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INTERPOLATION PROBLEMS FOR RANDOM FIELDS FROM OBSERVATIONS IN PERFORATED PLANE

The problem of estimation of linear functionals which depend on the unknown values of a homogeneous random field $\xi(k,j)$ in the region $K \subset Z^2$ from observations of the sum $\xi(k,j)+\eta(k,j)$ at points $(k,j) \in Z^2 \setminus K$ is investigated. Formulas for calculating the mean square errors and the spectral characteristics of the optimal linear estimate of functionals are derived in the case where the spectral densities are exactly known. Formulas that determine the least favourable spectral densities and the minimax (robust) spectral characteristics are proposed in the case where the spectral densities are not exactly known while a class of admissible spectral densities is given.

Key words: *random fields, estimation problem, minimax (robust) spectral characteristic.*

Introduction. In engineering, geology, automatic control etc., one often has a number of data points, obtained by sampling which represents values of stochastic processes or random fields for a limited number of values of the independent variable. It is often required to interpolate (i.e. estimate) the value of processes or fields for an intermediate value of the independent variable. Methods of solution of linear estimation problems for stochastic processes and random fields were developed by A. N. Kolmogorov [1], N. Wiener [2], A. M. Yaglom [3, 4], M. I. Yadrenko [5]. Traditional methods of solution of these problems are employed under the condition that spectral densities are known exactly. In practice, however, complete information on spectral densities is impossible in most cases. To solve the problem the parametric or nonparametric estimates of the unknown spectral densities are found or these densities are selected by other reasoning. Then the traditional estimation methods are applied provided that the estimated or selected densities are the true one. This procedure can result in a significant increasing of the value of the error as K. S. Vastola and H. V. Poor [6] have demonstrated with the help of some examples. This is a reason to search estimates which are optimal for all densities from a certain class of the admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error. Many investigators have been interested in minimax extrapolation, interpolation and filtering problems for stationary stochastic processes and random fields. A survey of results in minimax (robust) methods of data processing can be found in the paper by S. A. Kassam and H. V. Poor [7]. M. P. Moklyachuk [8–11] proposed the minimax approach to extrapolation, interpolation and filtering problems for functionals which depend on the unknown values of stationary processes and sequences. In papers by M. P. Moklyachuk and N. Yu. Shchestyuk [12–16], and M. P. Moklyachuk and S. V. Tatarinov [17] problems of extrapolation, interpolation and filtering of functionals which depend on the unknown values of homogeneous random fields were investigated. M. P. Moklyachuk and M. Sidey considerated the minimax approach to interpolation problems for functionals which depend on the unknown values of stationary sequences from observations with the missing intervals.

In this paper we deal with the problem of estimation of the unknown values of a mean-square homogeneous random field from observations on the perforated plane.

Optimal linear estimates. spectral densities are known. Let $\xi(k,j)$ be a homogeneous random field on Z^2. It means that $M\xi(k,j)=0$, $M\left|\xi(k,j)\right|^2<+\infty$ and $B(k-n,j-m)=M\xi(k,j)\xi(n,m)$ depends only on difference $(k-n,j-m)$. The correlation function $B(k-n,j-m)$ of the homogeneous random field of discrete arguments has the following spectral representation

$$B(k-n,j-m)=M\xi(k,j)\overline{\xi(n,m)}=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{i(k\lambda+j\mu)}F(d\lambda,d\mu),$$

where $F(d\lambda,d\mu)$ is the spectral measure of the field on $[-\pi;\pi)\times[-\pi;\pi)$. If this measure is absolutely continuous with respect to Lebesgue measure, then the correlation function can be represented in the form

$$B(k,j)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{i(k\lambda+j\mu)}f(\lambda,\mu)d\lambda\,d\mu,$$

where $f(\lambda,\mu)$ is the spectral density of the field.

Let the sum $\xi(k,j)+\eta(k,j)$ of the homogeneous random fields are observed in all points of perforated plane $(k,j)\in Z^2\setminus K$. Perforations are small holes in a thin material or web. In our case the holes are rectangles

$m^x \times m^y$. Assume that the number of rectangles on horizontal is s_x and number of rectangles on vertical is s_y (see picture 1). So we can formally define the set of perforations of the plane as $K = \bigcup_{t_1=0}^{s_x} \bigcup_{t_2=0}^{s_y} \left(m_{t_1}^x \times m_{t_2}^y \right)$. Let $m_1^x = m_2^x = ... = m_{s_x}^x$. Denote $l_x = m^x + n^x$, $l_y = m^y + n^y$, where n^x, n^y are distances between the rectangles on horizontal and vertical coordinates correspondently.

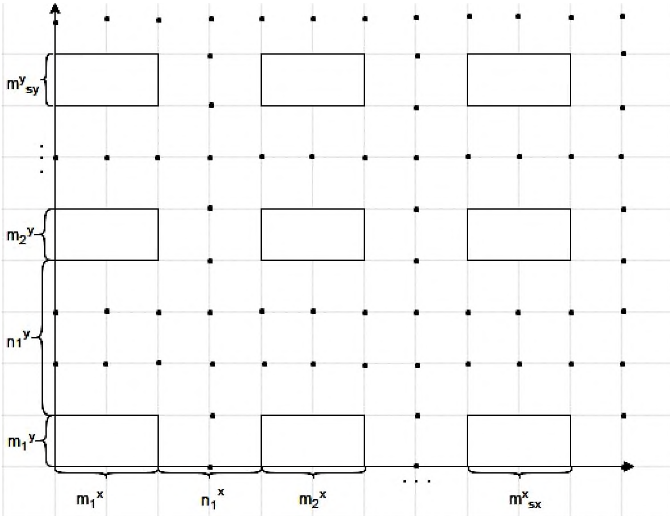

Fig. 1.

The problem is to find the optimal in the mean square sense estimate $\widehat{A}_K\xi$ of the linear functional

$$A_K\xi = \sum_{(k,l)\in K} a(k,l)\xi(k,l) = \sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1 l_x}^{t_1 l_x + m_x - 1} \sum_{j=t_2 l_y}^{t_2 l_y + m_y - 1} a(k,j)\xi(k,j) \right) \tag{1}$$

which depend on the unknown values of the field $\xi(k,j)$, $(k,j)\in K$ from observations of the sum $\xi(k,j)+\eta(k,j)$, where $(k,j)\in Z^2 \setminus K$, $K = \bigcup_{t_1=0}^{s_x} \bigcup_{t_2=0}^{s_y} \left(m_{t_1}^x \times m_{t_2}^y \right)$. This problem is reduced to the optimization problem:

$$\Delta = M\left|A_K\xi - \widehat{A}_K\xi\right|^2 \to \min.$$

Denote by $L_2(f_\xi + f_\eta)$ the Hilbert space of complex valued functions on $[-\pi,\pi)\times[-\pi,\pi)$ which are square integrated with respect to measure with the density $f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)$. Denote by $L_2^{K-}\left(f_\xi + f_\eta\right)$ the subspace of $L_2\left(f_\xi + f_\eta\right)$, generated by functions $e^{i(u\lambda+v\mu)}$, $(u,\upsilon)\in Z^2\setminus K$. Every linear estimate $\widehat{A}_K\xi$ of the unknown functional $\widehat{A}_K\,\xi$ is of the form

$$\widehat{A}_K\xi = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(\lambda,\mu)\left(Z_\xi(d\lambda,d\mu)+Z_\eta(d\lambda,d\mu)\right), \tag{2}$$

where $Z_\xi(d\lambda,d\mu)$ and $Z_\eta(d\lambda,d\mu)$ are orthogonal random measures of the fields $\xi(u,v)$ and $\eta(u,v)$ correspondingly, and $h(\lambda,\mu)$ is the spectral characteristic of the estimate $\widehat{A}_K\xi$. We will suppose that the condition of «minimality» holds true

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}\,d\lambda d\mu<\infty. \tag{3}$$

The spectral characteristic $h(\lambda,\mu)$ of the optimal linear estimate $\widehat{A}_K\xi$ is determined by the conditions [1]:

1) $h(\lambda,\mu)\in L_2^{K-}(f_\xi + f_\eta)$;

2) $\left(A_K(\lambda,\mu)-h(\lambda,\mu)\right)\perp L_2^{K-}(f_\xi + f_\eta)$.

These conditions will be used to derive formulas for the mean square errors and spectral characteristic of the optimal linear estimate $\widehat{A}_K\,\xi$.

From condition 2) we have

$$\left(A_K(\lambda,\mu)-h(\lambda,\mu)\right)\perp e^{i(k\lambda+j\mu)}.\ \forall(k,j)\in Z^2\setminus K,$$

where

$$A_K(\lambda,\mu)=\sum_{(k,j)\in K}a(k,j)e^{i(k\lambda+j\mu)}=\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x}\sum_{j=t_2l_y}^{t_2l_y+m_y}a(k,j)e^{i(k\lambda+j\mu)}\right).$$

This condition can be represented in the form:

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(A_K(\lambda,\mu)f_\xi(\lambda,\mu)-h(\lambda,\mu)\left(f_{\xi(\lambda,\mu)}+f_\eta(\lambda,\mu)\right)\right)e^{-i(k\lambda+j\mu)}d\lambda d\mu=0,$$

$$(k,j)\in Z^2\setminus K.$$

From the indicated condition we conclude that

$$\left(A_K(\lambda,\mu)\right)\left(f_\xi(\lambda,\mu)\right)-h(\lambda,\mu)\left(f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)\right)=C_K(\lambda,\mu),$$

$$C_K(\lambda,\mu)=\sum_{(k,j)\in K} c(k,j)e^{i(k\lambda+j\mu)}=\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x-1}\sum_{j=t_2l_y}^{t_2l_y+m_y-1} c(k,j)e^{i(k\lambda+j\mu)}\right),$$

where $c(k,j)$ are unknown coefficients which we need to determine.

From the derived relation we get the following formula for the spectral characteristic

$$h(\lambda,\mu)=\frac{A_K(\lambda,\mu)f_\xi(\lambda,\mu)}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}-\frac{C_K(\lambda,\mu)}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}.$$

From condition 1) we have

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(\lambda,\mu)e^{-i(u\lambda+v\mu)}d\lambda d\mu=0 \text{ for } (u,v)\in K\subset Z^2,$$

and we get that for any $(u,v)\in K\subset Z^2$:

$$\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x-1}\sum_{j=t_2l_y}^{t_2l_y+m_y-1} a(k,j)\right)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x-1}\sum_{j=t_2l_y}^{t_2l_y+m_y-1} a(k,j)\right)-$$

$$-\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x-1}\sum_{j=t_2l_y}^{t_2l_y+m_y-1} c(k,j)\right)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(\frac{e^{i((k-u)\lambda+(j-v)\mu)}}{f_\xi(\lambda\lambda\mu)+f_\eta(\lambda\lambda\mu)}\right)d\lambda d\mu=0.$$

Let us represent the functions $\dfrac{f_\xi(\lambda,\mu)}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}$, $\dfrac{1}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}$ in the Fourier series:

$$\frac{1}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} b_{pq}e^{i(p\lambda+q\mu)},$$
$$\frac{f_\xi(\lambda,\mu)}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}=\sum_{l=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} d_{lt}e^{i(l\lambda+t\mu)} \qquad (4)$$

and obtain the system for solving $c_{u,v}$, where $(k,j)\in K$:

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[\left(\sum_{(u,v\in K)} a_{uv}e^{i(u\lambda+v\mu)}\right)\left(\sum_{l,t=-\infty}^{\infty} d_{lt}e^{i(l\lambda+t\mu)}\right)-\right.$$

$$\left.-\left(\sum_{(m,n\in K)} C_{mn}e^{i(m\lambda+n\mu)}\right)\left(\sum_{p,q=-\infty}^{\infty} b_{pq}e^{i(p\lambda+q\mu)}\right)\right]e^{-i(k\lambda+j\mu)}d\lambda d\mu=0,$$

$$(k,j)\in K.$$

$$\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x-1}\sum_{j=t_2l_y}^{t_2l_y+m_y-1} d^K_{k-u,j-v}a(k,j)\right)=\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x-1}\sum_{j=t_2l_y}^{t_2l_y+m_y-1} d^K_{k-u,j-v}c(k,j)\right).$$

From this system we derive equations for all $(k,j)\in K$; or in the matrix form

$$D^K\vec{a}=B^K\vec{c},$$

where $\vec{a}$ is a vector composed from coefficients determining $A_K\xi$, $\vec{c}$ is a vector composed from the unknown coefficients $c(k,j)$, $(k,j)\in K$, D^K, B^K are operators, which are determined by matrices

$$D^{lt}_{kj}=d(l-k,t-j),\ B^{lt}_{kj}=b(l-k,t-j), \tag{5}$$

with elements that are the Fourier coefficients of the functions $\dfrac{f_\xi(\lambda,\mu)}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}$, $\dfrac{1}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}$ correspondingly. So the unknown coefficients $c(k,j)$ can be computed by formula

$$c(k,j)=\left(\left(B^K\right)^{-1}D^K\overline{a}_K\right)_{(k,j)},\ (k,j)\in K \tag{6}$$

and the spectral characteristic of the optimal linear estimate $\widehat{A}_K\xi$ may be calculated by the formula

$$h(\lambda,\mu)=\frac{A_K(\lambda,\mu)f_\xi(\lambda,\mu)}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}-$$

$$-\frac{\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x-1}\sum_{j=t_2l_y}^{t_2l_y+m_y-1}\left(\left(B^K\right)^{-1}D^K\overline{a}_K\right)_{(k,j)}e^{i(k\lambda+j\mu)}\right)}{f_\xi(\lambda,\mu)+f_\eta(\lambda,\mu)}. \tag{7}$$

The mean square error is calculated by the formula

$$\Delta\left(h;f_\xi,f_\eta\right)=$$

$$=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(A_KZ_\xi(d\lambda,d\mu)-hZ_{\xi+\eta}(d\lambda,d\mu)\right)\overline{\left(A_KZ_\xi(d\lambda,d\mu)-hZ_{\xi+\eta}(d\lambda,d\mu)\right)}=$$

$$=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(\left|A_K-h\right|^2f_\xi(\lambda,\mu)+\left|h\right|^2f_\eta(\lambda,\mu)\right)d\lambda d\mu,$$

where $A_K=A_K(\lambda,\mu)$, $h=h(\lambda,\mu)$. Making use of the form (7) of the spectral characteristic we can get formula for calculating the mean square error of the optimal linear estimate $\widehat{A}_K\xi$:

$$\Delta(h; f_\xi, f_\eta) = \tag{8}$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left| A_K(\lambda,\mu) f_\eta(\lambda,\mu) + \sum_{t_1=0}^{s_x-1} \sum_{t_2=0}^{s_y-1} \left(\sum_{k=t_1 l_x}^{t_1 l_x + m_x - 1} \sum_{j=t_2 l_y}^{t_2 l_y + m_y - 1} \left(\left(B^K\right)^{-1} D^K \overline{a}_K \right)_{(k,j)} e^{i(k\lambda + j\mu)} \right) \right|^2}{\left(f_\xi(\lambda,\mu) + f_\eta(\lambda,\mu) \right)^2} f_\xi(\lambda,\mu) d\lambda d\mu +$$

$$+ \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left| A_K(\lambda,\mu) f_\xi(\lambda,\mu) + \sum_{t_1=0}^{s_x-1} \sum_{t_2=0}^{s_y-1} \left(\sum_{k=t_1 l_x}^{t_1 l_x + m_x - 1} \sum_{j=t_2 l_y}^{t_2 l_y + m_y - 1} \left(\left(B^K\right)^{-1} D^K \overline{a}_K \right)_{(k,j)} e^{i(k\lambda + j\mu)} \right) \right|^2}{\left(f_\xi(\lambda,\mu) + f_\eta(\lambda,\mu) \right)^2} f_\eta(\lambda,\mu) d\lambda d\mu,$$

Proposition 1. The spectral characteristic $h(\lambda, \mu)$ and the mean square error $\Delta(h; f_\xi, f_\eta)$ of the optimal linear estimate $\widehat{A}_K\xi$ of the functional $A_K\xi$ from observations of the field $\xi(u,v) + \eta(u,v)$ at points $(u,\upsilon) \in Z^2 \setminus K$, $K = \bigcup_{t_1=0}^{s_x} \bigcup_{t_2=0}^{s_y} \left(m_{t_1}^x \times m_{t_2}^y \right)$ are calculated by formulas (7), (8).

Corollary 1. The spectral characteristic $h(\lambda, \mu)$ and the mean square error $\Delta(f)$ of the optimal linear estimate $\widehat{A}_K\xi$ of the functional $A_K\xi$ from observations of the field $\xi(u,v)$ at points $(u,\upsilon) \in Z^2 \setminus K$, $K = \bigcup_{t_1=0}^{s_x} \bigcup_{t_2=0}^{s_y} \left(m_{t_1}^x \times m_{t_2}^y \right)$ are calculated by formulas

$$h(\lambda, \mu) = A_K(\lambda, \mu) - \frac{\sum_{t_1=0}^{s_x-1} \sum_{t_2=0}^{s_y-1} \left(\sum_{k=t_1 l_x}^{t_1 l_x + m_x - 1} \sum_{j=t_2 l_y}^{t_2 l_y + m_y - 1} \left(\left(B^K\right)^{-1} D^K \overline{a}_K \right)_{(k,j)} e^{i(k\lambda + j\mu)} \right)}{f_\xi(\lambda, \mu)} \tag{9}$$

$$\Delta(f) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left| \sum_{t_1=0}^{s_x-1} \sum_{t_2=0}^{s_y-1} \left(\sum_{k=t_1 l_x}^{t_1 l_x + m_x - 1} \sum_{j=t_2 l_y}^{t_2 l_y + m_y - 1} \left(\left(B^K\right)^{-1} D^K \overline{a}_K \right)_{(k,j)} e^{i(k\lambda + j\mu)} \right) \right|^2}{f_\xi(\lambda, \mu)} d\lambda d\mu, \tag{10}$$

where B^K is operator determined by the matrix with elements $B_{kj}^{lt} = b(l-k, t-j)$ that are the Fourier coefficients of the function $\frac{1}{f_\xi(\lambda, \mu)}$.

Example 1. (perforated plane with 3 holes). Suppose that we observe the random field $\xi(u,v)$ without noise at points $(u,\upsilon)\in Z^2\setminus K$, where K is a union of 3 holes which are rectangles 3x2. Suppose that the distance between the rectangles is equal to 3. So, $s_x=3$, $s_y=1$, $m_x=3$, $m_y=2$, $n_x=3$, $l_x=m_x+n_x=6$. Suppose that the spectral density of the field can be represented in the form

$$f(\lambda,\mu)=f_1(\lambda)f_2(\mu)=B_0/\left|e^{i\lambda}-\beta_1\right|^2\left|e^{i\mu}-\beta_2\right|^2,\ \left|\beta_1\right|<1,\left|\beta_2\right|<1,$$

where $f_1(\lambda)=\sqrt{B_0}/\left|e^{i\lambda}-\beta_1\right|^2$, $f_2(\mu)=\sqrt{B_0}/\left|e^{i\mu}-\beta_2\right|^2$.

The problem is to estimate the functional

$$A_K\xi=\sum_{(k,l)\in K}a(k,l)\xi(k,l)=\sum_{t_1=0}^{2}\left(\sum_{k=6t_1}^{6t_1+2}\sum_{j=0}^{1}a(k,j)\xi(k,j)\right)+$$

$$=\sum_{k=0}^{2}\sum_{j=0}^{1}a(k,j)\xi(k,j)+\sum_{k=6}^{8}\sum_{j=0}^{1}a(k,j)\xi(k,j)+\sum_{k=12}^{14}\sum_{j=0}^{1}a(k,j)\xi,(k,j).$$

Let us extend the functions $\dfrac{1}{f(\lambda,\mu)}$ in Fourier series:

$$1/f(\lambda,\mu)=(1/B_0)\left|e^{i\lambda}-\alpha\right|^2\left|e^{i\mu}-\beta\right|^2=$$

$$=(1/B_0)\left(1+\alpha^2-\alpha e^{-i\lambda}-\alpha e^{i\lambda}\right)\left(1+\beta^2-\beta e^{-i\mu}-\beta e^{i\mu}\right)=$$

$$=\left(1+\alpha^2\right)\left(1+\beta^2\right)-\beta\left(1+\alpha^2\right)e^{-i\mu}-\beta\left(1+\alpha^2\right)e^{i\mu}-\alpha\left(1+\beta^2\right)e^{-i\lambda}+$$

$$+\alpha\beta e^{-i(\lambda+\mu)}+\alpha\beta e^{-i(\lambda-\mu)}-\alpha\left(1+\beta^2\right)e^{i\lambda}+\alpha\beta e^{i(\lambda-\mu)}+\alpha\beta e^{i(\lambda+\mu)}.$$

If we denote Fourier coefficients as

$$r_{00}=\left(1+\alpha^2\right)\left(1+\beta^2\right)=G,$$

$$r_{0,1}=r_{0,-1}=-\beta\left(1+\alpha^2\right)=D,\ r_{1,0}=r_{-1,0}-\alpha\left(1+\beta^2\right)=A,$$

then

$$1/f(\lambda,\mu)=G+De^{-i\mu}+De^{i\mu}+Ae^{-i\lambda}+Ae^{i\lambda}+$$

$$+\alpha\beta e^{-i(\lambda+\mu)}+\alpha\beta e^{-i(\lambda-\mu)}+\alpha\beta e^{i(\lambda-\mu)}+\alpha\beta e^{i(\lambda+\mu)}.$$

The unknown coefficients $c(k,j)$, $(k,j)\in K$ can be computed from the equation

$$\vec{a}=B^K\vec{c},$$

where

$$\vec{a} = \big(a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}, a_{6,0}, a_{6,1}, a_{7,0}, a_{7,1},$$

$$a_{8,0}, a_{8,1}, a_{12,0}, a_{12,1}, a_{13,0}, a_{13,1}, a_{14,0}, a_{14,1}\big)$$

$$\vec{c} = \big(c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}, \ldots c_{12,0}, c_{12,1}, c_{13,0}, c_{13,1}, c_{14,0}, c_{14,1}\big),$$

B^K is matrix from Fourier coefficients of $\dfrac{1}{f(\lambda, \mu)}$:

$$\begin{pmatrix} V & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & V \end{pmatrix},$$

where

$$V = \begin{pmatrix} G & D & A & \alpha\beta & 0 & 0 \\ D & G & \alpha\beta & A & 0 & 0 \\ A & \alpha\beta & G & D & A & \alpha\beta \\ \alpha\beta & A & D & G & \alpha\beta & A \\ 0 & 0 & A & \alpha\beta & G & D \\ 0 & 0 & \alpha\beta & A & D & G \end{pmatrix}.$$

The spectral characteristic $h(\lambda, \mu)$ can be computed by (10) and finally we get linear estimate of the functional $A_K\xi$:

$$\widehat{A}_K\xi = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(\lambda, \mu)\, Z_{\zeta}(d\lambda, d\mu) =$$

$$= \sum_{j=-1}^{2} \xi_{-1,j}\gamma_{-1,j} + \sum_{i=0}^{2}\xi_{i,2}\gamma_{i,2} + \sum_{j=-1}^{2} \xi_{3,j}\gamma_{3,j} + \sum_{i=0}^{2}\xi_{i,-1}\gamma_{i,-1} + \sum_{j=-1}^{2} \xi_{5,j}\gamma_{5,j} +$$

$$+\sum_{i=6}^{8}\xi_{i,2}\gamma_{i,2} + \sum_{j=-1}^{2} \xi_{9,j}\gamma_{9,j} + \sum_{i=6}^{8}\xi_{i,-1}\gamma_{i,-1} + \sum_{j=-1}^{2} \xi_{11,j}\gamma_{11,j} +$$

$$+\sum_{i=12}^{14} \xi_{i,2}\gamma_{i,2} + \sum_{j=-1}^{2} \xi_{15,j}\gamma_{15,j} + \sum_{i=12}^{14} \xi_{i,-1}\gamma_{i,-1},$$

where

$$\gamma_{-1,-1} = -\alpha\beta c_{0.0},\ \gamma_{-1,0} = -Ac_{0.0} - \alpha\beta c_{0.1},\ \gamma_{-1,1} = -Ac_{0.1} - \alpha\beta c_{0.0},$$

$$\gamma_{-1,2} = -\alpha\beta c_{0.1},\ \gamma_{0,2} = -Dc_{0,1} - \alpha\beta c_{1,1},$$

$$\gamma_{1,2} = -Dc_{1,1} - \alpha\beta c_{2,1} - \alpha\beta c_{0,1},\ \gamma_{2,2} = -Dc_{2,1} - \alpha\beta c_{1,1},$$

$$\gamma_{3,2} = -\alpha\beta c_{2,1},\ \gamma_{3,-1} = -\alpha\beta c_{2,0},$$

$$\gamma_{3,1}=-Ac_{2,1}-\alpha\beta c_{2,0},\ \gamma_{3,0}=-Ac_{2,0}-\alpha\beta c_{2,1},$$

$$\gamma_{2,-1}=-Dc_{2,0}-\alpha\beta c_{1,0},\ \gamma_{0,-1}=-Dc_{0,0}-\alpha\beta c_{1,0},$$

$$\gamma_{1,-1}=-Dc(1,0)-\alpha\beta c(2,0)-\alpha\beta c(0,0),$$

$$\gamma_{5,-1}=-\alpha\beta c(6,0),\ \gamma_{5,0}=-Ac(6,0)-\alpha\beta c(6,1),$$

$$\gamma_{5,2}=-\alpha\beta c(6,1),\ \ \gamma_{5,1}=-Ac(6,1)-\alpha\beta c(6,0),$$

$$\gamma_{6,2}=-Dc(6,1)-\alpha\beta c(7,1),\ \xi_{7,2}=-Dc(7,1)-\alpha\beta c(8,1)-\alpha\beta c(6,1),$$

$$\gamma_{8,2}=-Dc_{8,1}-\alpha\beta c_{7,1}\ ,$$

$$\gamma_{9,2}=-\alpha\beta c_{8,1},\ \xi_{9.0}=-Ac_{8,0}-\alpha\beta c_{8,1},\ \gamma_{9,1}=-Ac_{8,1}-\alpha\beta c_{8,0},\ \gamma_{9,-1}=-\alpha\beta c_{8,0},$$

$$\gamma_{6,-1}=-Dc(6,0)-\alpha\beta c(7,0),\ \gamma_{7,-1}=-Dc(7,0)-\alpha\beta c(8,0)-\alpha\beta c(6,0),$$

$$\gamma_{8,-1}=-Dc(8,0)-\alpha\beta c(7,0)$$

$$\gamma_{11,-1}=-\alpha\beta c(12,0),\ \gamma_{11,0}=-Ac(12,0)-\alpha\beta c(12,1),$$

$$\gamma_{11,1}=-Ac(12,1)-\alpha\beta c(12,0),\ \gamma_{11,2}=-\alpha\beta c(12,1),$$

$$\gamma_{12,2}=-Dc(12,1)-\alpha\beta c(13,1),\ \gamma_{13,2}=-Dc(13,1)-\alpha\beta c(14,1)-\alpha\beta c(12,1),$$

$$\gamma_{14,2}=-Dc(14,1)-\alpha\beta c(13,1),$$

$$\gamma_{15,2}=-\alpha\beta c(14,1),\ \gamma_{15,1}=-Ac(14,1)-\alpha\beta c(14,0),$$

$$\gamma_{15,0}=-\alpha\beta c(6,1),\ \gamma_{15,-1}=-\alpha\beta c(14,0),$$

$$\gamma_{12,-1}=-Dc(12,0)-\alpha\beta c(13,0),\gamma_{13,-1}=-Dc(13,0)-\alpha\beta c(14,0)-\alpha\beta c(12,0),$$

$$\gamma_{14,-1}=-Dc(14,0)-\alpha\beta c(13,0).$$

We can see that for estimating the functional were used only values in the neighbouring points (dots). It is naturally because we consider random field of the first order autoregression type for each argument. The mean square error $\Delta(h;f_\xi,f_\eta)$ may be calculated by the formula (9) after calculating vector $\overline{c}$.

Minimax (robust) approach to estimation problem. Formulas (1)–(10) can be applied to compute the spectral characteristic and the mean square error of the optimal linear estimate $\widehat{A}_K\xi$ of the functional $A_K\xi$ from observations of the field $\xi(u,v)+\eta(u,v)$ at points $(u,\upsilon)\in Z^2\setminus K$, $K=\bigcup_{t_1=0}^{s_x}\bigcup_{t_2=0}^{s_y}\left(m_{t_1}^x\times m_{t_2}^y\right)$ if the spectral densities $f_\xi=f$, $f_\eta=g$ are exactly known. In the case where spectral densities $f_\xi=f$, $f_\eta=g$ are not known exactly but sets $D_{(\xi,\eta)}=D_{f_\xi}\times D_{f_\eta}$ of possible spectral densities are given we apply the minimax (robust) approach to estimate the functional $A_K\xi$. With the help of this approach we can find an estimate that minimizes the mean square error for all spectral densities from a given class simultaneously. Such estimates are called minimax (robust).

Definition 1.1. For a given class of spectral densities $D = D_f \times D_g$ the spectral densities $f_0(\lambda,\mu) \in D_f$, $g_0(\lambda,\mu) \in D_g$ are called the *least favourable* in $D = D_f \times D_g$ for the optimal linear estimation of the functional $A_K\xi$ if

$$\Delta(f_0,g_0) = \Delta(h(f_0,g_0);f_0,g_0) = \max_{(f,g)\in D_f\times D_g} \Delta(h(f,g);f,g).$$

Definition 1.2. For a given class of spectral densities $D = D_f \times D_g$ the spectral characteristic $h^0(\lambda,\mu)$ of the optimal linear estimation of the $A_K\xi$ is called *minimax (robust)* if the following conditions holds true

$$h^0(\lambda,\mu) \in H_D = \bigcap_{(f,g)\in D_f\times D_g} L_2^{N-}(f+g),$$

$$\min_{h\in H_D} \max_{(f,g)\in D} \Delta(h;f,g) = \max_{(f,g)\in D} \Delta(h^0;f,g).$$

Spectral densities $f_0(\lambda,\mu), g_0(\lambda,\mu)$ are the *least favourable* in $D = D_f \times D_g$ for the optimal linear estimation of the functional $A_K\xi$, if Fourier coefficients (4), that correspond to the densities $f_0(\lambda,\mu), g_0(\lambda,\mu)$ determine operators $D_0{}^K, B_0{}^K$ by matrices (5) which give a solution to the constrained optimization problem

$$\sup_{(f,g)\in D_f\times D_g} \Delta(h(f_0,g_0);f,g) = \Delta(h(f_0,g_0);f_0,g_0), \qquad (11)$$

where

$$\Delta(h(f_0,g_0);f,g) =$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left| A_K(\lambda,\mu)g_0(\lambda,\mu)\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x-1}\sum_{j=t_2l_y}^{t_2l_y+m_y-1}\left(\left(B^K\right)^{-1}D^K\overline{a}_K\right)_{(k,j)} e^{i(k\lambda+j\mu)}\right)\right|^2}{\left(f_0(\lambda,\mu)+g_0(\lambda,\mu)\right)^2} f(\lambda,\mu)d\lambda d\mu +$$

$$+ \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left| A_K(\lambda,\mu)f_0(\lambda,\mu)\sum_{t_1=0}^{s_x-1}\sum_{t_2=0}^{s_y-1}\left(\sum_{k=t_1l_x}^{t_1l_x+m_x-1}\sum_{j=t_2l_y}^{t_2l_y+m_y-1}\left(\left(B^K\right)^{-1}D^K\overline{a}_K\right)_{(k,j)} e^{i(k\lambda+j\mu)}\right)\right|^2}{\left(f_0(\lambda,\mu)+g_0(\lambda,\mu)\right)^2} g(\lambda,\mu)d\lambda d\mu$$

The constrained optimization problem (11) is equivalent to the unconstrained optimization problem

$$\Delta_D(f,g) = -\Delta\big(h(f_0,g_0); f,g\big) + \delta\big((f,g) \mid D_f \times D_g\big) \to \inf, \quad (12)$$

where $\delta\big((f,g) \mid D_f \times D_g\big)$ is the indicator function of the set $D_f \times D_g$. A solution (f_0, g_0) of the problem (12) is characterized by the condition $0 \in \partial\Delta_D(f_0,g_0)$, where $\partial\Delta_D(f_0,g_0)$ is the subdifferential of the convex functional $\Delta_D(f,g)$ at point (f_0,g_0).

This condition gives us a possibility to determine the least favourable spectral densities for concrete classes of spectral densities.

Least favorable densities in the class $D = D_{2\varepsilon_1}(\lambda,\mu) \times D_{2\varepsilon_2}(\lambda,\mu)$**.** Consider the problem for the class of spectral densities $D = D_{2\varepsilon_1}(\lambda,\mu) \times D_{2\varepsilon_2}(\lambda,\mu)$, where

$$D_{2\varepsilon_1}(\lambda,\mu) = \left\{ f(\lambda,\mu) \mid \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (f(\lambda,\mu) - u_1(\lambda,\mu))^2 \, d\lambda d\mu \le \varepsilon_1 \right\},$$

$$D_{2\varepsilon_2}(\lambda,\mu) = \left\{ g(\lambda,\mu) \mid \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (g(\lambda,\mu) - u_2(\lambda,\mu))^2 \, d\lambda d\mu \le \varepsilon_2 \right\}.$$

Lema 1. Let $u(\lambda,\mu)$ be a non-negative function for $\forall \lambda \in [-\pi,\pi]$, $\forall \mu \in [-\pi,\pi]$, let $\varepsilon > 0$ and

$$D_{2\varepsilon} = \left\{ f(\lambda,\mu) \mid \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (f(\lambda,\mu) - u(\lambda,\mu))^2 \, d\lambda d\mu \le \varepsilon \right\}.$$

Then the subdifferential of the indicator function $\delta(f \mid D_{2\varepsilon})$ can be represented in the form

$$\partial\delta(f_0 \mid D_{2\varepsilon}) = \begin{cases} \{0\}, & \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (f_0(\lambda,\mu) - u(\lambda,\mu))^2 d\lambda \, d\mu < \varepsilon, \\ \{\gamma\varphi_0\}, & \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (f_0(\lambda,\mu) - u(\lambda,\mu))^2 \, d\lambda d\mu = \varepsilon, \end{cases}$$

where

$$\varphi_0(f) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (f_0(\lambda,\mu) - u(\lambda,\mu)) f(\lambda,\mu) \, d\lambda \, d\mu.$$

This lemma is a corollary of results proved in paper.

We can apply the condition $0 \in \partial\Delta_D(f_0,g_0)$ for this class of spectral densities and have the following statement.

Theorem 1. Let spectral densities $f_0(\lambda,\mu)\in D_{2\varepsilon_1}$, $g_0(\lambda,\mu)\in D_{2\varepsilon_2}$ satisfy condition (3) and suppose that functions $h_f\left(f_0,g_0\right)$, $h_g\left(f_0,g_0\right)$, computed by the formula

$$h_f\left(f_0,g_0\right)=\frac{\left|A_K\left(\lambda,\mu\right)g_0\left(\lambda,\mu\right)+C_K\left(\lambda,\mu\right)\right|}{f_0\left(\lambda,\mu\right)+g_0\left(\lambda,\mu\right)}, \qquad (13)$$

$$h_g\left(f_0,g_0\right)=\frac{\left|A_K\left(\lambda,\mu\right)f_0\left(\lambda,\mu\right)-C_K\left(\lambda,\mu\right)\right|}{f_0\left(\lambda,\mu\right)+g_0\left(\lambda,\mu\right)} \qquad (14)$$

are bounded. Spectral densities $f_0(\lambda,\mu)\in D_{2\varepsilon_1}$, $g_0(\lambda,\mu)\in D_{2\varepsilon_2}$ are the least favorable in the class $D_{2\varepsilon_1}\times D_{2\varepsilon_2}$ for the optimal linear estimation of the functional $A_K\xi$, if they satisfy relations

$$\left|A_K(\lambda,\mu)\frac{g_0(\lambda,\mu)}{f_0(\lambda,\mu)+g_0(\lambda,\mu)}+\frac{\sum\limits_{t_1=0}^{s_x-1}\sum\limits_{t_2=0}^{s_y-1}\left(\sum\limits_{k=t_1l_x}^{t_1l_x+m_x-1}\sum\limits_{j=t_2l_2}^{t_2l_2+m_2-1}((B_0^K)^{-1}D_0^K\overline{a}_K\right)}{f_0(\lambda,\mu)+g_0(\lambda,\mu)}\right|^2=$$

$$=(f_0(\lambda,\mu)-u_1(\lambda,\mu))\gamma_1(\lambda),$$

$$\left|A_K(\lambda,\mu)\frac{f_0(\lambda,\mu)}{f_0(\lambda,\mu)+g_0(\lambda,\mu)}-\frac{\sum\limits_{t_1=0}^{s_x-1}\sum\limits_{t_2=0}^{s_y-1}\left(\sum\limits_{k=t_1l_x}^{t_1l_x+m_x-1}\sum\limits_{j=t_2l_2}^{t_2l_2+m_2-1}((B_0^K)^{-1}D_0^K\overline{a}_K\right)}{f_0(\lambda,\mu)+g_0(\lambda,\mu)}\right|^2=$$

$$=(g_0(\lambda,\mu)-u_2(\lambda,\mu))\gamma_2(\lambda)$$

and determine a solution of the extremum problem (11), where $\gamma_1(\lambda)\geq 0,\gamma_2(\lambda)\geq 0$ can be computed by conditions

$$\frac{1}{4\pi^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}(f_0\left(\lambda,\mu\right)-u_1\left(\lambda,\mu\right))^2\,d\mu=\varepsilon_1\,,$$

$$\frac{1}{4\pi^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}(g_0\left(\lambda,\mu\right)-u_2\left(\lambda,\mu\right))^2\,d\mu=\varepsilon_2\,.$$

The minimax (robust) spectral characteristic $h\left(f_0,g_0\right)$ may be calculated by formula (7).

Conclusions. In the following papers we will propose formulas for the least favourable spectral densities in various classes of spectral densities and the minimax-robust spectral characteristic of the optimal linear estimates of a functional that depends on the unknown values of a random field based on observations in some regions of the plane.

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Досліджується задача оцінювання лінійних функціоналів від невідомих значень однорідного випадкового поля $\xi(k,j)$ для області $K \subset Z^2$ за спостереженями суми полів $\xi(k,j)+\eta(k,j)$ в точках $(k,j) \in Z^2 \setminus K$. Знайдено формули для обчислення середньоквадратичної похибки та спектральної характеристики оптимальної лінійної оцінки функціола у випадку відомих спектральних щільностей полів. Запропоновано формули для визначення найменш сприятливої спектральної щільності та мінімаксної (робастної) спектральної характеристики у випадку, коли спектральна характеристика точно не відома, але клас спектральних характеристик, до якого належить спектральна щільність визначено.

Ключові слова: *випадкове поле, задача оцінювання, мінімаксна (робастна) спектральна характеристика.*

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