \right)$
11	$\gamma = 1, \mu(\tau) = 1 + \frac{2+\ln \tau}{\tau \ln \tau}$	$F^{11}(t - \tau) \left(2t\partial_t + 2\tau\partial_\tau + x_i\partial_{x_i} + \left(\frac{2}{\tau \ln^2 \tau} - 2u \right) \partial_u \right)$

Consider equation (21). Since the parameter γ is arbitrary, one immediately obtains

$$\eta^0 = 0, \eta^1 = -\frac{\xi_t^t + \xi_\tau^t}{\gamma}, \zeta = (1 - \mu)\xi_\tau^\tau + \frac{\xi_{\tau\tau}^\tau}{\gamma}.$$

Solving the linear equations (17)–(20), we determine the coefficients of the operator (14):

$$\begin{aligned} \xi^t &= 2\beta t + F^1(t - \tau), \xi^\tau = 2\beta\tau + \beta_0, \\ \xi^i &= G^i(t - \tau) + \sum_{j=1}^n H^{ij}(t - \tau)x_j, \quad i = 1, \dots, n, \\ \eta &= -\frac{2\beta}{\gamma}u, \zeta = 2\beta(1 - \mu), \end{aligned} \tag{22}$$

where β_0 and β are arbitrary constants; F^1 and G^i are arbitrary smooth functions. The functions H with superscripts satisfy the conditions $H^{ij} + H^{ji} = 0$ for $i \neq j$, $H^{ii} = \beta$.

Finally, using the well-known formulae allowing to construct the Lie group, which corresponds to the Lie algebra generated by (14) with the coefficients (22), the Li group (13) has been identified. Because the Lie algebra obtained is infinite-dimensional, we present some details in Appendix A.

The proof is completed.

Theorem 2 Equation (11) with an arbitrary parameter γ and an arbitrary function μ is invariant under the infinity-dimensional principal algebra that is generated by the Lie symmetry operators:

$$F^1(t - \tau)\partial_t, G^i(t - \tau)\partial_{x_i}, H^{ij}(t - \tau)(x_i\partial_{x_j} - x_j\partial_{x_i}), \tag{23}$$

where F^1 , G^i and H^{ij} are arbitrary smooth functions of variable $t - \tau$, $i = 1, \dots, n$, $j = 1, \dots, n$, ($i < j$). In other words, operators (23) form the principal algebra of invariance of equation (11).

Theorem 3 Equation (11) with $\gamma(\gamma + 1) \neq 0$ depending on $\mu(\tau)$ and γ admits eleven extensions of the principal algebra (23) that are listed in Table 1. Any other equation of the form (11) is either invariant w.r.t. the principal algebra, or is reducible to one listed in Table 1 by the ETs (13).

Proof of Theorems 2 and 3. Because the nonlinear equation (11) involves only the single function μ as a parameter, we use a simplification of the algorithm that was suggested in [9, Chapter 2]. First of all, we note that the group of ETs (13) will be used at the final stage in order to reduce some arbitrary parameters to the values 0 or 1.

At the first step, we construct a system of determining equations. Applying the infinitesimal criterion of invariance to the most general form of Lie symmetries of equation (11) (see (14) without the last term), one immediately obtains

$$\xi_{x_i}^t = \xi_{x_i}^\tau = \xi_u^t = \xi_u^\tau = \xi_u^i = 0, \quad i = 1, \dots, n, \quad \eta = \eta^1(t, \tau, x)u + \eta^0(t, \tau, x).$$

So

$$X = \xi^t(t, \tau)\partial_t + \xi^\tau(t, \tau)\partial_\tau + \xi^i(t, \tau, x)\partial_{x_i} + \left(\eta^1(t, \tau, x)u + \eta^0(t, \tau, x)\right)\partial_u,$$

where ξ^t , ξ^τ , ξ^i , η^0 and η^1 are to-be-determined functions. Using the above form of Lie symmetries, the system of determining equations was derived:

$$\xi_t^t + \xi_\tau^t - 2\xi_{x_i}^i = 0, \tag{24}$$

$$\xi_{x_j}^i + \xi_{x_i}^j = 0, \quad i < j, \tag{25}$$

$$\xi_t^t + \xi_\tau^t = \xi_t^\tau + \xi_\tau^\tau, \tag{26}$$

$$2\eta_{x_i}^1 + \xi_t^i + \xi_\tau^i = 0, \tag{27}$$

$$\left(\xi_t^t + \xi_\tau^t + \gamma\eta^1\right)u^{\gamma+1} + (\gamma + 1)\eta^0u^\gamma + \left(\eta_t^1 + \eta_\tau^1 - \Delta\eta^1 + \mu'\xi^\tau - (1 - \mu)(\xi_t^t + \xi_\tau^t)\right)u + \eta_t^0 + \eta_\tau^0 - \Delta\eta^0 - (1 - \mu)\eta^0 = 0. \tag{28}$$

We remind the reader that $i, j = 1, \dots, n$.

At the second step, the parameters γ and μ in equation (11) are assumed to be arbitrary. In fact, according to the definition, the principal algebra of invariance consists of all Lie symmetries, which are symmetries of the equation in question independently on the form of γ and μ . In this case, the result presented in Theorem 2 follows straightforwardly. Indeed, we immediately obtain from equation (28) the following conditions:

$$\eta^1 = 0, \quad \eta^0 = 0, \quad \xi^\tau = 0, \quad \xi_t^t + \xi_\tau^t = 0. \tag{29}$$

Solving the linear equations (24)–(27) and the last one from (29), we derive exactly the coefficients of the Lie symmetries (23). Thus, the principal algebra of invariance is identified.

At the third step, we need to find all possible specifications of parameters γ and μ that can lead to inequivalent extensions of the principal algebra (23). Firstly, we split equation (28) with respect to the different exponents 1, γ and $\gamma + 1$ of u . Taking into account that $\gamma(\gamma + 1) \neq 0$, we conclude that $\gamma = 1$ is the only special case. Thus, one needs to consider two cases: **1)** γ is an arbitrary parameter and **2)** $\gamma = 1$.

Let us examine Case **1)** in detail. Splitting equation (28) with respect to the above exponents, we obtain $\eta^0 = 0$ and the equations

$$\xi_t^t + \xi_\tau^t + \gamma\eta^1 = 0, \tag{30}$$

$$\eta_t^1 + \eta_\tau^1 - \Delta\eta^1 + \mu'\xi^\tau - (1 - \mu)(\xi_t^t + \xi_\tau^t) = 0. \tag{31}$$

Equation (30) immediately gives $\eta_{x_i}^1 = 0$. So, solving equations (24)–(27), we identify the functions ξ^t , ξ^τ and ξ^i as follows:

$$\begin{aligned} \xi^t &= 2t F^3(t - \tau) + F^1(t - \tau), \quad \xi^\tau = 2\tau F^3(t - \tau) + F^2(t - \tau), \\ \xi^i &= G^i(t - \tau) + H^{ij}(t - \tau)x_j, \end{aligned}$$

where $H^{ij} + H^{ji} = 0$ for $i < j$. Here, the functions F^k ($k = 1, 2, 3$) and G^i are arbitrary smooth functions. Substituting the functions ξ^t and ξ^τ derived above into (30), we find the function $\eta^1 = -\frac{2}{\gamma} F^3$.

Thus, to complete the examination of Case I), one needs to solve the classification equation (31) that now can be rewritten in the following form

$$\mu' (2\tau F^3 + F^2) = 2(1 - \mu)F^3. \tag{32}$$

To do this, one needs to consider two different subcases: $\mu' F^3 = 0$ and $\mu' F^3 \neq 0$.

The first subcase leads to the following results: Case 1 of Table 1 if $\mu' = 0$ and $\mu \neq 1$; Case 3 of Table 1 if $\mu = 1$. If $\mu' \neq 0$ then only the principal algebra (23) is obtained.

In the second subcase, equation (32) can be rewritten in the form

$$\frac{F^2}{2F^3} = \frac{1 - \mu}{\mu'} - \tau. \tag{33}$$

Since μ does not depend on the variable t , equation (33) leads to the linear ODE

$$\frac{1 - \mu}{\mu'} - \tau = \tau_0.$$

Solving the above equation, one obtains

$$\mu = 1 + \frac{\alpha}{\tau + \tau_0}, \quad \alpha \neq 0,$$

where τ_0 and α is an arbitrary constants. Using the ET $\tau + \tau_0 \rightarrow \tau$ (see the first line in (13)) the above functions reduces to the form $\mu = 1 + \frac{\alpha}{\tau}$. Moreover, it means that $F^2 = 0$ (see (33)), while the function F^3 remains arbitrary (refer to formula (32)). As a result, the additional operator

$$F^3(t - \tau) \left(2t\partial_t + 2\tau\partial_\tau + x_i\partial_{x_i} - \frac{2}{\gamma} u\partial_u \right)$$

is derived. Thus, Case 2 of Table 1 is identified.

In Case 2), a quite similar analysis leads to the results presented in Cases 4–11 of Table 1, in which $\alpha \neq 0$ and β are arbitrary constants.

Notably, the group of ETs (13) (more complicated transformations then the τ -translation mentioned above) was used for simplifications of the function $\mu(\tau)$ in Cases 4–6 and 8–11 of Table 1. In Cases 4–6, the ET

$$t^* = \frac{|\alpha|}{2} t, \quad \tau^* = \frac{|\alpha|}{2} \tau, \quad x^* = \sqrt{\frac{|\alpha|}{2}} x, \quad u^* = \frac{2}{|\alpha|} u \tag{34}$$

were used in order to remove the parameter α , which naturally arises in the expressions obtained for function $\mu(\tau)$, namely: $\mu(\tau) = 1 - \alpha \tan \frac{\alpha\tau}{2}$ (Case 4), $\mu(\tau) = 1 + \alpha \tanh \frac{\alpha\tau}{2}$ (Case 5) and $\mu(\tau) = 1 + \alpha \coth \frac{\alpha\tau}{2}$ (Case 6). In Cases 8–10 and Case 11, the ETs

$$t^* = e^{\frac{\beta}{\alpha}} t, \tau^* = e^{\frac{\beta}{\alpha}} \tau, x^* = e^{\frac{\beta}{2\alpha}} x, u^* = e^{-\frac{\beta}{\alpha}} u$$

and

$$t^* = e^{\beta} t, \tau^* = e^{\beta} \tau, x^* = e^{\frac{\beta}{2}} x, u^* = e^{-\beta} u,$$

removing the parameter β , where used for the same purposes. As a result, Table 1 is derived.

The proof is completed. □

Remark 1 The nonlinear equation (11) is reduced to equation

$$u_{\tau^*} = \Delta u + (1 - \mu(\tau^*))u - u^{\gamma+1} \tag{35}$$

by the substitution

$$t^* = t - \tau, \tau^* = \tau,$$

in which the function u should be still considered as function of $n + 2$ independent variables. The Lie symmetry analysis is a little bit simpler for (35). However, we would still go back to the original variables because the variable $t^* = \tau - t$ does not have any biological meaning in contrast to τ and t . Taking into account that one of the main goals of this work is application of the results for modelling of epidemics, we prefer to keep the biologically motivated variables.

In conclusion of this section, we present the following observation. Equation (11) with $\gamma = 1$, i.e.

$$u_t + u_{\tau} = \Delta u + (1 - \mu(\tau))u - u^2 \tag{36}$$

can be rewritten in the form

$$u_t + u_{\tau} = \Delta u - u^2 + \frac{\mu'}{2} + \frac{(1 - \mu)^2}{4}$$

by using the substitution

$$u \rightarrow u + \frac{1 - \mu}{2}. \tag{37}$$

In particular, equation (36) with $\mu = 1 + \frac{2}{\tau}$ (see Case 7 in Table 1) is reduced to the equation

$$u_t + u_\tau = \Delta u - u^2$$

by the transformation

$$u \rightarrow u - \frac{1}{\tau}. \quad (38)$$

Now one notes that the above equation corresponds to Case 3 of Table 1. Obviously, (38) is not a ET, therefore it is a form-preserving transformation (FPT). We note that FPTs (equivalent terminology is ‘admissible transformations’ [13]) were introduced in [21, 22] for study nonlinear PDEs. Later it was shown that FPTs play essential role in solving LSC of PDEs belonging to a given class (see [9, Chapter 2] and papers cited therein).

Notably, a sophisticated approach for so-called enhanced group analysis based on admissible transformations was independently developed in [33]. A further Lie symmetry classification of (11) based on form-preserving/admissible transformations lies beyond the specific scopes of this study.

4 Exact Solutions

Let us construct exact solutions of Eq. (11) in the case $n = 1$:

$$u_t + u_\tau = u_{xx} + (1 - \mu(\tau))u - u^{\gamma+1}. \quad (39)$$

It can be noted that the nonlinear equation (39) is reduced to equation

$$u_{t^*} = u_{xx} + (1 - \mu(t^* - \tau^*))u - u^{\gamma+1} \quad (40)$$

by the substitution

$$t^* = \frac{t + \tau}{2}, \quad \tau^* = \frac{t - \tau}{2}. \quad (41)$$

Equation (40) is nothing else but a nonlinear reaction-diffusion (RD) equation involving the variable τ^* as a parameter in the reaction term.

4.1 Travelling Wave Type Solutions

Firstly, we examine equation (40) with a constant function $\mu(\tau)$, i.e. the RD equation

$$u_{t^*} = u_{xx} + \lambda u - u^{\gamma+1}, \quad (42)$$

where $\lambda = 1 - \mu = \text{const}$. Exact solutions of equation (42) with arbitrary and correctly-specified γ were found in several studies. In particular, known travelling waves (TWs) are summarized in [14]. So, taking a known TW solution and replacing arbitrary constants by arbitrary functions of the variable $\tau^* = \frac{t-\tau}{2}$, one readily obtains an exact solution of equation (39) with $\mu(\tau) = 1 - \lambda$. In particular, using the travelling wave presented in [6] (see (3.6) therein) one arrives at the following solution of equation (39)

$$u(t, \tau, x) = \lambda^{1/\gamma} \left(1 + C(t - \tau) \exp \left(\pm \frac{\gamma\sqrt{\lambda}}{\sqrt{2(\gamma+2)}} x - \frac{\lambda\gamma(\gamma+4)}{4(\gamma+2)} (t + \tau) \right) \right)^{-2/\gamma}, \tag{43}$$

where $\lambda = 1 - \mu > 0$ and $C(t - \tau)$ is an arbitrary smooth function. Obviously, (43) with $C = \text{const} > 0$ and $\gamma > 0$ is a typical TW.

Equation (42) with $\gamma = 2$:

$$u_{t^*} = u_{xx} + \lambda u - u^3,$$

possesses more complicated solutions that were constructed for the first time in [20] and later were identified using Q -conditional (nonclassical) symmetries in [3, 10] (see also [9, Chapter 4]). Assuming $\lambda > 0$, the relevant solution has the form

$$u = \frac{\sqrt{\lambda} C_2 \exp\left[\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{2}t^*\right] - \sqrt{\lambda} C_3 \exp\left[-\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{2}t^*\right]}{C_1 + C_2 \exp\left[\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{2}t^*\right] + C_3 \exp\left[-\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{2}t^*\right]}.$$

Thus, equation (39) with $\mu(\tau) = 1 - \lambda$ and $\gamma = 2$ possesses the following family of exact solutions involving three arbitrary functions:

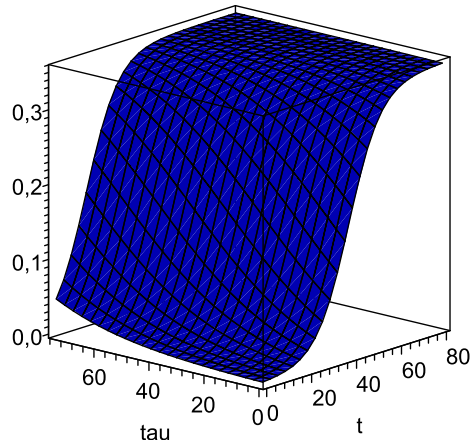
$$u = \frac{\sqrt{\lambda} C_2 (t - \tau) \exp\left[\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{4}(t + \tau)\right] - \sqrt{\lambda} C_3 (t - \tau) \exp\left[-\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{4}(t + \tau)\right]}{C_1 (t - \tau) + C_2 (t - \tau) \exp\left[\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{4}(t + \tau)\right] + C_3 (t - \tau) \exp\left[-\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{4}(t + \tau)\right]}. \tag{44}$$

In the special case, stationary (time-independent) solutions of the form

$$u = \frac{\sqrt{\lambda} c_2 \exp\left[\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{2}\tau\right] - \sqrt{\lambda} c_3 \exp\left[-\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{2}\tau\right]}{c_1 + c_2 \exp\left[\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{2}\tau\right] + c_3 \exp\left[-\frac{\sqrt{2\lambda}}{2}x + \frac{3\lambda}{2}\tau\right]} \tag{45}$$

can be identified from the above family of solutions by choosing the functions $C_1 = c_1$, $C_i = c_i \exp\left(\frac{3\lambda}{4}(\tau - t)\right)$, $i = 2, 3$ (c_i are arbitrary constants). Interestingly, the exact solution (45) with $c_3 = 0$ and $c_1 c_2 > 0$ is a TW if one considers τ as an analog of time (the same occurs in the case $c_2 = 0$ and $c_1 c_3 > 0$).

Fig. 1 3D plot of the function $U(t, \tau)$ using formula (46) with $x_0 = 1, \lambda = 1/8, c_1 = 25, c_2 = 1/10$



Example. Let us examine the properties of the exact solution (44) in the domain $G = \{(t, \tau, x) \in (0, +\infty) \times (0, \tau_{\max}) \times (0, x_0)\}$. The total number of infected individuals of the age τ up to time t is calculated by the formula

$$U(t, \tau) = \int_0^{x_0} u(t, x, \tau) dx = \sqrt{2} \ln \left(\frac{C_2 e^{\frac{\sqrt{2\lambda}}{2} x_0} + C_3 e^{-\frac{\sqrt{2\lambda}}{2} x_0} + C_1 e^{-\frac{3\lambda(t+\tau)}{4}}}{C_2 + C_3 + C_1 e^{-\frac{3\lambda(t+\tau)}{4}}} \right).$$

In order to provide meaningful interpretation for modelling of an epidemic, one requires that the function $U(t, \tau)$ is nonnegative, nondecreasing and bounded for all $t > 0$ and $\tau \in (0, \tau_{\max})$. These requirements can be easily satisfied taking into account arbitrariness of C_1, C_2 and C_3 . Setting, for example, $C_1 = c_1 e^{-\frac{3\lambda(t-\tau)}{8}}, C_2 = c_2$ and $C_3 = 0$ ($c_1 > 0$ and $c_2 > 0$ are arbitrary constants), the function $U(t, \tau)$ takes the form

$$U(t, \tau) = \sqrt{2} \ln \left(\frac{c_2 e^{\frac{\sqrt{2\lambda}}{2} x_0} + c_1 e^{-\frac{3\lambda(3t+\tau)}{8}}}{c_2 + c_1 e^{-\frac{3\lambda(3t+\tau)}{8}}} \right). \tag{46}$$

This function is nonnegative, nondecreasing, and bounded for any positive parameters λ, x_0, c_1 and c_2 . An example of the $U(t, \tau)$ plot of is presented in Fig. 1 (we remind the reader that equation (39) is presented in the nondimensional form, see formulae (9)). It can be easily seen that the function $U(t, \tau)$ is increasing with time and this is in agreement with its meaning, i.e. the total number of the infected individuals of the age τ should grow with time. Moreover, the total number is growing with the age τ as well because typically older individuals are more affected by epidemics than younger.

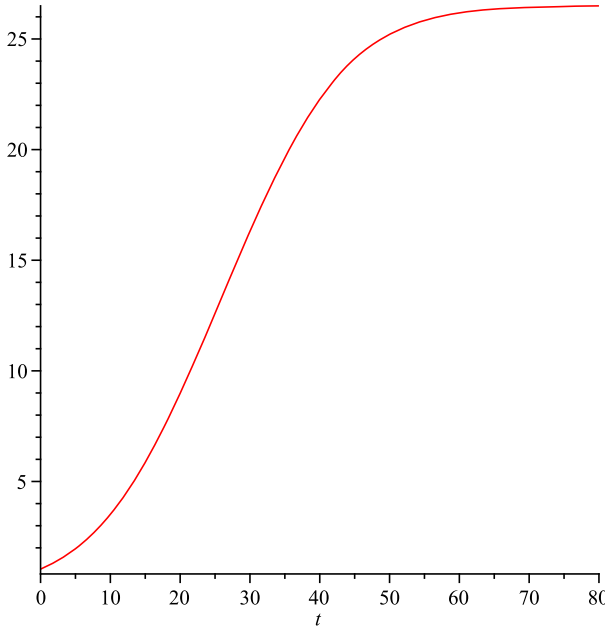


Fig. 2 Plot of the function $U_{\text{total}}(t)$ using formula (47) with $x_0 = 1$, $\tau_{\text{max}} = 75$, $\lambda = 1/8$, $c_1 = 25$, $c_2 = 1/10$

The total number of infected individuals up to time t is calculated using formula (6), hence we obtain :

$$U_{\text{total}}(t) = \sqrt{2} \int_0^{\tau_{\text{max}}} \ln \left(\frac{c_2 e^{\frac{\sqrt{2\lambda}}{2} x_0} + c_1 e^{-\frac{3\lambda(3t+\tau)}{8}}}{c_2 + c_1 e^{-\frac{3\lambda(3t+\tau)}{8}}} \right) d\tau. \tag{47}$$

The integral arising in (47) can be expressed in the terms of the special function dilogarithm, however the formula obtained is rather cumbersome and is omitted here. The relevant plot was drawn and is presented in Fig. 2. Interestingly, the curve presented in Fig. 2 has the form that is typical for modelling of the total number of infected individuals during each wave of COVID-19 pandemic (see, e.g., examples in [5]). Notably such curve is often called sigmoid. It should be stressed that the curves obtained from official data in many countries qualitatively have the form of sigmoid as well.

4.2 Lie Solutions

Now we construct exact solutions of equation (39) using Lie symmetries presented in Theorem 2. Depending on the function μ , there are several cases when equation (39) admits additional Lie symmetries (see Table 1). Because this function means the death rate and should satisfy natural restrictions, in particular $\mu > 0$ and $\frac{d\mu}{dt} > 0$, let

us consider Case 5 of Table 1. From the applicability point of view, one expects that μ involves a parameter because the death rate cannot be fixed, therefore we introduce $\alpha > 0$ using ET (34). Thus, our aim is to solve the equation

$$u_t + u_\tau = u_{xx} - \alpha \tanh \frac{\alpha\tau}{2} u - u^2 \tag{48}$$

using its Lie symmetry

$$X = F^1(t - \tau)\partial_t + F^5(t - \tau)\partial_\tau + G(t - \tau)\partial_x - \frac{\alpha^2 F^5(t - \tau)}{4} \operatorname{sech}^2 \frac{\alpha\tau}{2} \partial_u. \tag{49}$$

Applying transformation (41) to (48)–(49), we arrive at the equation

$$u_{t^*} = u_{xx} - \alpha \tanh \frac{\alpha(t^* - \tau^*)}{2} u - u^2. \tag{50}$$

Thus, according to Case 5 of Table 1 (taking into account ET (34)!) the most general form of Lie symmetry of the above equation reads as

$$\begin{aligned} X^* = & \frac{F^1(\tau^*) + F^5(\tau^*)}{2} \partial_{t^*} + \frac{F^1(\tau^*) - F^5(\tau^*)}{2} \partial_{\tau^*} + G(\tau^*)\partial_x \\ & - \frac{\alpha^2}{4} F^5(\tau^*) \operatorname{sech}^2 \frac{\alpha(t^* - \tau^*)}{2} \partial_u, \end{aligned} \tag{51}$$

where F^1 , F^5 and G are arbitrary functions. Different reductions of (50) to ODEs can be derived depending on these functions. A complete list of the reductions and their analyses lies beyond the scopes of this study. Here we present an interesting particular case.

Let us consider a particular case of (51), setting $F^1 = F^5 = 1$, $G = g = \text{const}$, that is

$$X^* = \partial_{t^*} + g\partial_x - \frac{\alpha^2}{4} \operatorname{sech}^2 \frac{\alpha(t^* - \tau^*)}{2} \partial_u. \tag{52}$$

So, the ansatz generated by the Lie symmetry operator (52) is

$$u = \Phi(\omega_1, \omega_2) - \frac{\alpha}{2} \tanh \frac{\alpha(t^* - \tau^*)}{2}, \quad \omega_1 = \tau^*, \quad \omega_2 = x - gt^*. \tag{53}$$

Substituting ansatz (53) into equation (50), we obtain the second-order ODE

$$\Phi_{\omega_2\omega_2} + g \Phi_{\omega_2} - \Phi^2 + \frac{\alpha^2}{4} = 0,$$

which takes the form

$$\Phi_{\omega_2\omega_2} + g \Phi_{\omega_2} + \Phi(\alpha - \Phi) = 0, \tag{54}$$

by applying the substitution

$$\Phi \rightarrow \Phi - \frac{\alpha}{2}. \tag{55}$$

Equation (54) corresponds to the known ODE that arises when one seeks for TWs of the famous Fisher equation

$$u_t = u_{xx} + u(\alpha - u),$$

using the ansatz $u = \Phi(\omega_2)$, $\omega_2 = x - gt$. The well-known exact solution of equation (54) has the form

$$\Phi = \frac{\beta^2}{\left(1 + C_1(\omega_1) e^{\frac{\beta}{\sqrt{6}}\omega_2}\right)^2}, \tag{56}$$

in the case $\alpha = \beta^2$, $g = \frac{5\beta}{\sqrt{6}}$, and the form

$$\Phi = \frac{\beta^2}{\left(1 + C_1(\omega_1) e^{\frac{\beta}{\sqrt{6}}\omega_2}\right)^2} - \beta^2, \tag{57}$$

in the case $\alpha = -\beta^2$, $g = \frac{5\beta}{\sqrt{6}}$. Here $C_1(\omega_1)$ is an arbitrary smooth function. In particular, setting $C_1 = 1$ in (56), one obtains the well-known TW of the Fisher equation that was identified for the first time in [1].

Thus, taking into account (41), (53), (55), and (56)–(57) and applying the translation $t \rightarrow t + \tau_0$, $\tau \rightarrow \tau - \tau_0$, we arrive at the exact solution

$$u = \beta^2 \left[1 + C_1(t - \tau) \exp\left(\frac{\beta}{\sqrt{6}}\left(x - \frac{5\beta}{\sqrt{6}}\frac{t + \tau}{2}\right)\right) \right]^{-2} - \frac{\beta^2}{2} \tanh \frac{\beta^2}{2} (\tau - \tau_0) - \frac{\beta^2}{2} \tag{58}$$

of equation

$$u_t + u_\tau = u_{xx} - \beta^2 \tanh \frac{\beta^2(\tau - \tau_0)}{2} u - u^2,$$

where β and τ_0 are arbitrary constants.

Examples of solution (58) are presented in Fig. 3. One may conclude that the exact solution is an increasing function of the time t and the age τ . Such behaviour is

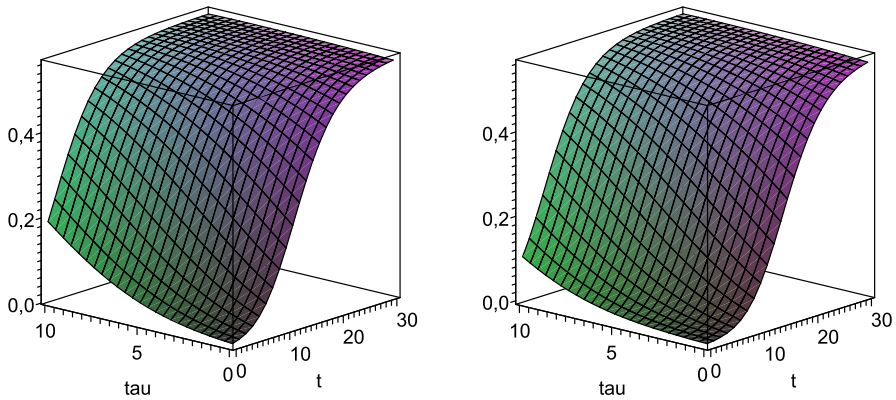


Fig. 3 3D plots of the exact solution $u(t, \tau, x)$ (58) for the fixed points $x = -1$ (left) and $x = 1$ (right). The parameters are fixed as follows: $\beta = 3/4$, $C_1 = 10$, $\tau_0 = 20$

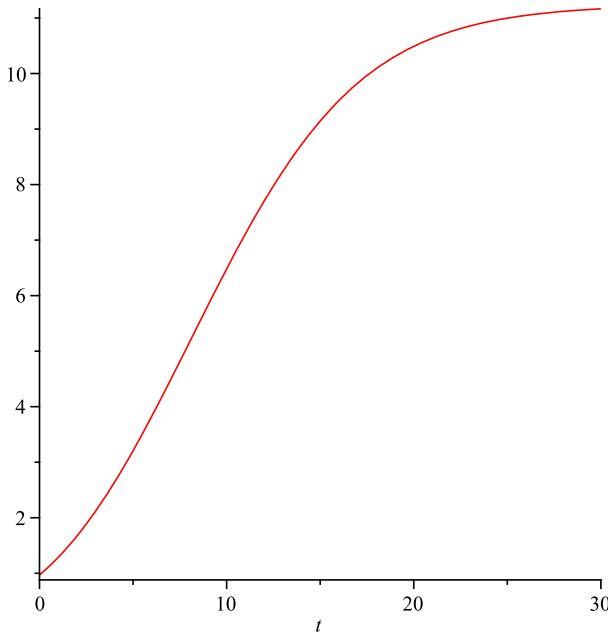


Fig. 4 Plot of the function $U_{\text{total}}(t)$ (see formula (6) with $I = [0, 10]$ and $\Omega = [-1, 1]$) derived using the exact solution (58) with $\beta = 3/4$, $C_1 = 10$, $\tau_0 = 20$

in agreement with the biological interpretation because u means the total density of infected individuals during the epidemic spread. In Fig. 4, the curve illustrating the total number of infected population is drawn. The function $U_{\text{total}}(t)$ was built using formula (6) with $I = [0, 10]$ and $\Omega = [-1, 1]$. As one may easily note, the curve is again a sigmoid.

To the best of our knowledge, the general solution of equation (54) is unknown provided parameters g and α are arbitrary. In the case $g = \frac{5\beta}{\sqrt{6}}$, the general solution of this equation has the form:

$$\Phi = \begin{cases} 6 e^{-\frac{2\beta}{\sqrt{6}} \omega_2} \wp \left(-\frac{\sqrt{6}}{\beta} e^{-\frac{\beta}{\sqrt{6}} \omega_2} + C_1(\omega_1); 0, C_2(\omega_1) \right), & \alpha = \beta^2, \\ 6 e^{-\frac{2\beta}{\sqrt{6}} \omega_2} \wp \left(-\frac{\sqrt{6}}{\beta} e^{-\frac{\beta}{\sqrt{6}} \omega_2} + C_1(\omega_1); 0, C_2(\omega_1) \right) - \beta^2, & \alpha = -\beta^2, \end{cases} \tag{59}$$

where \wp is the Weierstrass function, C_1 and C_2 are arbitrary smooth functions.

In the case $g = 0$, the general solution of equation (54) is known in the form :

$$\Phi = 6\wp \left(\omega_2 + C_1(\omega_1); \frac{\alpha^2}{12}, C_2(\omega_1) \right) + \frac{\alpha}{2},$$

therefore the exact solution

$$u(t^*, \tau^*, x) = 6\wp \left(x + C_1(\tau^*); \frac{\alpha^2}{12}, C_2(\tau^*) \right) - \frac{\alpha}{2} \tanh \frac{\alpha(t^* - \tau^*)}{2}$$

of equation (50) identified.

Finally, applying substitution (41), one obtains the solution

$$u(t, \tau, x) = 6\wp \left(x + C_1(t - \tau); \frac{\alpha^2}{12}, C_2(t - \tau) \right) - \frac{\alpha}{2} \tanh \frac{\alpha \tau}{2} \tag{60}$$

of the initial equation (48). Notably, setting $C_2 = \frac{\alpha^3}{216}$, solution (60) can be expressed in terms of elementary functions:

$$u(t, \tau, x) = \frac{\alpha}{2} \left(-1 + 3 \csc^2 \left[\frac{\sqrt{\alpha}}{2} x + C(t - \tau) \right] - \tanh \frac{\alpha \tau}{2} \right),$$

where C is an arbitrary smooth function.

Remark 2 All exact solutions presented in this subsection can be identified from the known solutions of the Fisher equation by a chain of substitutions. In fact, equation (48) is related with the Fisher equation via substitution $u^* = u + \alpha - \frac{\alpha}{1+e^{\alpha\tau}}$ and (41).

5 Conclusions

In this study, the age-structured diffusive model with the governing equation (5) is suggested for the mathematical modelling of epidemics. The model is a generalization of the known age-structured model based on the linear equation (3). To construct the model, the recently developed model [6] was used. A further generalization is

suggested in order to take into account space dependence of the parameters describing the effectiveness of the government restrictions (quarantine rules) and the virus transmission mechanism.

The LSC problem for the nonlinear equation (5) was solved. As a result, eleven inequivalent cases (up to equivalence transformations) were identified depending on the death rate $\mu(\tau)$ and the parameter γ , which is related to restrictions introduced by authorities during the epidemic spread. As follows from Table 1, the extensions of the principal algebra (23) are highly nontrivial and cannot be intuitively predicted (excepting Cases 1–3). We want to stress that all the Lie algebras obtained are infinite-dimensional because the Lie symmetry operators involve arbitrary functions of the variable $t - \tau$. It is rather unusual situation because Lie algebras of invariance of nonlinear evolution equations arising in real world applications typically are finite-dimensional. There are not many exceptions and probably the best known one is the nonlinear fast diffusion equation

$$u_t = (u^{-1}u_x)_x + (u^{-1}u_y)_y,$$

admitting an infinite-dimensional Lie algebra, which was firstly identified in [30]. Several other examples concern nonlinear systems of PDEs (not single equations!), including the classical Navier–Stokes system. Very recently, it was proved that two-component nonlinear evolution systems related to Ricci flows admit an infinite-dimensional Lie algebra [8, 26].

In order to construct exact solutions, the simplest form, $\mu(\tau) = \text{const}$, and the form arising in Case 5 of Table 1 were used. Notably, the latter form is plausible because one takes into account the typical behaviour of the function $\mu(\tau)$ (the death rate increases with age). The TW type solutions were constructed in the case $\mu(\tau) = \text{const}$. In particular, arbitrary functions arising in the exact solution (44) were specified in order to obtain the total number of infected individuals with meaningful interpretation for modelling of an epidemic (see Figs. 1 and 2). In Case 5, another type of exact solutions was constructed. It was demonstrated that some solutions with the correctly-specified parameters produce qualitatively correct distributions of infected population depending on time and age (see Figs. 3 and 4).

In a forthcoming paper, we are going to construct exact solutions in the case of two space variables, which is the most important from applicability point of view.

A Details for the Proof of Theorem 1

To simplify calculations, we consider the case of two space variables $(x_1, x_2) = (x, y)$. Let us show that the Lie algebra generated by (14) with the coefficients (22), generates the infinite-parameter Li group (13).

Formulae (22) with two space variables take the form

$$\begin{aligned} \xi^t &= 2\beta t + F^1(t - \tau), \quad \xi^\tau = 2\beta\tau + \beta_0, \\ \xi^x &= \beta x + H^{12}(t - \tau)y + G^1(t - \tau), \quad \xi^y = \beta y - H^{12}(t - \tau)x + G^2(t - \tau), \\ \eta &= -\frac{2\beta}{\gamma}u, \quad \zeta = 2\beta(1 - \mu). \end{aligned} \quad (61)$$

The Lie symmetry corresponding to the parameter β has the form

$$X_1 = 2t\partial_t + 2\tau\partial_\tau + x\partial_x + y\partial_y - \frac{2}{\gamma}u\partial_u + 2(1 - \mu)\partial_\mu.$$

So, according to the well-known algorithm, one needs to solve the initial problem

$$\begin{aligned} \frac{dt^*}{d\varepsilon_1} &= 2t^*, \quad \frac{d\tau^*}{d\varepsilon_1} = 2\tau^*, \quad \frac{dx^*}{d\varepsilon_1} = x^*, \quad \frac{dy^*}{d\varepsilon_1} = y^*, \quad \frac{du^*}{d\varepsilon_1} = -\frac{2}{\gamma}u^*, \quad \frac{d\mu^*}{d\varepsilon_1} = 2(1 - \mu^*), \\ t^*(0) &= t, \quad \tau^*(0) = \tau, \quad x^*(0) = x, \quad y^*(0) = y, \quad u^*(0) = u, \quad \mu^*(0) = \mu. \end{aligned} \tag{62}$$

Hereafter ε with a lower subscript means a group parameter. All the equations in (62) are simple linear ODEs, therefore one easily obtains:

$$t^* = te^{2\varepsilon_1}, \quad \tau^* = \tau e^{2\varepsilon_1}, \quad x^* = xe^{\varepsilon_1}, \quad y^* = ye^{\varepsilon_1}, \quad u^* = ue^{-\frac{2}{\gamma}\varepsilon_1}, \quad \mu^* = (\mu - 1)e^{-2\varepsilon_1} + 1, \tag{63}$$

where the group parameter ε_1 can be replaced by $\alpha = e^{\varepsilon_1} > 0$.

The Lie symmetry corresponding the parameter β_0 has the form $X_2 = \partial_\tau$. So, the corresponding Lie group is

$$t^* = t, \quad \tau^* = \tau + \varepsilon_2, \quad x^* = x, \quad y^* = y, \quad u^* = u, \quad \mu^* = \mu. \tag{64}$$

All other Lie symmetries generated by (14) with the coefficients (61) involve arbitrary functions. Consider the simplest symmetry

$$X_F = F^1(t - \tau)\partial_t,$$

where $F^1 \neq 0$ is an arbitrary smooth function. For each function F^1 , we need to solve the initial problem

$$\frac{dt^*}{d\varepsilon_F} = F^1(t^* - \tau^*), \quad t^*(0) = t.$$

Obviously $\tau^* = \tau$, because $\frac{d\tau^*}{d\varepsilon_F} = 0$ and $\tau^*(0) = \tau$, therefore

$$H(t^* - \tau) = \varepsilon_F + C,$$

i.e.

$$t^* = \tau + H^{-1}(\varepsilon_F + C),$$

where H is a primary function for $\frac{1}{F^1}$ and H^{-1} is the inverse function to H . Finally, using the initial condition, one obtains $C = H(t - \tau)$, hence

$$t^* = \tau + H^{-1}(\varepsilon_F + H(t - \tau)).$$

Because there is an infinite number of functions F^1 , the above expression can be rewritten as $t^* = t + T(t - \tau)$, where $H^{-1}(\varepsilon_F + H(t - \tau)) = T(t - \tau) + t - \tau$. Simultaneously, the restriction $T(t - \tau) \neq -(t - \tau)$ springs up. Thus, the following group of ET involving an arbitrary function $T(t - \tau)$ is derived:

$$t^* = t + T(t - \tau), \tau^* = \tau, x^* = x, y^* = y, u^* = u, \mu^* = \mu. \quad (65)$$

In a quite similar way, the equivalence transformations corresponding to the Lie symmetries

$$X_{G1} = G^1(t - \tau)\partial_x, \quad X_{G2} = G^2(t - \tau)\partial_y$$

and

$$X_H = H^{12}(t - \tau)y\partial_x - H^{12}(t - \tau)x\partial_y,$$

were constructed. As a result, the following ETs were derived

$$t^* = t, \tau^* = \tau, x^* = x + \varepsilon_{G1}G^1(t - \tau), y^* = y, u^* = u, \mu^* = \mu; \quad (66)$$

$$t^* = t, \tau^* = \tau, x^* = x, y^* = y + \varepsilon_{G2}G^2(t - \tau), u^* = u, \mu^* = \mu; \quad (67)$$

and

$$\begin{aligned} t^* &= t, \tau^* = \tau, x^* = x \cos(\varepsilon_H H^{12}(t - \tau)) + y \sin(\varepsilon_H H^{12}(t - \tau)), \\ y^* &= -x \sin(\varepsilon_H H^{12}(t - \tau)) + y \cos(\varepsilon_H H^{12}(t - \tau)), u^* = u, \mu^* = \mu. \end{aligned} \quad (68)$$

Finally, taking a superposition of of ETs (63)–(68) and introducing new notations for arbitrary functions, we arrive at the infinite-parameter Li group (13) in the case of two space variables.

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Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors contributed equally to this work and declare no competing interests.

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