ORDERED STATES AND NONLINEAR LARGE-SCALE EXCITATIONS IN A PLANAR MAGNET OF SPIN $s = 1$

J.M. BERNATSKA, P.I. HOLOD

UDC 538.221

National University "Kyiv-Mohyla Academy"
(2, Skovoroda Str., Kyiv 04070, Ukraine),

M.M. Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03143, Ukraine)

We study ordered states and topological excitations in a quasi-two-dimensional magnet modeled by a square lattice with spins $s = 1$ at all sites, and a Hamiltonian with the biquadratic exchange interaction between the nearest neighbor sites. We propose two effective Hamiltonians for the description of large-scale excitations in the strictly two-dimensional case. They describe excitations of the mean field in the nematic phase and the mixed ferromagnetic-nematic phase. It is shown that the effective Hamiltonians are minimized on configurations with fixed topological charge. These topological excitations can arise at low temperatures and cause the destruction of a long-range order in the strictly two-dimensional system.

1. Introduction

Quasi-two-dimensional magnets have various technological applications. They serve as magnetic films used for the recording of information, thin ferromagnetic layers in Josephson semiconductor junctions, layered resistive systems, *etc.*

Here, we will not deal with applied aspects of the theory of magnetism. However, we note that the study of two-dimensional systems has also a significant value. Studying the ordered states, their stability, and excitation spectra, we obtain model scenarios of the self-organization of a substance with decrease in a temperature or under the action of external fields. To support the above-presented assertion, it is worth to recall an important role played by the Onsager's results [1] on the two-dimensional Ising model, or the Kosterlitz–Thouless theory of topological phase transitions [2, 3]. This is also related to the study of the two-dimensional O(3)-sigma model or a planar Heisenberg magnet [4]. As a natural continuation of this trend, we mention numerous papers devoted to investigations of two-dimensional continuous or lattice systems with high spins at sites.

In the present paper, we consider a generalized Heisenberg magnet taking the bilinear and biquadratic interactions at the nearest-neighbor sites of a square or cubic lattice into account. The Hamiltonian for magnets of spin $s = 1$ was proposed and studied long ago [5, 6] without any restriction on dimensionality. At the beginning of the 1970s, the existence of ordered phases different from the ferromagnetic or antiferromagnetic one was established by the mean-field methods. In particular, if the constant of biquadratic interaction is larger than that of bilinear interaction, then a pure quadrupole ordering or a spin nematic state can be realized in the system [7, 8].

It is known that a two-dimensional system with a continuous group of symmetry has no long-range order at $T > 0$ (the Mermin–Wagner theorem). In many cases, the instability of ordered phases in two-dimensional systems implies the existence of nonlinear topological excitations caused by small fluctuations of the temperature. The role of such excitations in the destruction of a long-range order is proved within the model of plane rotators [2,3] and for the two-dimensional Heisenberg ferromagnet [4].

The main result of our work is the proof of the existence of topological excitations in the model with biquadratic interaction between the nearest spins $s = 1$ at sites of a square lattice. We will consider the boundary case of the nematic phase, where the constants of the bilinear and biquadratic interactions are identical. It is known that, in this case, the energies of both possible phases (nematic and ferromagnetic-quadrupole ones) are equal. In order to study excitations of the nematic phase, we assume that $K - J = \varepsilon$, and ε is a small positive value. It is obvious that topological excitations exist at these parameters and differ slightly from those in the case of $\varepsilon = 0$. If $\varepsilon > 0$, the manifold of degeneration of the ground state of the system is deformed, but the topology is not sensitive to smooth deformations. Therefore, our conclusion about the existence of topological excitations at $J = K$ remains valid also in the case of $K > J > 0$.

The present paper contains two parts. The first part is a survey. In the mean-field approximation, we reveal the existence conditions for ordered phases and solve the self-consistent relations for order parameters. Comparing with results of other researchers on this topic, we obtain the conditions of occurrence of the nematic state. In the second part, by averaging the equations of motion over coherent states and by passing from a plane square lattice to a continuous plane, we obtain formulas for the free energy in the case of an inhomogeneous distribution of the mean field. Topological excitations are described in terms of the inhomogeneous distribution. Depending on a choice of the equilibrium state and a way of the averaging, we get two formulas for the free energy: the first corresponds to excitations of the pure nematic state, and the second is related to excitations of a state with nonzero magnetization and quadrupole moment. In the case of SU(3)-symmetry, we obtain self-dual solutions of the problem of minimization. The obtained topological excitations give the absolute minimum for the free energy, and its value is proportional to a topological charge.

2. The Quantum Model of a Planar Magnet

Let us consider a plane square lattice containing atoms of spin s at each site. Each atom is assigned by three spin operators $\{\hat{S}^1_n, \hat{S}^2_n, \hat{S}^3_n\}$ obeying the standard commutation relations

$$[\hat{S}^\alpha_n, \hat{S}^\beta_n] = i\varepsilon^{\alpha\beta\gamma}\hat{S}^\gamma_n\delta_{nm},$$

where the indices α, β, and γ run the values $\{1, 2, 3\}$ for each site n, and δ_{nm} is the Kronecker symbol.

As usual, such a system is described by the Heisenberg Hamiltonian. As $s \geqslant 1$, we can include higher orders of the exchange interaction in the Hamiltonian. In particular, magnets with the biquadratic interaction were studied in the 1970s [9, 10]. The relevant Hamiltonian will be considered in what follows. Let

$$\hat{\mathcal{H}} = -\sum_{n,\delta}\{J(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta}) + K(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta})^2\}, \qquad (1)$$

where $\hat{\boldsymbol{S}}_n$ denotes the vector $(\hat{S}^1_n, \hat{S}^2_n, \hat{S}^3_n)$ of the spin operators at site n, and δ runs over the nearest-neighbor sites. We assume that the exchange integrals J and K are positive, i.e. we consider mainly the ferromagnetic interaction.

The operators $\{\hat{S}^\alpha\}$ are defined over the $(2s+1)$-dimensional space of an irreducible representation of the group SU(2). The spin operators generate the complete matrix algebra over this space (the Burnside theorem). With respect to the adjoint action $\mathrm{ad}_{\hat{S}^\alpha}$, the complete matrix algebra is divided into a direct sum of irreducible collections of tensor operators. For example, let us consider the case of $s = 1$. Then, for the complete matrix algebra over the representation space, we have $\dim \mathrm{Mat}_{3\times3} \simeq [9] = [1] + [3] + [5]$. Obviously, the bases in three- and five-dimensional irreducible collections are formed, respectively, by the operators $\hat{S}^\alpha$ and by the tensor operators of weight 2. The latter are the operators of quadrupole moment chosen in the form

$$\hat{Q}^{\alpha\beta}_n = \hat{S}^\alpha_n\hat{S}^\beta_n + \hat{S}^\beta_n\hat{S}^\alpha_n,\ \alpha \neq \beta,$$

$$\hat{Q}^{[2,2]}_n = (\hat{S}^1_n)^2 - (\hat{S}^2_n)^2,$$

$$\hat{Q}^{[2,0]}_n = \sqrt{3}\big((\hat{S}^3_n)^2 - \tfrac{2}{3}\big).$$

A normalization of the operators $\hat{S}^\alpha$ is defined by the relation

$$(\hat{S}^1)^2 + (\hat{S}^2)^2 + (\hat{S}^3)^2 = s(s+1)\mathbb{I}_3,$$

which yields $\mathrm{Tr}(\hat{S}^\alpha)^2 = \frac{1}{3}\, s\,(s+1)(2s+1)$. For $s = 1$, we have $\mathrm{Tr}(\hat{S}^\alpha)^2 = 2$. We extend such a normalization for all other operators.

Now we fix the canonical basis $\{|{+1}\rangle, |{-1}\rangle, |0\rangle\}$ in the representation space. Then a matrix representation of the operators of spin and quadrupole moment is as follows:

$$\hat{S}^1_n = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 0\end{pmatrix},\ \hat{S}^2_n = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & i\\ i & -i & 0\end{pmatrix},$$

$$\hat{S}^3_n = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \hat{Q}^{[2,0]}_n = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix},$$

$$\hat{Q}^{12}_n = \begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ \hat{Q}^{13}_n = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -1\\ 1 & -1 & 0\end{pmatrix},$$

$$\hat{Q}^{23}_n = \frac{1}{\sqrt{3}}\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & -i\\ i & i & 0\end{pmatrix},\ \hat{Q}^{[2,2]}_n = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

It is easy to see that the proposed matrices are connected with the Gell-Mann matrices $\hat{\Lambda}_a$, $a = 1, 2, \ldots, 8$, which also form a basis in $i\mathfrak{su}(3)$. The connection is given by the linear transformations

$$\hat{S}^1_n = \tfrac{1}{\sqrt{2}}(\hat{\Lambda}_4 + \hat{\Lambda}_6),\ \hat{S}^2_n = \tfrac{1}{\sqrt{2}}(\hat{\Lambda}_5 - \hat{\Lambda}_7),\ \hat{S}^3_n = \hat{\Lambda}_3,$$

$\hat{Q}_n^{12} = \hat{\Lambda}_2, \quad \hat{Q}_n^{[2,0]} = \hat{\Lambda}_8, \quad \hat{Q}_n^{[2,2]} = \hat{\Lambda}_1,$

$\hat{Q}_n^{13} = \frac{1}{\sqrt{2}}(\hat{\Lambda}_5 + \hat{\Lambda}_7), \quad \hat{Q}_n^{23} = \frac{1}{\sqrt{2}}(\hat{\Lambda}_4 - \hat{\Lambda}_6).$

By $\{\hat{P}^a\}$, we denote the collection of operators $\{\hat{S}_n^1, \hat{S}_n^2, \hat{S}_n^3, \hat{Q}_n^{12}, \hat{Q}_n^{13}, \hat{Q}_n^{23}, \hat{Q}_n^{[2,2]}, \hat{Q}_n^{[2,0]}\}$. It is easy to prove that the commutation relations

$$[\hat{P}_n^a, \hat{P}_m^b] = iC_{abc}\hat{P}_n^c\delta_{nm}$$

hold true. Here, the tensor of structure constants C_{abc} is antisymmetric under a permutation of any pair of indices, and its nonzero components are

$$C_{123} = C_{145} = C_{167} = C_{264} = C_{257} = C_{356} = 1,$$

$$C_{168} = C_{528} = \sqrt{3}, \quad C_{437} = 2.$$

In terms of the operators of spin and quadrupole moment, Hamiltonian (1) takes the form

$$\hat{\mathcal{H}} = -\left(J - \tfrac{1}{2}K\right)\sum_{n,\delta}\sum_{\alpha}\hat{S}_n^\alpha, \hat{S}_{n+\delta}^\alpha - -\tfrac{1}{2}K\sum_{n,\delta}\sum_{a}\hat{Q}_n^a\hat{Q}_{n+\delta}^a - \tfrac{4}{3}KN, \quad (2)$$

where N denotes the total number of sites of the lattice. Obviously, the Hamiltonian remains SU(2)-invariant; hence, the operators $\hat{S}_n^\alpha$ and $\hat{Q}_n^a$ are transformed by formulas of the adjoint representation

$$\hat{U}\hat{S}_n^\alpha\hat{U}^{-1} = \sum_{\beta}\hat{D}^{\alpha\beta}(\hat{U})\hat{S}_n^\beta,$$

$$\hat{U}\hat{Q}_n^a\hat{U}^{-1} = \sum_{b}\hat{D}^{ab}(\hat{U})\hat{Q}_n^b, \quad \forall\hat{U}\in \mathrm{SU}(2),$$

where $\hat{D}^{\alpha\beta}(\hat{U})$ and $\hat{D}^{ab}(\hat{U})$ are matrices of the real irreducible representations of SU(2) with dimensions 3 and 5, respectively. If $K = J$, then the SU(2)-symmetry can be extended to the group SU(3). In this case, Hamiltonian (2) looks as

$$\hat{\mathcal{H}} = -\tfrac{1}{2}J\sum_{n,\delta}\sum_{a}\hat{P}_n^a\hat{P}_{n+\delta}^a - \tfrac{4}{3}JN. \quad (3)$$

To study the possible ordered phases of such a spin system, we use the mean-field approximation.

3. Mean-Field Approximation. Ordered States

Now we replace the interaction between spin and quadrupole operators that is described by Hamiltonian (2) by an effective interaction between the operators $\hat{P}_n^a$ and the classical mean field. Components of the mean field are considered proportional to averages (quasiaverages) of the quantum operators $\{\hat{P}_n^a\}$. The Hamiltonian in the mean-field approximation has the form

$$\hat{\mathcal{H}}_{\mathrm{MF}} = -(J - \tfrac{1}{2}K)\sum_{n,\delta}\sum_{\alpha}\hat{S}_n^\alpha\langle\hat{S}_{n+\delta}^\alpha\rangle - -\tfrac{1}{2}K\sum_{n,\delta}\sum_{a}\hat{Q}_n^a\langle\hat{Q}_{n+\delta}^a\rangle - \tfrac{4}{3}KN. \quad (4)$$

It is worth to give a warning that a direct calculation of the averages $\{\langle\hat{S}_n^\alpha\rangle\}$ and $\{\langle\hat{Q}_n^a\rangle\}$, for example by means of the density matrix $\hat{\rho}(T) = \exp\{-\frac{\mathcal{H}}{kT}\}$, results in the zero values. This follows from the SU(2)-symmetry of Hamiltonian (2). Nonzero values of the averages appear if the symmetry is broken. Symmetry breaking can be stimulated by an external field which vanishes after specifying an order in the magnetic system. The quantities calculated in this way are called "quasiaverages" [11].

Hence, we assume that the nonzero quasiaverages $\{\langle\hat{S}_n^\alpha\rangle\}$ and $\{\langle\hat{Q}_n^a\rangle\}$ exist in our system and form a classical 8-component vector field $\mu_a(x_n)$, $a = 1, 2, \ldots, 8$. In order to obtain nonzero values of $\{\langle\hat{Q}_n^a\rangle\}$, the external field must have nonzero gradient. If the mean field is homogeneous, the action of the group SU(2) transforms Hamiltonian (4) to

$$\hat{\mathcal{H}}_{\mathrm{MF}} = -(J - \tfrac{1}{2}K)\sum_{n}\hat{S}_n^3\langle\hat{S}^3\rangle - -\tfrac{1}{2}K\sum_{n}\hat{Q}_n^{[2,0]}\langle\hat{Q}^{[2,0]}\rangle - \tfrac{4}{3}KN = -\tfrac{4}{3}KN - -\sum_{n}\left\{(J - \tfrac{1}{2}K)\hat{S}_n^3\mu_3 + \tfrac{1}{2}K\hat{Q}_n^{[2,0]}\mu_8\right\}.$$

In the case of thermodynamic equilibrium and an infinite lattice, the fields $\mu_3 = \langle\hat{S}^3\rangle$ and $\mu_8 = \langle\hat{Q}^{[2,0]}\rangle$ are constant, i.e. they have the same values at all points x_n (a homogeneous mean field). These quantities serve as *order parameters*. Obviously, μ_3 describes a normalized magnetization (the ratio of the z-projection of a magnetic moment to the saturation magnetization), and μ_8 is analogously related to a quadrupole moment.

In the mean-field approximation, it is easy to calculate the partition function for the homogeneous case,

$$NZ(\mu_3, \mu_8, T) = \mathrm{Tr}\, e^{-\frac{\mathcal{H}_{\mathrm{MF}}}{kT}} = \mathrm{Tr}\, e^{-\frac{Nh_{\mathrm{MF}}}{kT}},$$

where $h_{\rm MF}$ denotes a one-site Hamiltonian

$$h_{\rm MF} = -(J - \tfrac{1}{2}K)\mu_3 \hat{S}^3 - \tfrac{1}{2}K\mu_8 \hat{Q}^{[2,0]} - \tfrac{4}{3}K. \qquad (5)$$

The introduced mean field makes sense if the following *self-consistent relations* are held:

$$\mu_3 = \langle \hat{S}^3 \rangle_{MF} = \frac{\mathrm{Tr}\,\hat{S}^3 e^{-\frac{N h_{\rm MF}}{kT}}}{\mathrm{Tr}\,e^{-\frac{N h_{\rm MF}}{kT}}},$$

$$\mu_8 = \langle \hat{Q}^{[2,0]} \rangle_{MF} = \frac{\mathrm{Tr}\,\hat{Q}^{[2,0]} e^{-\frac{N h_{\rm MF}}{kT}}}{\mathrm{Tr}\,e^{-\frac{N h_{\rm MF}}{kT}}}.$$

These relations serve as an analog of the Weiss equation from the theory of ferromagnetism. The averages of operators can be presented via the partition function:

$$\mu_3 = \frac{kT}{(J - \frac{K}{2})} \frac{\partial}{\partial \mu_3} \ln Z(\mu_3, \mu_8, T),$$

$$\mu_8 = \frac{2kT}{K} \frac{\partial}{\partial \mu_8} \ln Z(\mu_3, \mu_8, T).$$

For the system described by the one-site Hamiltonian (5), the self-consistent relations get the form

$$\mu_3 = \frac{2\,\mathrm{sh}\,\frac{(J-\frac{K}{2})\mu_3}{kT}}{\exp\Big\{-\frac{\sqrt{3}\,K\mu_8}{2kT}\Big\} + 2\,\mathrm{ch}\,\frac{(J-\frac{K}{2})\mu_3}{kT}},$$

$$\mu_8 = \frac{2}{\sqrt{3}} \frac{\mathrm{ch}\,\frac{(J-\frac{K}{2})\mu_3}{kT} - \exp\Big\{-\frac{\sqrt{3}\,K\mu_8}{2kT}\Big\}}{\exp\Big\{-\frac{\sqrt{3}\,K\mu_8}{2kT}\Big\} + 2\,\mathrm{ch}\,\frac{(J-\frac{K}{2})\mu_3}{kT}}. \qquad (6)$$

We note that the true averages are always less than their expectation values calculated from the self-consistent relations. Therefore, solutions of (6) have a qualitative sense only.

Here, we will analyze Eqs. (6) and make comparison with results described in the literature. The obvious solution corresponds to the paramagnetic state with $\mu_3 = 0$ and $\mu_8 = 0$; this state is realized at temperatures $kT > J - K/2$. At the same time, this inequality shows that the model oriented to ferromagnetic materials gives a meaningful result only in the region $J - K/2 < 0$ which contains areas with the ferromagnetic and quadrupole orderings, according to the well-known phase diagram (Fig. 1) for the bilinear-biquadratic $s = 1$ one-dimensional spin model [12].

In the case of $K < 0$, the self-consistent relations have a unique nontrivial solution corresponding to the ferromagnetic ordering, because this solution tends to the limiting values $\mu_3 = 1$ and $\mu_8 = \frac{1}{\sqrt{3}}$ as the temperature decreases to zero. The critical temperature

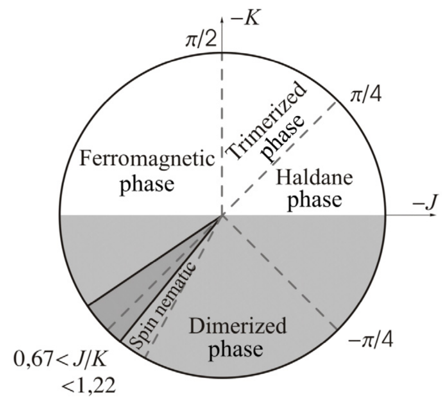

Fig. 1. Phase diagram of a one-dimensional system of spins $s = 1$

of the transition from a ferromagnetic state into a paramagnetic one is determined in terms of the constants J and K as $T_c = \frac{2}{3k}(J - K/2)$.

In the case of $K > 0$ (the light-grey region in Fig. 1), Eqs. (6) have more than one nontrivial solution: two solutions corresponding to ferromagnetic states with the boundary values $\mu_3^{(1)} = 1$ and $\mu_3^{(2)} = 2/3$ (and the corresponding values of μ_8) and two solutions describing nematic states ($\mu_3 = 0$) with the boundary values $\mu_8^{(1)} = \frac{-2}{\sqrt{3}}$ and $\mu_8^{(2)} = \frac{1}{\sqrt{3}}$. The existence of four ordered states in ferromagnets is also reported in [7]: they are a ferromagnetic state with $\mu_3^{(1)} = s$, a quadrupole (or nematic) state with $\mu_3 = 0$, $\mu_8^{(1)} = -s(s+1)/\sqrt{3}$, a partially ordered quadrupole state with $\mu_3 = 0$, $\mu_8^{(1)} > 0$, and a partially ordered ferromagnetic state with $\mu_3^{(1)} < s$. Partially ordered states are unstable [7].

Analyzing the temperature evolution of solutions of (6) as $K > 0$, $J > 0$, we revealed two critical temperatures which are solutions of the equation

$$2\Big(\frac{J - K/2}{kT} - 1\Big) = \exp\Big\{\frac{K}{kT}\Big(1 - \frac{3kT}{2(J - K/2)}\Big)\Big\}.$$

The obvious solution is $T_{c1} = \frac{2}{3k}(J - K/2)$. The other solution T_{c2} is calculated numerically. In the region $J > K$, i.e. for ferromagnetic materials, the temperature T_{c2} is less than T_{c1}, whereas the reversed situation takes place for nematics in the region $J < K$. At a smaller critical temperature, the solution $\mu_3^{(2)}$ disappears. Then only the solution $\mu_3^{(1)}$ exists in a certain interval of temperatures. The comparison with results of work [8] shows that, at a higher critical temperature, we have a second-order phase transition from a ferromagnetic state into a paramagnetic one.

A somewhat different behavior is observed in materials with $J \approx K$ or, more precisely, $0.67 < J/K < 1.22$, i.e. on the boundary between the ferromagnetic and nematic regions (the dark-grey region in Fig. 1). Disappearing at a lower critical temperature, the solution $\mu_3^{(2)}$ arises again at a higher critical temperature and grows continuously from zero toward $\mu_3^{(1)}$. When two solutions coincide at a certain temperature T_0, they disappear by jump. We may conclude that the phase transition from a ferromagnetic state to a paramagnetic one is a transition of the first order. This well agrees with results of work [8], where the change of a second-order phase transition into a first-order one in the region $2J/3 < K < J$ was considered (for ferromagnetic materials).

Further, we consider the case $J = K$ corresponding to the boundary between the ferromagnetic and quadrupole regions. Moreover, as $J = K$, the quantum Hamiltonian (2) and the Hamiltonian in the mean-field approximation are SU(3)-invariant. The latter has the form

$$\hat{\mathcal{H}}_{\mathrm{MF}} = -\tfrac{1}{2} J \sum_n \sum_a \hat{P}^a_n \langle \hat{P}^a \rangle - \tfrac{4}{3} JN =$$

$$= -\tfrac{1}{2} J \sum_n \sum_a \hat{P}^a_n \mu_a - \tfrac{4}{3} JN. \qquad (7)$$

4. Equations of Motion for Large-Scale Fluctuations of the Mean Field

We now return to the quantum SU(3)-invariant spin model with Hamiltonian (3). The Heisenberg evolution equations for the operators $\hat{P}^a_n$ have the form

$$i\hbar \frac{d\hat{P}^a_n}{dt} = [\hat{P}^a_n, \hat{\mathcal{H}}]. \qquad (8)$$

We assume that the system is in an ordered state and take the average of the both sides of Eq. (8) over the Heisenberg (time independent) coherent states

$$|\psi(n)\rangle = \frac{1}{\sqrt{N}} \big(c_1(n)|1\rangle + c_{-1}(n)|{-1}\rangle + c_0(n)|0\rangle \big),$$

$$|c_1|^2 + |c_{-1}|^2 + |c_0|^2 = 1.$$

On the other hand, the averaging can be performed by means of the density matrix (thermodynamical averaging) as $T > 0$ [13]. In both cases, we suppose

$$\langle \hat{P}^a_n \hat{P}^b_m \rangle \approx \langle \hat{P}^a_n \rangle \langle \hat{P}^b_m \rangle = \mu_a(n)\mu_b(m), \qquad (9)$$

i.e. we neglect correlations between fluctuations of the quantum fields $\hat{P}^a_n$. Then we obtain the following system of *Hamiltonian equations* for the averages $\mu_a(n)$:

$$\hbar \frac{d\mu_a(n)}{dt} = C_{abc}\mu_b(n) \frac{\partial \langle \mathcal{H} \rangle}{\partial \mu_c(n)} = \{\mu_a(n), \langle \mathcal{H} \rangle\}. \qquad (10)$$

Taking (9) into account, we have $\langle \mathcal{H} \rangle = \langle \mathcal{H}_{\mathrm{MF}} \rangle$.

In order to investigate large-scale fluctuations of the field $\mu_a(n)$, we take a two-dimensional continuum instead of the discrete lattice. Such a transition is well known for an SU(2)-magnet [14] and underlies the macroscopic phenomenological theory of magnetism [15]. In the continuous theory, the dynamical variables are the densities of averaged spin and quadrupole moments:

$$M_a(\boldsymbol{x}) = \lim_{S \to 0} \frac{1}{S} \sum_{n \in S} \mu_a(n) \delta_{\boldsymbol{x}, \boldsymbol{x}_n} = \sum_{n \in S} \mu_a(n) \delta(\boldsymbol{x} - \boldsymbol{x}_n).$$

Here, S is a “physically” infinitesimal region of the lattice, and $\delta(\boldsymbol{x} - \boldsymbol{x}_n)$ is the Dirac delta-function which has the dimension of reciprocal area. A Poisson bracket on the space of $\{M_a(\boldsymbol{x})\}$ is calculated by the formula

$$\{M_a(\boldsymbol{x}), M_b(\boldsymbol{y})\} = C_{abc} M_c(\boldsymbol{x}) \delta(\boldsymbol{x} - \boldsymbol{y}).$$

In what follows, we deal with the dimensionless field $\mu_a(\boldsymbol{x}) = l^2 M_a(\boldsymbol{x})$, where l is a distance between the nearest neighbors of the square lattice.

Considering $(j \pm 1, k)$, $(j, k \pm 1)$ as the nearest neighbors of the site $n = (j, k)$, we perform the transition in Eqs. (10) from the discrete variable $\boldsymbol{x}_n$ to a continuous one $\boldsymbol{x}$. Then the field $\{\mu_a(\boldsymbol{x})\}$ satisfies the equations

$$\hbar \frac{\partial \mu^a(\boldsymbol{x})}{\partial t} = \{\mu_a(\boldsymbol{x}), \mathcal{H}_{\mathrm{eff}}\} = -C_{abc}\mu_b \frac{\delta \mathcal{H}_{\mathrm{eff}}}{\delta \mu_c}, \qquad (11)$$

where

$$\mathcal{H}_{\mathrm{eff}} = J \int \sum_a \Big(\frac{\partial \mu_a}{\partial \boldsymbol{x}} \Big)^2 d^2\boldsymbol{x}.$$

A suitable representation of the system of Hamiltonian equations (11) is the matrix equation

$$\frac{\partial \hat{\mu}}{\partial t} = \frac{2Jl^2}{\hbar} [\hat{\mu}, \Delta \hat{\mu}], \qquad (12)$$

where

$$\hat{\mu} = -\frac{i}{2} \sum_a \mu_a(\boldsymbol{x}) \hat{P}^a.$$

Obviously, $\hat{\mu}$ is a Hermitian 3×3 matrix, the bracket $[\cdot, \cdot]$ denotes a matrix commutator, and $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$ is the Laplace operator over a two-dimensional space.

Equation (11) generalizes the Landau–Lifshits equation [16] for isotropic ferromagnets to the case of the 8-component mean field $\mu_a(\boldsymbol{x})$. The Landau–Lifshits equation is well-known in the macroscopic theory of magnetism and suitable to describe large-scale excitations in planar magnets and to explore the ferromagnetic resonance.

5. Free Energy and Topological Charge

As mentioned above, the Hamiltonian coincides with the free energy at constant temperature and volume. Therefore, we use the notion of free energy in what follows. As shown in Appendix (Section 6) by means of an algebraic approach, Eq. (12) coincides with the two-dimensional generalization of Eq. (25) on a degenerate orbit. This equation is Hamiltonian, though it is nonintegrable in the two-dimensional case; and its Hamiltonian can be used as an effective free energy for the spin system in question, namely:

$$\mathcal{F}_1^{\text{eff}} = \frac{2}{3h_0} \iint \left(\mu_{a,x}^2 + \mu_{a,y}^2\right) dxdy.$$

Obviously, $\mathcal{F}_1^{\text{eff}}$ is a part of the total free energy of a magnet and arises from an inhomogeneous distribution of the average values $\{\mu_a(x)\}$.

The algebraic approach yields one more equation of motion, associated with a generic orbit. Evidently, this equation can be obtained from the quantum-mechanical approach, by performing a relevant averaging that takes correlations into account. Therefore, we consider another effective free energy

$$\mathcal{F}_2^{\text{eff}} = \frac{1}{2(3f_0^2 - h_0^3)} \iint \Big(h_0^2(\mu_{a,x}^2 + \mu_{a,y}^2) -$$

$$-6f_0(\mu_{a,x}T_{a,x} + \mu_{a,y}T_{a,y}) + 3h_0(T_{a,x}^2 + T_{a,y}^2)\Big)\, dxdy.$$

The equations of extremals for the functionals of free energy are a two-dimensional generalization of Eqs. (25) and (26).

If the equations for constraints determining orbits are solved or, in other words, orbits are parametrized, then the formulas for the free energy can be simplified. The orbits of a co-adjoint representation of semisimple compact Lie groups are compact Kählerian manifolds. Therefore, it is suitable to use a complex parametrization. In order to parametrize a generic orbit, one requires three complex variables w_1, w_2, and w_3. Explicit formulas for the parametrization of a generic orbit are the following:

$$\mu_1 = \frac{m-\sqrt{3}\,q}{2\sqrt{2}} \cdot \frac{w_2 + w_3 + \bar{w}_2 + \bar{w}_3}{1+|w_2|^2+|w_3|^2} - \frac{m}{\sqrt{2}} \frac{(1-w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2) + (1-\bar{w}_1)(w_3 - w_1 w_2)}{1+|w_1|^2+|w_3-w_1w_2|^2},$$

$$\mu_2 = \frac{m-\sqrt{3}\,q}{2i\sqrt{2}} \cdot \frac{w_3 - w_2 - \bar{w}_3 + \bar{w}_2}{1+|w_2|^2+|w_3|^2} - \frac{im}{\sqrt{2}} \frac{(1+w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2) - (1+\bar{w}_1)(w_3 - w_1 w_2)}{1+|w_1|^2+|w_3-w_1w_2|^2},$$

$$\mu_3 = -\frac{m-\sqrt{3}\,q}{2} \cdot \frac{|w_2|^2 - |w_3|^2}{1+|w_2|^2+|w_3|^2} + \frac{m(1-|w_1|^2)}{1+|w_1|^2+|w_3-w_1w_2|^2},$$

$$\mu_4 = \frac{m-\sqrt{3}\,q}{2i} \cdot \frac{\bar{w}_2 w_3 - w_2\bar{w}_3}{1+|w_2|^2+|w_3|^2} + \frac{im(w_1-\bar{w}_1)}{1+|w_1|^2+|w_3-w_1w_2|^2},$$

$$\mu_5 = \frac{m-\sqrt{3}\,q}{2\sqrt{2}} \cdot \frac{w_3 - w_2 + \bar{w}_3 - \bar{w}_2}{1+|w_2|^2+|w_3|^2} - \frac{m}{\sqrt{2}} \frac{(1+w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2) + (1+\bar{w}_1)(w_3 - w_1 w_2)}{1+|w_1|^2+|w_3-w_1w_2|^2},$$

$$\mu_6 = \frac{m-\sqrt{3}\,q}{2i\sqrt{2}} \cdot \frac{w_2 + w_3 - \bar{w}_2 - \bar{w}_3}{1+|w_2|^2+|w_3|^2} + \frac{im}{\sqrt{2}} \frac{(1-\bar{w}_1)(w_3 - w_1 w_2) - (1-w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2)}{1+|w_1|^2+|w_3-w_1w_2|^2},$$

$$\mu_7 = \frac{m-\sqrt{3}\,q}{2} \cdot \frac{\bar{w}_2 w_3 + w_2\bar{w}_3}{1+|w_2|^2+|w_3|^2} - \frac{m(w_1+\bar{w}_1)}{1+|w_1|^2+|w_3-w_1w_2|^2},$$

$$\mu_8 = -\frac{m-\sqrt{3}\,q}{2\sqrt{3}} \cdot \frac{2-|w_2|^2-|w_3|^2}{1+|w_2|^2+|w_3|^2} + \frac{m}{\sqrt{3}} \cdot \frac{1+|w_1|^2 - 2|w_3-w_1w_2|^2}{1+|w_1|^2+|w_3-w_1w_2|^2}. \tag{13}$$

Here, m and q denote boundary values of the quantities μ_3 and μ_8, respectively. For a degenerate orbit, it is sufficient to have two complex variables, for instance w_2 and w_3; in this case, we assign $w_1 = 0$.

After the restriction onto an orbit by formulas (13), the expressions for free energy become

$$\mathcal{F}^{\text{eff}} = \int \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} \left(\frac{\partial w_\alpha}{\partial z} \frac{\partial \bar{w}_\beta}{\partial \bar{z}} + \frac{\partial w_\alpha}{\partial \bar{z}} \frac{\partial \bar{w}_\beta}{\partial z} \right) dz d\bar{z}, \quad (14)$$

where $g_{\alpha\bar{\beta}}$ denote components of a metrics on the orbit, and the real coordinates x and y on a plane are changed into complex-valued ones z and $\bar{z}$.

While co-adjoint orbits are Kählerian manifolds, they possess Kähler potentials which generate a related metric tensor g and a 2-form h; their components are calculated by the formulas

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \Phi}{\partial w_\alpha \partial \bar{w}_\beta}, \qquad h_{\alpha\bar{\beta}} = i \frac{\partial^2 \Phi}{\partial w_\alpha \partial \bar{w}_\beta}.$$

A 2-form gives rise to a topological charge

$$Q = \frac{1}{4\pi} \int \sum_{\alpha\beta} h_{\alpha\bar{\beta}} \left(\frac{\partial w_\alpha}{\partial z} \frac{\partial \bar{w}_\beta}{\partial \bar{z}} - \frac{\partial w_\alpha}{\partial \bar{z}} \frac{\partial \bar{w}_\beta}{\partial z} \right) dz \wedge d\bar{z},$$

which means a degree of the mapping of a plane into an orbit, which is realized by a function of w_1, w_2, and w_3.

On a degenerate orbit of SU(3), the function $\Phi_2 = \ln(1 + |w_2|^2 + |w_3|^2)$ serves as a Kählerian potential, and the metric tensor from (14) is a Kählerian one. Then the formulas for the topological charge and the free energy differ only in a sign ('+' for the free energy, and '−' for the topological charge), hence,

$$\mathcal{F}[\xi] \geqslant 4\pi |Q|.$$

The equality holds if the second term in the brackets vanishes, which happens if the functions $\{w_\alpha\}$ are holomorphic or antiholomorphic.

Consequently, the holomorphic functions form a class of solutions with quadrupole ordering ($m = 0$) that correspond to the minimum of the free energy. The same takes place for antiholomorphic functions.

We now consider the case of a generic orbit. The cohomology class of rank 2 for the orbit is two-dimensional, then there exist two basis 2-forms generated by the Kählerian potentials Φ_2 and $\Phi_1 = \ln(1 + |w_1|^2 + |w_3 - w_1 w_2|^2)$. As a unique Kählerian potential, we take the one corresponding to the Kirillov–Kostant–Souriau form

$$\Phi = m\Phi_1 - \frac{m - \sqrt{3}\, q}{2}\, \Phi_2. \quad (15)$$

Generally speaking, the metric tensor of the free energy (14) is not Kählerian. However, we can construct a Kählerian metrics of the form

$$\mathcal{F}_2^{\text{eff}} = \frac{1}{2(3f_0^2 - h_0^3)} \iint \Big(C_1(\mu_{a,x}^2 + \mu_{a,y}^2) -$$
$$+ C_2(\mu_{a,x} T_{a,x} + \mu_{a,y} T_{a,y}) + C_3(T_{a,x}^2 + T_{a,y}^2) \Big)\, dxdy,$$

by choosing the appropriate coefficients C_1, C_2, and C_3.

Then all conclusions relative to a degenerate orbit can be extended to a generic one. That is, the class of solutions with ferromagnetic ordering that correspond to a minimum of the free energy are realized by holomorphic (or antiholomorphic) functions.

5.1. Large-scale topological excitations

Excitations of a state with quadrupole ordering are described by spatially inhomogeneous distributions of the 8-component vector field $\mu_a(x)$ living on a degenerate orbit

$$\mathcal{O}(\mu_3 = 0, \mu_8) \simeq \frac{\text{SU}(3)}{\text{SU}(2) \times \text{U}(1)}.$$

Mappings of topological charge 2 can be modeled by the holomorphic functions

$$w_2(z) = \frac{a_1}{z - z_1}, \qquad w_3(z) = \frac{a_2}{z - z_2},$$

where a_1, a_2, z_1, and z_2 are fixed complex numbers.

The components μ_3 and μ_8 of the mean field, whose boundary values are called order parameters, are presented in Figs. 2 and 3 (we show a cut along the straight line joining poles of the functions $w_2(z)$ and $w_3(z)$, the origin of coordinates being at a middle of the interval $z_1 z_2$). In Figs. 2 and 3, q is a value of the component μ_8 at the initial point of a degenerate orbit ($\mu_3 = 0$).

These excitations are analogous to Belavin–Polyakov solitons well-known for planar Heisenberg ferromagnets (the quantities a_1 and a_2 define the widths of solitons, and the quantities z_1, z_2 give their positions). It is easy to see that the energy of a configuration does not depend on the width of a soliton, which proves the scale invariance of the energy in the two-dimensional case. Hence, topological perturbations can have arbitrary large size. Such an instability (an unrestricted increase of the soliton width without energy pumping) can cause the destruction of a nematic order in the considered model.

6. Appendix. Integrability of SU(3)-Symmetric Equations of the Landau–Lifshits Type in a One-Dimensional Space

It is known that the system of equations (12) in the one-dimensional case is an integrable Hamiltonian system on a degenerate orbit of the group SU(3) [17]. We generalize Eq. (12) to the case of a generic orbit. The algebraic-geometric nature of equations like (12) is revealed in the frame of the so-called orbit approach to nonlinear Hamiltonian systems. Below, we construct integrable Hamiltonian equations on orbits of the group SU(3).

Consider polynomials in a complex variable λ, whose coefficients are anti-Hermitian matrices of the algebra $\mathfrak{su}(3)$. We denote the set of polynomials by $\widetilde{\mathfrak{g}}_+ \simeq \mathfrak{su}(3) \otimes \mathcal{P}(\lambda)$, where $\mathcal{P}(\lambda)$ is a ring of polynomials with the standard multiplication operation. If $A, B \in \widetilde{\mathfrak{g}}_+$ have the form $A(\lambda) = \sum_n \hat{A}_n \lambda^n$, $B(\lambda) = \sum_k \hat{B}_k \lambda^k$, then

$$[A, B] = \sum_{n,k} [\hat{A}_n, \hat{B}_k] \lambda^{n+k} \in \widetilde{\mathfrak{g}}_+. \tag{16}$$

Operation (16) shows the structure of a graded Lie algebra in $\widetilde{\mathfrak{g}}_+$. Let $X_a^n = \lambda^n \hat{X}_a$ be a basis in $\widetilde{\mathfrak{g}}_+$, where $\hat{X}_a = -\frac{i}{2}\hat{\Lambda}_a$, $a = 1, 2, \ldots, 8$, $\{\hat{\Lambda}_a\}_{a=1}^8$ denote the Gell-Mann matrices.

In $\widetilde{\mathfrak{g}}_+$, we introduce a bilinear ad-invariant form

$$\langle A, B \rangle = -2 \operatorname{res} \lambda^{-N-2} \operatorname{Tr} A(\lambda) B(\lambda), \tag{17}$$

where $N+1$ is the maximum degree of the matrix polynomials A and B. Let $\mathcal{M} = (\widetilde{\mathfrak{g}}_+)^*$ be a space dual to $\widetilde{\mathfrak{g}}_+$ with respect to form (17). The collection of linear forms

$$\xi(\lambda) = \sum_{n=0}^{N} \sum_{a=1}^{8} \xi_a^n \lambda^n \hat{X}_a + \left(\xi_3 \hat{X}_3 + \xi_8 \hat{X}_8\right) \lambda^{N+1}$$

creates a closed ad-invariant subspace $\mathcal{M}^N$ in $\mathcal{M}$. The coordinates ξ_a^n of $\xi(\lambda) \in \mathcal{M}^N$ are calculated by the formula

$$\xi_a^n = \langle \xi(\lambda), X_a^{-n+N+1} \rangle.$$

In the linear space $\mathcal{M}^N$ with coordinates ξ_a^n, $n = 0, 1, \ldots, N$, we define the Lie–Poisson bracket

$$\{f_1, f_2\} = \sum_{m,n} \sum_{a,b}^{8} W_{ab}^{mn} \frac{\partial f_1}{\partial \xi_a^m} \frac{\partial f_2}{\partial \xi_b^n} \tag{18}$$

with the Poisson tensor field

$$W_{ab}^{mn} = \langle \xi(\lambda), [X_a^{-m+N+1}, X_b^{-n+N+1}] \rangle.$$

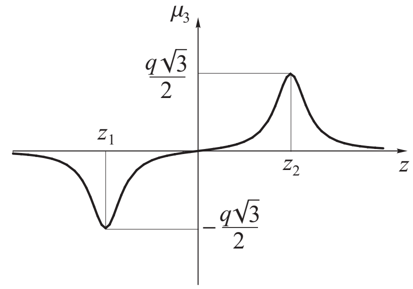

Fig. 2. Contour of $\mu_3(z, \bar{z})$, $\mu_3(\infty) \to 0$

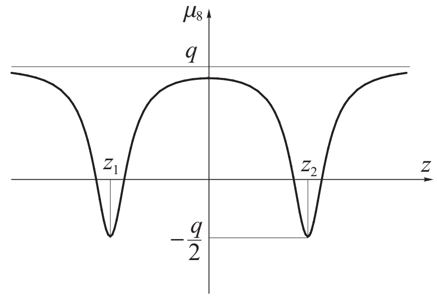

Fig. 3. Contour of $\mu_8(z, \bar{z})$, $\mu_8(\infty) \to q$

We also define two ad-invariant functions $I_2(\lambda)$ and $I_3(\lambda)$ by the formulas

$$I_2(\lambda) = -2 \operatorname{Tr} \xi^2(\lambda) = \sum_a \xi_a^2(\lambda),$$

$$I_3(\lambda) = -4i \operatorname{Tr} \xi^3(\lambda) = d_{abc} \xi_a(\lambda) \xi_b(\lambda) \xi_c(\lambda).$$

Here, $d_{abc} = -2i \operatorname{Tr}(X_a X_b X_c + X_b X_a X_c)$, and $\xi_a(\lambda)$ is a polynomial

$$\xi_a(\lambda) = \xi_a^0 + \xi_a^1 \lambda + \xi_a^2 \lambda^2 + \cdots + \xi_a^{N+1} \lambda^{N+1}.$$

The invariant functions are also polynomials in the complex parameter λ:

$$I_2(\lambda) = h_0 + h_1 \lambda + \cdots + h_{2N+2} \lambda^{2N+2},$$

$$I_3(\lambda) = f_0 + f_1 \lambda + \cdots + f_{3N+3} \lambda^{3N+3}.$$

It is easy to prove that the coefficients $h_0, \ldots, h_{N+1}$, $f_0, \ldots f_{N+1}$ are annihilators of bracket (18). Fixing them, we obtain the system of algebraic equations

$$h_n = \text{const}, \; f_n = \text{const}, \; n = 0, \ldots, N+1, \tag{19}$$

which gives an embedding of the orbit $\mathcal{O}^{N+1}$ of dimension $6(N+1)$ into the linear space $\mathcal{M}^{N+1}$. The remaining coefficients $\{h_{N+2}, \ldots, h_{2N+2}, f_{N+2}, \ldots, f_{3N+3}\}$ form a pairwise commutative collection of

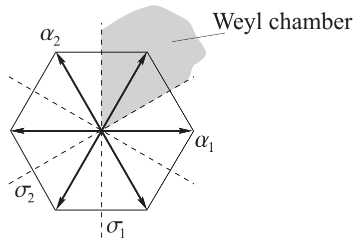

Fig. 4. Root diagram of the group SU(3)

integrals of motion, which is necessary to integrate the Hamiltonian system. We are interested in the functions h_{N+2} and h_{N+3} and the corresponding Hamiltonian equations. In particular, the Hamiltonian h_{N+2} gives rise to the so-called stationary equations. In terms of the coordinates ξ_a^n, they are

$$\frac{\partial \xi_a^n}{\partial x} = 2 f_{abc} \xi_c^0 \xi_b^{n+1}, \tag{20}$$

where $\{f_{abc}\}$ are the antisymmetric structure constants of the algebra $\mathfrak{su}(3)$ in the basis of Gell-Mann matrices:

$$[X_a, X_b] = f_{abc} X_c,$$

$$f_{123} = 1,\ f_{458} = f_{786} = \tfrac{\sqrt{3}}{2},$$

$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \tfrac{1}{2}.$$

By x, we denote a "time" with respect to the Hamiltonian h_{N+2}, which corresponds to the evolution equations

$$\frac{\partial \xi_a^n}{\partial t} = 2 f_{abc} (\xi_c^0 \xi_b^{n+2} + \xi_c^1 \xi_b^{n+1}). \tag{21}$$

Equations (20) and (21) are consistent, because the corresponding Hamiltonians commute: $\{h_{N+2}, h_{N+3}\} = 0$. Hence, we can determine evolution (21) on trajectories of system (20), i.e. we suppose the dynamical variables ξ_a^n in Eq. (21) to be dependent on x. Combining (20) and (21), we have

$$\frac{\partial \xi_a^0}{\partial t} = 2 f_{abc} \xi_c^0 \xi_b^2 = \frac{\partial \xi_a^1}{\partial x}. \tag{22}$$

The variables $\{\xi_a^1\}$ can be expressed in terms of $\{\xi_a^0\}$ and $\{\frac{\partial}{\partial x}\xi_a^0\}$. Then (22) becomes a closed system of partial differential equations for $\{\xi_a^0\}$. For $\{\xi_a^1\}$, it is necessary to solve the degenerate system of equations

$$\frac{\partial \xi_a^0}{\partial x} = 2 f_{abc} \xi_c^0 \xi_b^1, \tag{23}$$

which becomes possible after the restriction to the orbit $\mathcal{O}^{N+1} \subset \mathcal{M}^{N+1}$.

6.1. Classification of orbits

It follows from Eqs. (19) that the orbit $\mathcal{O}^{N+1}$ in $\mathcal{M}^{N+1}$ has the structure of a vector bundle over a co-adjoint orbit of the group SU(3). Hence, a classification of the orbits $\mathcal{O}^{N+1}$ is reduced to that of orbits of SU(3).

Since the group SU(3) is simple, we have $\mathfrak{g}^* = \mathfrak{g}$. Therefore, we consider $\{\xi_a^0\}$ also as coordinates in the algebra $\mathfrak{g} \simeq \mathfrak{su}(3)$. Then a generic element $\hat{\xi} \in \mathfrak{su}(3)$ is represented by the matrix

$$\widehat{\xi} = -\frac{i}{2} \begin{pmatrix} \xi_3^0 + \frac{1}{\sqrt{3}}\xi_8^0 & \xi_1^0 - i\xi_2^0 & \xi_4^0 - i\xi_5^0 \\ \xi_1^0 + i\xi_2^0 & -\xi_3 + \frac{1}{\sqrt{3}}\xi_8 & \xi_6^0 - i\xi_7^0 \\ \xi_4^0 + i\xi_5^0 & \xi_6^0 + i\xi_7^0 & -\frac{2}{\sqrt{3}}\xi_8 \end{pmatrix}. \tag{24}$$

Let $\hat{\xi}(0)$ be a fixed element of $\mathfrak{su}(3)$. By definition, the set $\mathcal{O}_{\hat{\xi}(0)} = \{g\hat{\xi}(0)g^{-1}, \forall g \in \mathrm{SU}(3)\}$ is an orbit of SU(3) through the point $\hat{\xi}(0)$. Elements g' of the group SU(3) such that $g'\hat{\xi}(0)g'^{-1} = \hat{\xi}(0)$ form the stationary subgroup at the point $\hat{\xi}(0)$. The orbit $\mathcal{O}_{\hat{\xi}(0)}$, being a homogeneous space, is a coset space $\mathrm{SU}(3)/\mathrm{G}' \simeq \mathcal{O}_{\hat{\xi}(0)}$, where G' denotes the stationary subgroup.

The maximal commutative subalgebra of the semisimple algebra $\mathfrak{g}$, which is called Cartan, can be diagonalized. The Cartan subalgebra of $\mathfrak{su}(3)$ is formed from diagonal matrices depending on two parameters ξ_3^0 and ξ_8^0. It is well known that a proper transformation puts any element $\hat{\xi} \in \mathfrak{g}$ into the Cartan subalgebra of $\mathfrak{g}$. This implies that *each orbit intersects the Cartan subalgebra at least once*. In fact, there is more than one intersection point number, more precisely as many as the *order* of the Weyl group W(G). We discuss this in what follows.

The nontrivial similarity transformations $\hat{\xi} \to g\hat{\xi}g^{-1}$ that preserve the subalgebra $\mathfrak{h}$ form a discrete subgroup $\mathrm{W(G)} \subset \mathrm{G}$, which is called a Weyl group [18]. The action of the group W(G) on the subalgebra $\mathfrak{h} = \mathfrak{h}^*$ is generated by reflections in planes orthogonal to simple roots. The Weyl group of SU(3) is generated by two reflections σ_1 and σ_2 in the planes shown in Fig. 4 by dotted lines. The full Weyl group consists of six elements $\{e, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1 \simeq \sigma_2\sigma_1\sigma_2\}$, and is isomorphic to the group of permutations S_3, $\mathrm{ord}\, S_3 = 3!$.

Since a Weyl group acts over a Cartan subalgebra, every point $\sigma\hat{\xi}(0)\sigma^{-1}$, $\sigma \in \mathrm{W(G)}$, belongs to an orbit through $\hat{\xi}(0) \in \mathfrak{h}$. The group W(G) acts efficiently (it changes an element $\hat{\xi}$ into another one, if the element does not belong to reflection planes). An open domain in a Cartan subalgebra, where a Weyl group acts efficiently, is called *a Weyl chamber* (see Fig. 4). Elements of different Weyl chambers are adjoint by elements $\sigma \in \mathrm{W(G)}$. This implies that each orbit through a point $\hat{\xi}(0)$ of a Weyl chamber intersects the Cartan subalgebra as many times as the order of W(G). If $\hat{\xi}(0)$ is an interior point of a Weyl chamber, we call the orbit a *generic* one and call the points $\sigma\hat{\xi}(0)\sigma^{-1}$, $\forall\sigma \in \mathrm{W(G)}$ *poles of an orbit*.

For a generic orbit, a stationary subgroup G′ coincides with a maximum torus T^r for a group G of rank r. In the case of the group SU(3), we have $\mathrm{T}^2 = \mathrm{U(1)} \times \mathrm{U(1)}$. Hence, the generic orbits are coset spaces $\mathcal{O}_{\mathrm{gen}} \simeq \mathrm{SU(3)/U(1)\times U(1)}$.

If an initial point $\hat{\xi}(0)$ belongs to the wall of a Weyl chamber (in the case of SU(3), it belongs to one of the reflection lines), then we deal with a degenerate orbit. In this case, a stationary subgroup G′ contains a semisimple subgroup generated by roots orthogonal to the initial point $\hat{\xi}(0)$. Consider the group SU(3). If $\hat{\xi}(0)$ lies on the vertical reflection line, α_1 and $-\alpha_1$ are orthogonal to this element. The corresponding $\mathfrak{sl}(2)$-triple $\{X_{\alpha_1}, X_{-\alpha_1}, H_{\alpha_1} = [X_{\alpha_1}, X_{-\alpha_1}]\}$ generates a subgroup $\mathrm{SU(2)} \subset \mathrm{SU(3)}$. Obviously, the element $\hat{\xi}(0) = \frac{-i}{2\sqrt{3}}\,\xi_8^0\,\mathrm{diag}(1,1,-2)$ is invariant under a transformation $g'\hat{\xi}(0)g'^{-1}$, where g' is the unitary matrix

$$g' = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta^* & \alpha^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 & 0 \\ 0 & e^{i\varphi/2} & 0 \\ 0 & 0 & e^{-i\varphi} \end{pmatrix}.$$

Hence, $g' \in \mathrm{SU(2)\times U(1)}$, and a degenerate orbit is a coset space $\mathcal{O}_{\mathrm{deg}} \simeq \mathrm{SU(3)/SU(2)\times U(1)}$.

6.2. Equations on orbits and their Hamiltonians

Let us return to the construction of integrable systems on the orbits of loop groups and consider Eq. (22). In order to solve this equation, we have to restrict the degenerate system (23) to an orbit.

If $\xi_a^0 \in \mathcal{O}_{\mathrm{deg}}$, then the matrix $2f_{abc}\xi_c^0$ has rank 4, and its inversion gives a solution

$$\xi_a^1 = \tfrac{2}{3h_0}\, f_{abc}\xi_b^0\xi_{c,x}^0 + \tfrac{h_1}{2h_0}\,\xi_a^0,$$

where the constants h_0 and h_1 define an orbit by Eqs. (19).

If $\xi_a^0 \in \mathcal{O}_{\mathrm{gen}}$, then the matrix $2f_{abc}\xi_c^0$ has rank 6 and is invertible on a generic orbit. Then we have

$$\xi_a^1 = \tfrac{1}{2(h_0^3-3f_0^2)}\Big(h_0^2 f_{abc}\xi_b^0\xi_{c,x}^0 +$$

$$+3h_0 f_{abc}\eta_b^0\eta_{c,x}^0 - 6f_0 f_{abc}\xi_b^0\eta_{c,x}^0\Big) +$$

$$+\tfrac{2f_0f_1-3h_0^2h_1}{6(f_0^2-h_0^3)}\,\xi_a^0 + \tfrac{3f_0h_1-2h_0f_1}{6(f_0^2-h_0^3)}\,\eta_a^0,$$

where $\eta_a^0 = d_{abc}\xi_b^0\xi_c^0$, and the constants h_0, h_1, f_0, and f_1 come from Eqs. (19).

Substituting the obtained expressions in the right-hand side of (22), we get two equations for the functions $\xi_a(x,t) \equiv \xi_a^0$:

$$\frac{\partial \xi_a}{\partial t} = \tfrac{2}{3h_0}\, f_{abc}\xi_b\xi_{c,xx} + \tfrac{h_1}{h_0}\,\xi_{a,x}, \quad \xi_a \in \mathcal{O}_{\mathrm{deg}}, \tag{25}$$

$$\frac{\partial \xi_a}{\partial t} = \tfrac{1}{2(h_0^3-3f_0^2)}\Big(h_0^2 f_{abc}\xi_b\xi_{c,xx} - 3f_0 f_{abc}\xi_b\eta_{c,xx} +$$

$$+3h_0 f_{abc}\eta_b\eta_{c,xx} - 3f_0 f_{abc}\eta_b\xi_{c,xx}\Big) +$$

$$+\tfrac{2f_0f_1-3h_0^2h_1}{6(f_0^2-h_0^3)}\,\xi_{a,x} + \tfrac{3f_0h_1-2h_0f_1}{6(f_0^2-h_0^3)}\,\eta_{a,x}, \qquad \xi_a \in \mathcal{O}_{\mathrm{gen}}. \tag{26}$$

We set $h_1 = 0$ in Eq. (25) and replace the variables ξ_a by μ_a. Then its generalization to the two-dimensional case gets the form

$$\frac{\partial \mu_a}{\partial t} = \tfrac{1}{6h_0}\, C_{abc}\mu_b\Delta\mu_c,$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Obviously, this equation has the Hamiltonian

$$\mathcal{H}^{\mathrm{eff}} = \tfrac{1}{12h_0}\iint \big(\mu_{a,x}^2 + \mu_{a,y}^2\big)\, dxdy.$$

It is easy to see that (25) coincides with a one-dimensional analog of (12). In other words, Eq. (12) can be considered as a two-dimensional generalization of the integrable equation (25).

In the same way, we treat Eq. (26), namely, we replace ξ_a by μ_a and assign $f_1 = h_1 = 0$. Generalized to two dimensions, the obtained equations get the form

$$\frac{\partial \mu_a}{\partial t} = \tfrac{1}{8(h_0^3-3f_0^2)}\Big(h_0^2 C_{abc}\mu_b\Delta\mu_c - 3f_0 C_{abc}\widetilde{\eta}_b\Delta\mu_c +$$

$$+3h_0 C_{abc}\widetilde{\eta}_b\Delta\widetilde{\eta}_c - 3f_0 C_{abc}\mu_b\Delta\widetilde{\eta}_c\Big). \tag{27}$$

Here, $\widetilde{\eta}_a$ are quadratic forms in μ_a: $\widetilde{\eta}_a = \widetilde{d}_{abc}\mu_b\mu_c$, where $\widetilde{d}_{abc} = \frac{1}{4}\,\mathrm{Tr}(P_aP_bP_c + P_bP_aP_c)$. Obviously, Eq. (27) is Hamiltonian with the following effective Hamiltonian:

$$\mathcal{H}^{\mathrm{eff}} = \tfrac{1}{16(h_0^3-3f_0^2)}\iint \big(h_0^2\mu_{a,x}^2 + h_0^2\mu_{a,y}^2 -$$

$$-6f_0\mu_{a,x}\widetilde{\eta}_{a,x} - 6f_0\mu_{a,y}\widetilde{\eta}_{a,y} + 3h_0\widetilde{\eta}_{a,x}^2 + 3h_0\widetilde{\eta}_{a,y}^2\big)\, dxdy.$$

7. Conclusions

In the present paper, we have constructed nonlinear stationary excitations appearing in the nematic phase of a planar magnet of spin $s = 1$ modeled by a square lattice with the biquadratic interaction between the nearest-neighbor sites. These excitations are characterized by an integer topological charge and reveal themselves as regions with nonzero magnetization and mean quadrupole moment. The topological excitations in a two-dimensional system (without taking an anisotropy and a demagnetizing field into account) can increase unrestrictedly without a pumping of energy. This destroys the nematic state in the system, according to the Mermin–Wagner theorem on the absence of a long-range order in one- and two-dimensional systems.

This work is partly supported by the grant of DFFD UkrF16/457-2007 and the grant of the International Charitable Fund for Renaissance of Kyiv-Mohyla Academy.

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Received 02.04.08.
Translated from Ukrainian by V.V. Kukhtin

ВПОРЯДКОВАНІ СТАНИ ТА НЕЛІНІЙНІ КРУПНОМАСШТАБНІ ЗБУДЖЕННЯ У ПЛОСКОМУ МАГНЕТИКУ ЗІ СПІНОМ $s = 1$

Ю.М. Бернацька, П.І. Голод

Р е з ю м е

Досліджено впорядковані стани та топологічні збудження у квазідвовимірному магнетику, змодельованому квадратною ґраткою зі спінами $s = 1$ у вузлах та гамільтоніаном з біквадратною обмінною взаємодією найближчих сусідів. Запропоновано два ефективних гамільтоніани для опису крупномасштабних збуджень у строго двовимірному випадку. Один з них описує збудження середнього поля в нематичній фазі, інший — у змішаній феромагнітно-нематичній фазі. Показано, що ефективні гамільтоніани мінімізуються на конфігураціях, які мають фіксований топологічний заряд. Ці топологічні збудження можуть виникати при низьких температурах і бути причиною руйнування далекого порядку у строго двовимірній системі.