УДК 513.6

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UNITARY REPRESENTATIONS OF THE THREE DIMENSIONAL UNITRIANGULAR POLINOMIAL TRANSFORMATION GROUP

A series of representations of a group of unitriangular polynomial transformations of affine space is constructed[1].

Let U_n be a subgroup of triangular polynomial transformations of the affine Cremona group GA_n – the group of biregular transformations of affine space $A_n = A_n(\mathbb{F})$ over a field $\mathbb{F}$ of the zero characteristic. The elements of U_3 can be represented in the form of tuples

$$\langle x_1 + a_1, x_2 + a_2(x_1), x_3 + a_3(x_1, x_2)\rangle, \quad (1)$$

where $a_1 \in \mathbb{F}$ and $a_2(x_1)$, $a_3(x_1, x_2)$ are polynomials.

As shown in [1, 2], GA_n has the structure of an ∞-dimensional algebraic group and U_n is a closed subgroup of GA_n (in the ∞-Zarissky topology). Remark that U_3 can be considered as iterated wreath algebraic product $\mathbb{F}^+ wr_a(\mathbb{F}^+ wr_a \mathbb{F}^+)$, where algebraic means that we use only polynomial functions as elements of the base of such kind of wreath product. As for finite dimensional algebraic groups Lie algebra $\mathrm{Lie}(GA_3)$ can be defined as algebra of derivations of the polynomial algebra which have the form

$$a_1(x_1, x_2, x_3)\frac{\partial}{\partial x_1} + a_2(x_1, x_2, x_3)\frac{\partial}{\partial x_2} + \\ + a_3(x_1, x_2, x_3)\frac{\partial}{\partial x_3};$$

$\mathrm{Lie}(U_n)$ consists of derivations of the form

$$c_1\frac{\partial}{\partial x_1} + c_2(x_1)\frac{\partial}{\partial x_2} + c_3(x_1, x_2)\frac{\partial}{\partial x_3}. \quad (2)$$

Remark that U_3 has the infinite analog of an upper central series. To describe this series one should consider an inverse lexicographical ordering on the polynomial algebra $\mathbb{F}[x_2, x_3]$, i.e. $x_2 \prec x_3$. Let $\mathbf{m} = x_2^{\alpha} x_3^{\beta}$ be a monomial then we have the series of subgroups

$$F_{\mathbf{m}} = \{\langle x_1, x_2, x_3 + a(x_2, x_3)\rangle \mid a(x_2, x_3) \prec \mathbf{m}\},$$

in particular $F_{x_1} = \langle x_1, x_2, x_3 + c\rangle$, $c \in \mathbb{F}$ is a center of U_3,

$$\{h \in U_n \mid \forall g \in U_n, [g, h] \in F_{\mathbf{m}}\} = F_{\mathbf{m}^+},$$

where $[g, h]$ is a commutator in the group U_n, and $\mathbf{m}^+$ is the next monomial ($\mathbf{m} \prec \mathbf{m}^+$). Moreover, for the subgroup

$$F(x_2, x_3) = \\ = \{\langle x_1, x_2, x_3 + a_3(x_1, x_2)\rangle \mid a_3 \in \mathbb{F}[x_2, x_3]\}$$

we get $F(x_2, x_3) = \cup_{\mathbf{m}} F_{\mathbf{m}}$. The series can be prolonged in such manner:

$$H_k = \{\langle x_1, x_2 + a_2(x_1), x_3 + a_3(x_2, x_3)\rangle \mid \\ \deg a_2(x_1) < k\} \supseteq F(x_2, x_3),$$

$$\{h \in U_n \mid \forall g \in U_n, [g, h] \in H_k\} = H_{k+1}.$$

Thus U_3 is a locally nilpotent ∞-dimensional algebraic group. To take it into account it is naturally to generalize the Kirillov's method of describing unitary representations of nilpotent Lie groups (see [3]) on the class of these locally nilpotent groups.

Remember that in accordance of the Kirillov's method we should to construct a representation of the Lie algebra $\mathrm{Lie}(U_3)$ by skew-Hermitian operators on a space of functions. For a correspondent Lie algebra $\mathrm{Lie}(U_n)$ we have the structure of a locally nilpotent Lie algebra. Let

$$Z = \{c\frac{\partial}{\partial x_3} \mid c \in \mathbb{F}\}, \quad Y = \{cx_1\frac{\partial}{\partial x_3} \mid c \in \mathbb{F}\}$$

[1] Робота частково підтримана Державним фондом фундаментальних досліджень Ф25./157-2008 № ДР 0107U010499 та Міжнародним благодійним фондом відродження Києво-Могилянської академії.

be a one-dimensional center of $\mathrm{Lie}(U_n)$, and $Y : Z+Y = \mathrm{Lie}(F_{x_1^2})$ is the second two-dimensional hyper center. Let us put

$$X = \{c\frac{\partial}{\partial x_1} | c \in \mathbb{F}\},$$
$$W = \left\{a_2(x_1)\frac{\partial}{\partial x_2} + a_3(x_1,x_2)\frac{\partial}{\partial x_3}\right\} \setminus (Z+Y)$$

then we have a decomposition $\mathrm{Lie}(U_n) = X+Y+ +Z+W$ as a vector space with a natural counting basis:

$$\epsilon = \frac{\partial}{\partial x_1}, \quad \varepsilon(k) = x_1^k\frac{\partial}{\partial x_2},$$
$$\xi(k,l) = x_1^k x_2^l \frac{\partial}{\partial x_3}, k,l = 0,1,2\ldots.$$

In accordance with [3], we should describe representations of the Lie subalgebra $L_0 = Y+ +Z+W$ and obtain ones for all $\mathrm{Lie}(U_n)$ by the induction procedure.

Basis elements of the subalgebra L_0 satisfies next conditions:

$$[\varepsilon(k), \varepsilon(k')] = 0, \quad (3)$$
$$[\xi(r_1,r_2), \xi(r_1',r_2')] = 0, \quad (4)$$
$$[\varepsilon(k), \xi(r_1,r_2)] = r_2\xi(r_1+k, r_2-1), \quad (5)$$

In particular, $[T_\epsilon, T_{\xi(1,0)}] = T_{\xi(0,0)} = i\lambda E$. For $\lambda \neq 0$ let us put $iT_\epsilon = P$, $\frac{1}{i\lambda}T_{\xi(1,0)} = Q$, then we get a well known operator's equality

$$PQ - QP = iE.$$

In accordance to the Stoune-Von Neuman theorem (see [3]) all irreducible representations of such pairs can be describing as linear operators

$$P = i\frac{\partial}{\partial t}, \qquad Q = t,$$

i.e. operators of derivation and multiplication by t which act on the appropriate space of functions $f(t)$. It follows that $T_\epsilon = \frac{\partial}{\partial t}$, $T_{\xi(1,0)} = i\lambda t$.

Let us introduce series of representations of L_0, which act on the next basis elements in such a manner:

$$T_{\varepsilon(k)} = Ab^k\frac{\partial}{\partial t}, \quad (6)$$
$$T_{\xi(r_1,0)} = b^{r_1}c_0, \quad (7)$$
$$T_{\xi(0,1)} = c_1 t, \quad (8)$$

where A, b, c_1 are parameters.

Let us choose the space of polynomials $\mathbb{C}[t]$ as a representation infinite dimensional one. Then an action of linear operators on this space, in particular $T_{\xi(r_1,r_2)}$ can be described in such a manner:

$$T_{\xi(r_1,r_2)} = \sum_{i=0}^{\infty} c_{r_1,r_2}^i(t)\frac{\partial^i}{\partial t^i}, \; c_{r_1,r_2}^i(t) \in K[t]. \quad (9)$$

In accordance with (4) we get

$$[T_{\xi(r_1,r_2)}, T_{\xi(0,1)}] = 0, \quad (10)$$
$$[T_{\xi(r_1,r_2)}, t] = 0, \quad (11)$$
$$[T_{\xi(r_1,r_2)}, t] = \sum_{i=0}^{\infty} c_{r_1,r_2}^{i+1}(t)(i+1)\frac{\partial^i}{\partial t^i} = 0. \quad (12)$$

Thus, for all i

$$c_{r_1,r_2}^{i+1}(t) = 0;$$

so we obtain

$$T_{\xi(r_1,r_2)} = c_{r_1,r_2}^0(t), \forall r_1, r_2 \geq 0. \quad (13)$$

In accordance to (5) the condition should be satisfied:

$$[T_{\varepsilon(k)}, T_{\xi(r_1,r_2)}] = r_2 T_{\xi(r_1+k, r_2-1)},$$

therefore

$$[Ab^k\frac{\partial}{\partial t}, c_{r_1,r_2}^0(t)] = r_2 c_{r_1+k,r_2-1}^0(t),$$
$$Ab^k\frac{\partial c_{r_1,r_2}^0(t)}{\partial t} = r_2 c_{r_1+k,r_2-1}^0(t).$$

If one puts $f(k) = Ab^k$,

$$g(r_1) = \frac{\partial c_{r_1,r_2}^0(t)}{\partial t}, \; h(r_1) = r_2 c_{r_1+k,r_2-1}^0(t),$$

one gets the equalities

$$c_{r_1,r_2}^0(t) = b^{r_1}c_{0,r_2}^0(t),$$
$$f(0)g(0) = h(0)$$

and a differential equation

$$A\frac{\partial c_{0,r_2}^0(t)}{\partial t} = r_2 c_{0,r_2-1}^0(t),$$

which solutions can be easily obtained:

$$c^0_{0,r_2}(t) = \sum_{j=0}^{r_2} d_j \binom{r_2}{j} (\frac{t}{A})^{r_2-j}, \ d_j \in K, \quad (14)$$

here $\binom{r_2}{j}$ is a binomial coefficient.

So, for abelian subalgebra $\xi(r_1, r_2)$ we get:

$$T_{\xi(r_1,r_2)} = b^{r_1} \sum_{j=0}^{r_2} d_j \binom{r_2}{j} (\frac{t}{A})^{r_2-j}. \quad (15)$$

To get the representations of the all algebra $\mathrm{Lie}(U_3)$ one should consider the representations which are induced by ones of L_0 constructed bellow.

Theorem 1. *1. There is a series of the representations of the subalgebra L_0 which are defined in such manner:*

$$T_{\varepsilon(k)} = Ab^k \frac{\partial}{\partial t},$$

$$T_{\xi(r_1,r_2)} = b^{r_1} \sum_{j=0}^{r_2} d_j \binom{r_2}{j} (\frac{t}{A})^{r_2-j}.$$

2. Via exponent map there is a series of representations of the normal subgroup

$$N_0 = \{\langle x_1, x_2 + a_2(x_1), x_3 + a_3(x_1, x_2)\rangle\},$$

$$g = \langle x_1, x_2 + x_1^k, x_3\rangle \to U_g : f(t) \to b^k f(t),$$

$$g = \langle x_1, x_2, x_3 + x_1^{r_1} x_2^{r_2}\rangle \to U_g : f(t) \to$$

$$\sum_m \frac{b^{mr_1}}{m!} \left(\sum_{j=0}^{r_2} d_j \binom{r_2}{j} (\frac{t}{A})^{r_2-j} \right)^m f(t).$$

There is an induced representation of U_3 on the space of functions in two variables: if

$$g = g_1 \cdot g_0, \ g_1 = \langle x_1 + a_1, x_2, x_3\rangle, \ g_0 \in N_0$$

then

$$U_g : f(u, t) \to U_{g_0} f(u + a_1, t).$$

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УНІТАРНІ ЗОБРАЖЕННЯ ТРИВИМІРНОЇ ГРУПИ УНІТРИКУТНИХ ПОЛІНОМІАЛЬНИХ ПЕРЕТВОРЕНЬ

Побудовано серію зображень тривимірної групи унітрикутних поліноміальних перетворень афінного простору.