STRONG CONSISTENCY OF DISCRETE ESTIMATORS FOR DRIFT PARAMETERS IN THE COX–INGERSOLL–ROSS MODEL

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In the paper [1], a new model for analyzing interest rates, which is called the Cox–Ingersoll–Ross process, was introduced. This model is described by the following stochastic differential equation:

$$r_t = r_0 + \int_0^t (a - br_s)\, ds + \int_0^t \sigma\sqrt{r_s}\, dW_s, \quad t \geq 0, \tag{1}$$

where r_0, a, b and σ are positive constants and $W = \{W_t, t \geq 0\}$ is a Wiener process.

The Cox–Ingersoll–Ross model is important and popular. It is used in many areas, for example, in the pricing of interest rate derivatives and as a model of stochastic volatility. Thus parameter estimation is significant for the practical application of this model.

We are interested in developing theoretical estimators for parameters (a, b) when σ is known for models based on discrete observations. Additionally, we aim to explore how estimators designed for continuous data can be adapted for the case of discrete observations. Therefore in this study we investigate the discrete counterparts of the strongly consistent estimators introduced in [2, Theorem 5].

To construct the estimators, we use an equidistant partition of the interval $[0, n]$, $n > 0$, with a step size $\delta_n = n^{-\beta}$, where $\beta > 0$. Also, two additional notations are introduced: the number of observations, denoted as $m_n = n^{\beta+1}$, and the observation moments, denoted as $t_k = \frac{k}{n^\beta}$.

Our estimates for parameters a and b on the interval $[0, n]$ are the following:

$$\hat{a}_n = \frac{\frac{\sigma^2}{2}\left(\sum\limits_{k=0}^{m_n-1} r_{t_k}\delta_n\right)^2}{n\sum\limits_{k=0}^{m_n-1} r_{t_k}^2\delta_n - \left(\sum\limits_{k=0}^{m_n-1} r_{t_k}\delta_n\right)^2}, \quad \hat{b}_n = \frac{\frac{\sigma^2}{2}\cdot n\sum\limits_{k=0}^{m_n-1} r_{t_k}\delta_n}{n\sum\limits_{k=0}^{m_n-1} r_{t_k}^2\delta_n - \left(\sum\limits_{k=0}^{m_n-1} r_{t_k}\delta_n\right)^2}.$$

We have acquired the rate of convergence of certain stochastic integrals and sums from estimators $\hat{a}_n$ and $\hat{b}_n$. The main results are summarized in the following two theorems.

Theorem 1. *For $2a > \sigma^2$ and $\beta > 0$ the following convergence is true*

$$\frac{1}{n}\sum_{k=0}^{m_n-1} r_{t_k}^2 \delta_n \to \frac{a^2}{b^2} + \frac{a\sigma^2}{2b^2} \ \text{a.s., as } n \to \infty. \tag{2}$$

Moreover, for any $q \geq 1$

$$\mathsf{E}\left|\frac{1}{n}\sum_{k=0}^{m_n-1} r_{t_k}^2 \delta_n - \left(\frac{a^2}{b^2} + \frac{a\sigma^2}{2b^2}\right)\right|^q \leq Cn^{-\frac{(\beta\wedge 1)q}{2}}. \tag{3}$$

Theorem 2. *Assume that $2a > \sigma^2$. Then the following convergences hold a.s. as $n \to \infty$:*

$$\hat{a}_n \to a, \quad \hat{b}_n \to b,$$

i.e. $\hat{a}_n, \hat{b}_n$ are strongly consistent estimators of the parameters a, b respectively.

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References

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