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MORE ON LINEAR AND METRIC TREE MAPS

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Abstract. We consider linear and metric self-maps on vertex sets of finite combinatorial trees. Linear maps are maps which preserve intervals between pairs of vertices whereas metric maps are maps which do not increase distances between pairs of vertices. We obtain criteria for a given linear or a metric map to be a positive (negative) under some orientation of the edges in a tree, we characterize trees which admit maps with Markov graphs being paths and prove that the converse of any partial functional digraph is isomorphic to a Markov graph for some suitable map on a tree.

Keywords: tree, Markov graph, metric map, non-expanding map, linear map, graph homomorphism.

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1. INTRODUCTION

A map $f : V(G_1) \to V(G_2)$ between vertex sets of two connected finite undirected simple graphs G_1 and G_2 is called *metric* (or *non-expanding*, or *1-Lipschitz*) provided $d_{G_2}(f(u), f(v)) \leq d_{G_1}(u, v)$ for all pairs $u, v \in V(G_1)$ (here $d_G(u, v)$ denotes the standard distance between two vertices u and v in a connected graph G). It can be easily seen that a map f is metric if and only if $d_{G_2}(f(u), f(v)) \leq 1$ for all edges $uv \in V(G_1)$. Thus, metric maps provide natural generalization of graph homomorphisms. The properties of metric self-maps on vertex sets of general connected graphs can be found, for example, in [1, 7–11]. In [7] it was proved that trees can be characterized as connected graphs having the property that each their metric self-map has a fixed edge. The existence of a fixed hypercube for every metric self-map on a median graph is proven in [1]. In [8, 9] the problem of existence of metric retractions for various types of subgraphs was studied. The complete characterization of (connected) graphs having regular semigroups of metric self-maps was obtained in [10]. In particular, we note that stars are the only trees with regular semigroups of metric self-maps.

A map $f : V(G_1) \to V(G_2)$ is called *linear* if $f([u,v]_{G_1}) \subset [f(u), f(v)]_{G_2}$ for all pairs $u, v \in V(G_1)$ (here $[u,v]_G$ denotes the metric interval between u and v in a connected graph G). Linear and metric self-maps on vertex sets of trees were studied in [6] using the notion of Markov graphs. Namely, having arbitrary self-map $\sigma : V(X) \to V(X)$ on the vertex set of a tree X, the corresponding *Markov graph* $\Gamma = \Gamma(X, \sigma)$ is a directed graph with the vertex set $V(\Gamma) = E(X)$ and the arc set $A(\Gamma) = \{(u_1v_1, u_2v_2) : u_2, v_2 \in [\sigma(u_1), \sigma(v_1)]_X\}$. Thus, the vertices of Γ are the edges of X and there is an arc $e_1 \to e_2$ in Γ if the edge e_1 "covers" e_2 under the map σ. A digraph is called *partial functional* if the outdegrees of its vertices are bounded by one. With these definitions, it is easy to see that $\sigma : V(X) \to V(X)$ is a metric map on a tree X if and only if its Markov graph $\Gamma(X, \sigma)$ is partial functional. The dual result for linear maps was obtained in [6]. By Γ^{co} we denote the *converse* digraph of a given digraph Γ (which is obtained from Γ by reversing orientations of the arcs).

Theorem 1.1 ([6]). *A map $\sigma : V(X) \to V(X)$ on a tree X is linear if and only if the converse digraph $(\Gamma(X, \sigma))^{co}$ is partial functional.*

Corollary 1.2 ([6]). *A map $\sigma : V(X) \to V(X)$ on a tree X is a linear metric map if and only if each weak component of $\Gamma(X, \sigma)$ is a path or a cycle.*

Moreover, the Markov graph $\Gamma(X, \sigma)$ is a disjoint union of cycles if and only if σ is an automorphism of X (see [2]). Thus if $\Gamma(X, \sigma)$ is a cycle, then X is a star. Similarly, in [6] it was proved that if $\Gamma(X, \sigma)$ is a path, then X is a *spider* (a tree with at most one vertex of degree at least three).

In this paper we provide criteria for linear and metric maps on trees X to be τ-positive (τ-negative) for some orientations τ of X. Further, we characterize all spiders X which admit maps σ with Markov graphs $\Gamma(X, \sigma)$ being paths. Finally, we prove that the converse of a partial functional digraph is isomorphic to a Markov graph $\Gamma(X, \sigma)$ for some suitable pair (X, σ).

2. PRELIMINARIES

2.1. GRAPHS AND DIGRAPHS

In this paper we consider undirected as well as directed graphs. All graphs assumed to be simple and finite. By a *graph* G we mean a pair (V, E), where $V = V(G)$ is the set of *vertices* and $E = E(G)$ is the set of *edges* (which are unordered pairs of vertices) in G. The existence of an edge $\{u, v\}$ in G will be shortly denoted as $uv \in E(G)$. For a set of vertices $U \subset V(G)$ the corresponding *induced subgraph* $G[U]$ is a graph G' with $V(G') = U$ and $E(G') = \{uv \in E(G) : u, v \in U\}$. We define $G - U$ to be the subgraph induced by the set of vertices $V(G) \backslash U$.

For a vertex $u \in V(G)$ put $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ for its *neighborhood* and $N_G[u] = N_G(u) \cup \{u\}$ for the *closed neighborhood*. The cardinality of $N_G(u)$ is called the *degree* of a vertex u. A vertex of degree one is called a *leaf*. The corresponding unique edge incident to it is called a *leaf edge*. By $L(G)$ we denote the set of leaf vertices in G.

A graph is *connected* provided each pair of its vertices can be joined by a path. A subset $U \subset V(G)$ is called *connected* if its induced subgraph $G[U]$ is connected. In a connected graph G by $d_G(u,v)$ we denote the *distance* between two vertices $u, v \in V(G)$ which equals the number of edges on a shortest $u-v$ path in G. Denote by

$$[u,v]_G = \{w \in V(G) : d_X(u,w) + d_X(w,v) = d_X(u,v)\}$$

the metric interval between a pair of vertices $u, v \in V(G)$. A set $U \subset V(G)$ is called *convex* provided $[u,v]_X \subset U$ whenever $u, v \in U$. Observe that a convex set is always connected. For a set of vertices $U \subset V(G)$ by $\mathrm{Conv}_X(U)$ we denote its *convex hull* which is the smallest convex set containing U.

A connected graph G is called *median* if for any triple of its vertices $u, v, w \in V(G)$ the intersection of three intervals $[u,v]_G \cap [u,w]_G \cap [v,w]_G$ is a singleton. The corresponding unique vertex is called the *median* of a triple u, v, w and denoted by $m_G(u,v,w)$.

A set of vertices $F \subset V(G)$ in a connected graph G is called *Chebyshev* provided for any vertex $u \in V(G)$ there is a unique vertex $v \in F$ with

$$d_G(u,v) = d_G(u,F) = \min\{d_G(u,w) : w \in F\}.$$

The corresponding vertex $\mathrm{pr}_F(u) = v$ is called the *projection* of a vertex u on a Chebyshev set F.

A *tree* is a connected acyclic graph. Paths (connected graphs G with diameter $|V(G)| - 1$) and stars (connected graphs G with $|L(G)| \geq |V(G)| - 1$) are prime examples of trees. A *spider* is a tree having at most one vertex of degree at least three. If such a vertex does not exist, then the spider has to be a path. In this case, the spider will be called *trivial*. Thus, a non-trivial spider (also, a *starlike* tree) has a unique vertex of degree at least three which is called its *center*. For a collection of natural numbers $a_1, \ldots, a_m \in \mathbb{N}$ denote by $\mathrm{Sp}(a_1, \ldots, a_m)$ a (unique up to isomorphism) spider X centered at $u \in V(X)$ such that the multiset of distances $\{d_X(u,v) : v \in L(X)\}$ equals $\{a_1, \ldots, a_m\}$. For example, $\mathrm{Sp}(n)$ is a path with $n+1$ vertices and $\mathrm{Sp}(1, 1, \ldots, 1)$ is just a star. An *orientation* of a tree X is a map $\tau : E(X) \to V(X)$ such that $\tau(e)$ is incident to e for all edges $e \in E(X)$.

A *digraph* D is a pair (V, A), where $V = V(D)$ is the *vertex set* and $A = A(D) \subset V \times V$ is the *arc set* of D. If there is an arc $(u,v) \in A(D)$, then we will write $u \to v$ in D. An arc of the form $u \to u$ is called a *loop*. Put

$$N_D^+(u) = \{v \in V(D) : (u,v) \in A(D)\}$$

and

$$N_D^-(u) = \{v \in V(D) : (v,u) \in A(D)\}.$$

The numbers $d_D^+(u) = |N_D^+(u)|$ and $d_D^-(u) = |N_D^-(u)|$ are called the *outdegree* and the *indegree* of a vertex u in a digraph D, respectively. Denote by $V_0^+(D)$ and $V_0^-(D)$ the sets of vertices with zero outdegrees and zero indegrees in D, respectively.

A digraph D is called *weakly connected* provided its underlying graph (which is obtained from D by ignoring loops and arc orientations) is connected. By $D_1 \sqcup D_2$ we denote the *disjoint union* of two digraphs D_1 and D_2.

If two digraphs D_1 and D_2 are isomorphic, we write $D_1 \simeq D_2$.

For a linear ordering of the vertex set $V(D) = \{u_1, \dots, u_n\}$ of a digraph D denote by $M_D = (a_{ij})$ the corresponding *adjacency matrix*, where $a_{ij} = 1$ if $u_i \to u_j$ in D and $a_{ij} = 0$ otherwise.

2.2. MARKOV GRAPHS FOR MAPS ON TREES

Denote by $\mathcal{T}(X)$ the full transformation semigroup on the vertex set $V(X)$ of a given n-vertex tree X and by $\mathrm{Mat}_{n-1}(\mathbb{F}_2)$ the semigroup of $(n-1) \times (n-1)$-matrices over two-element field.

Theorem 2.1 ([2]). *Let X be a tree with n vertices and suppose that some linear ordering of $E(X)$ is fixed. Then the function $M : \mathcal{T}(X) \to \mathrm{Mat}_{n-1}(\mathbb{F}_2)$, $M(\sigma) = M_{\Gamma(X,\sigma)}$ is a semigroup homomorphism.*

The next result shows that the set of edges in a tree X having non-zero indegrees in $\Gamma(X, \sigma)$ always induces a subtree (i.e. a connected subgraph) in X. By $\mathrm{Im}\,\sigma$ we denote the image of a map σ.

Proposition 2.2 ([5]). *Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. Put $E(\sigma) = \{e \in E(X) : d^-_\Gamma(e) \geq 1\}$. Then $E(\sigma) = E(\mathrm{Conv}_X(\mathrm{Im}\,\sigma))$.*

As was noted above, automorphisms of trees can be easily characterized in terms of their Markov graphs.

Proposition 2.3 ([2]). *For a tree X and its map $\sigma : V(X) \to V(X)$ the Markov graph $\Gamma(X, \sigma)$ is a disjoint union of cycles if and only if σ is an automorphism of X.*

By an *edge labeling* on a tree X we will mean any map of the form

$$\tau : E(X) \to V(X) \cup \{1, -1\}.$$

Any such a pair (X, τ) will be called *mixed tree* and denoted by $X(\tau)$.

Let X be a tree and $e = uv \in E(X)$ be its edge. Put

$$A_X(u, v) = \{w \in V(X) : d_X(u, w) \leq d_X(v, w)\}$$

for the set of vertices which are closer to u than to v in X. In a similar fashion one can define the second set

$$A_X(v, u) = \{w \in V(X) : d_X(v, w) \leq d_X(u, w)\}.$$

A map $\sigma : V(X) \to V(X)$ naturally induces the *edge labeling* $\tau_\sigma : E(X) \to V(X) \cup \{-1,1\}$ defined as follows: $\tau_\sigma(e) = u$ if $\sigma(u), \sigma(v) \in A_X(u, v)$; $\tau_\sigma(e) = v$ if $\sigma(u), \sigma(v) \in A_X(v, u)$; $\tau_\sigma(e) = 1$ if $\sigma(u) \in A_X(u, v)$ and $\sigma(v) \in A_X(v, u)$; $\tau_\sigma(e) = -1$ if $\sigma(u) \in A_X(v, u)$ and $\sigma(v) \in A_X(u, v)$ for every edge $e = uv \in E(X)$. If $\tau_\sigma(e) = u$ (or $\tau_\sigma(e) = v$), then the edge e gets an orientation $v \to u$ (or $u \to v$). In other cases, e is *σ-positive* or *σ-negative* depending on the sign of $\tau_\sigma(e)$. A labeling $\tau : E(X) \to V(X) \cup \{1, -1\}$ is called *admissible* if $\tau = \tau_\sigma$ for some map σ.

Theorem 2.4 ([3])**.** *Let X be a tree and $\tau : E(X) \to V(X) \cup \{1, -1\}$ be an edge labeling such that the restriction $\tau|_{\tau^{-1}(V(X))}$ is an orientation of X. Then τ is admissible if and only if:*

(1) *the outdegree of each vertex from $X(\tau)$ is at most one,*
(2) *each vertex from $X(\tau)$ is incident to at most one τ-negative edge,*
(3) *if a vertex from $X(\tau)$ is incident to a τ-negative edge, then it has zero outdegree.*

Any admissible edge labeling τ naturally defines the corresponding map $\sigma_\tau : V(X) \to V(X)$ as follows: $\sigma_\tau(u) = v$ if $uv \in E(G)$ and $\tau(uv) \in \{v, -1\}$ and $\sigma_\tau(u) = u$ otherwise.

For a map σ by $\operatorname{fix} \sigma$ we denote the set of its fixed points and by $p(X, \sigma)$, $n(X, \sigma)$ the number of σ-positive, σ-negative edges in X, respectively.

Theorem 2.5 ([3])**.** *For any tree X and its map $\sigma : V(X) \to V(X)$ we have*

$$n(X, \sigma) + |\operatorname{fix} \sigma| = p(X, \sigma) + 1.$$

Note that from Theorem 2.5 it follows that for a map σ on a tree X without fixed points we always have $n(X, \sigma) \geq 1$. If, additionally, σ is metric, then every σ-negative edge is fixed by σ.

Given an orientation $\tau : E(X) \to V(X)$ of a tree X and a map $\sigma : V(X) \to V(X)$, an arc $e_1 \to e_2$ in $\Gamma(X, \sigma)$ is called *τ-positive* provided $\mathrm{pr}_{e_2}(\sigma(\tau(e_1))) = \tau(e_2)$. Otherwise, $e_1 \to e_2$ is a *τ-negative* arc. A map σ is *τ-positive* (or *τ-negative*) if all the arcs in $\Gamma(X, \sigma)$ are τ-positive (or τ-negative). Note that a composition of two τ-positive maps as well as two τ-negative maps is always τ-positive.

A digraph Γ is called an *M-graph* if there exists a tree X and its map $\sigma : V(X) \to V(X)$ such that $\Gamma \simeq \Gamma(X, \sigma)$. Each such a pair (X, σ) will be called a *realization* of an M-graph Γ. The following simple observation will be used later in the paper.

Remark 2.6. Let Γ_1 and Γ_2 be a pair of M-graphs such that there exist their realizations (X_i, σ_i), $i = 1, 2$ with $\operatorname{fix} \sigma_i \neq \emptyset$. Then the disjoint union $\Gamma_1 \sqcup \Gamma_2$ is also an M-graph.

2.3. LINEAR AND METRIC MAPS ON TREES

The next result shows that linear maps between median graphs are exactly the maps which preserve medians.

Proposition 2.7 ([6])**.** *Let G_1, G_2 be two median graphs and $f : V(G_1) \to V(G_2)$ be some map. Then f is linear if and only if*

$$f(m_{G_1}(u, v, w)) = m_{G_2}(f(u), f(v), f(w))$$

for all triplets of vertices $u, v, w \in V(G_1)$.

We note that projection on a Chebyshev set need not to be a linear or a metric map as the following example suggests.

Example 2.8. Consider a graph G with the vertex set $V(G) = \{1, \dots, 8\}$ and the edge set $E(G) = \{12, 14, 23, 25, 37, 45, 48, 56, 67, 78\}$. Clearly, G is connected and the set $A = \{4, \dots, 8\}$ is Chebyshev in G. We have $2 \in [1, 3]_G$, but

$$\mathrm{pr}_A(2) = 5 \notin [4, 7]_G = [\mathrm{pr}_A(1), \mathrm{pr}_A(3)]_G.$$

Thus pr_A is not linear. Similarly,

$$d_G(2, 3) = 1 < 2 = d_G(5, 7) = d_G(\mathrm{pr}_A(2), \mathrm{pr}_A(3)).$$

Hence, pr_A is not metric.

However, if we restrict ourselves to trees, we can guarantee that projections on connected (and thus Chebyshev) sets are always linear and metric maps. In fact, projections whose images contain at least three vertices can be characterized as maps having Markov graphs with every arc being a loop (see [2]).

It should be also noted that the image of a metric map is always a connected set. In fact, this condition ensures that a given linear map on a tree is metric.

Proposition 2.9 ([6])**.** *A linear map $\sigma : V(X) \to V(X)$ on a tree X is metric if and only if its image* $\mathrm{Im}\,\sigma$ *is a connected set.*

One can observe that an automorphism of a tree is a linear metric permutation of its vertices. The converse statement is also true.

Corollary 2.10. *For a tree X and its $\sigma : V(X) \to V(X)$ the following statements are equivalent:*

(1) *σ is an automorphism of X,*
(2) *σ is a metric permutation,*
(3) *σ is a linear permutation.*

Proof. Trivially, the first statement implies the third (see Proposition 2.3 and Theorem 1.1). If σ is a linear permutation, then $\mathrm{Im}\,\sigma = V(X)$ is a connected set. Hence, by Proposition 2.9, σ is also a metric permutation. This means that the third statement implies the second. Finally, let σ be a metric permutation. Then the Markov graph $\Gamma(X, \sigma)$ is partial functional. Since σ is a permutation, then the inequality $d^-_{\Gamma(X,\sigma)}(e) \geq 1$ holds for every edge $e \in E(X)$. Therefore, $\Gamma(X, \sigma)$ is a disjoint union of cycles. By Proposition 2.3, σ is an automorphism of X. Thus, the second statement implies the first. $\square$

3. LINEAR AND METRIC MAPS AS τ-POSITIVE AND τ-NEGATIVE MAPS

We start by showing that a metric map on a tree can have at most one negative edge.

Proposition 3.1. *Let $\sigma : V(X) \to V(X)$ be a metric map on a tree X. Then*

(1) *$n(X, \sigma) \leq 1$ and the equality $n(X, \sigma) = 1$ implies $p(X, \sigma) = 0$,*
(2) *if $n(X, \sigma) = 0$, then the set of σ-positive edges in X induces a subtree X' such that for any vertex $u \in V(X')$ and an arc $v \to w$ in $X(\tau_\sigma)$ it holds $w \in [u, v]_X$.*

Proof. Let $e_i = u_i v_i \in E(X)$, $i = 1, 2$ be a pair of different edges in X. Without loss of generality, we can assume that $v_1 \in [u_1, u_2]_X$ and $u_2 \in [v_1, v_2]_X$. If both e_1 and e_2 are σ-negative, then from Theorem 2.4 we obtain $v_1 \neq u_2$. Since σ is a metric map, $\sigma(v_1) = u_1$ and $\sigma(u_2) = v_2$. Thus,

$$d_X(\sigma(v_1), \sigma(u_2)) = d_X(u_1, v_2) = d_X(v_1, u_2) + 2 > d_X(v_1, u_2)$$

which is a contradiction. Therefore, $n(X, \sigma) \leq 1$.

If the edge e_1 is σ-negative and the edge e_2 is σ-positive, then we obtain equalities $\sigma(v_1) = u_1$ and $\sigma(u_2) = u_2$. Note that in this case $v_1 \neq u_2$ as well. Moreover, we have the inequality

$$d_X(\sigma(v_1), \sigma(u_2)) = d_X(v_1, u_2) + 1 > d_X(v_1, u_2)$$

which is a contradiction again. Therefore, the first statement of the proposition holds.

Now let $n(X, \sigma) = 0$. From Theorem 2.5 it follows that in this case $|\operatorname{fix} \sigma| = p(X, \sigma) + 1$. However, σ is metric implying that each vertex which is incident to a σ-positive edge is a fixed point for σ. Therefore, the set of σ-positive edges induces a subtree X' in X. Furthermore, if there exists a vertex $u \in V(X')$ and an arc $v \to w$ in $X(\tau_\sigma)$ with $w \notin [u, v]_X$, then $v \in [u, w]_X$ (since $vw \in E(X)$). This would imply

$$d_X(\sigma(u), \sigma(v)) = d_X(u, \sigma(v)) \geq d_X(u, w) = d_X(u, v) + 1 > d_X(u, v)$$

which is a contradiction. Therefore, the second statement of the proposition holds. □

A similar result holds for linear maps.

Proposition 3.2. *Let $\sigma : V(X) \to V(X)$ be a linear map on a tree X. Then*

(1) *$n(X, \sigma) \leq 1$ and the equality $n(X, \sigma) = 1$ implies $p(X, \sigma) = 0$,*
(2) *if $n(X, \sigma) = 0$, then each vertex with a non-zero outdegree in $X(\tau_\sigma)$ is incident to at most one σ-positive edge.*

Proof. Let $e_i = u_i v_i \in E(X)$, $i = 1, 2$ be a pair of different edges in X. Without loss of generality, we can assume that $v_1 \in [u_1, u_2]_X$ and $u_2 \in [v_1, v_2]_X$. If both e_1 and e_2 are σ-negative, then $\sigma(u_1) \in A_X(v_1, u_1), \sigma(v_1) \in A_X(u_1, v_1)$ and $\sigma(u_2) \in A_X(v_2, u_2)$. This means that for the vertex $v_1 \in [u_1, u_2]_X$ we have $\sigma(v_1) \notin [\sigma(u_1), \sigma(u_2)]_X$. A contradiction with linearity of σ. Similarly, if the edge e_1 is σ-negative and the edge e_2 is σ-positive, then for the vertex $v_1 \in [u_1, v_2]_X$ it holds $\sigma(v_1) \notin [\sigma(u_1), \sigma(v_2)]_X$ which is a contradiction again. Therefore, the first statement of the proposition holds.

Now let $n(X, \sigma) = 0$ and suppose that for an edge $uv \in E(X)$ we have $\tau_\sigma(uv) = v$. If the edges $e_i = uw_i \in E(X)$, $i = 1, 2$ are both σ-positive, then there is a pair of arcs $uw_i \to uv$, $i = 1, 2$ in $\Gamma(X, \sigma)$. From Theorem 1.1 it follows that σ is not a linear map. The obtained contradiction proves the second statement of the proposition. □

From the definition of a τ-positive map it clearly follows that each such a map σ cannot have σ-negative edges. We show that this condition is sufficient for a given linear or a metric map to be τ-positive for some orientation τ of a tree.

Theorem 3.3. *Let $\sigma : V(X) \to V(X)$ be a linear or a metric map on a tree X with $|V(X)| \geq 2$. Then there exists an orientation τ of X such that σ is τ-positive if and only if $n(X, \sigma) = 0$.*

Proof. The necessity of the condition is clear. Thus we must only prove its sufficiency. At first, suppose σ is a linear map. Since $n(X, \sigma) = 0$, from Theorem 2.5 we can conclude that there exists a fixed point $u_0 \in \operatorname{fix} \sigma$. Consider an orientation τ of X such that $X(\tau)$ is an in-tree rooted at u_0. We want to prove that σ is τ-positive. To the contrary, assume there is a τ-negative arc $e_1 \to e_2$ in $\Gamma(X, \sigma)$. This means that

$$\mathrm{pr}_{e_2}(\sigma(\tau(e_1))) = \tau^{-1}(e_2) \quad \text{and} \quad \mathrm{pr}_{e_2}(\sigma(\tau^{-1}(e_1))) = \tau(e_2).$$

We have $\tau(e_1) \in [\tau^{-1}(e_1), u_0]_X$, however

$$\sigma(\tau(e_1)) \notin [\sigma(\tau^{-1}(e_1)), u_0]_X = [\sigma(\tau^{-1}(e_1)), \sigma(u_0)]_X$$

which contradicts the linearity of σ.

Further, we use induction on $|V(X)| \geq 2$. The case $|V(X)| = 2$ is obvious. Therefore, let $|V(X)| \geq 3$ and σ be a non-linear metric map. Then for all edges $e \in E(X)$ we have $d^+_{\Gamma(X,\sigma)}(e) \leq 1$ and there exists an edge $e' \in E(X)$ with $d^-_{\Gamma(X,\sigma)}(e') \geq 2$ (see Theorem 1.1). Using the equality

$$|A(\Gamma(X, \sigma))| = \sum_{e \in E(X)} d^+_{\Gamma(X,\sigma)}(e) = \sum_{e \in E(X)} d^-_{\Gamma(X,\sigma)}(e),$$

we can conclude that there exists an edge $uv \in E(X)$ with zero indegree in $\Gamma(X, \sigma)$. In light of Proposition 2.2 we can also assume that uv is a leaf edge. Thus, let $u \in L(X)$. Consider the tree $X' = X - \{u\}$ and the map $\sigma' = \mathrm{pr}_{V(X) \setminus \{u\}} \circ \sigma$. Clearly, σ' is a metric map on X'. By induction assumption, there exists an orientation τ' of X' such that σ' is τ'-positive. If $\sigma(u) = \sigma(v)$ or $u, v \in \operatorname{fix} \sigma$, then put

$$\tau(e) = \begin{cases} \tau'(e), & \text{if } e \neq uv, \\ v, & \text{if } e = uv \end{cases}$$

for all $e \in E(X)$. Otherwise, let $\sigma(u) \neq \sigma(v)$ and $\sigma(u) \neq u$ or $\sigma(v) \neq v$. Since $d^-_{\Gamma(X,\sigma)}(uv) = 0$, then $\sigma(u)\sigma(v) \in E(X')$ and therefore $\sigma^{-1}(\tau'(\sigma(u)\sigma(v))) \cap \{u, v\}$ is a singleton set. In this case we put

$$\tau(e) = \begin{cases} \tau'(e), & \text{if } e \neq uv, \\ w, & \text{if } e = uv \text{ and } \sigma^{-1}(\tau'(\sigma(u)\sigma(v))) \cap \{u, v\} = \{w\} \end{cases}$$

for all $e \in E(X)$. From the construction of τ it clearly follows that σ is a τ-positive map. □

It is also clear that a τ-negative map σ cannot have σ-positive edges. However, the condition $p(X, \sigma) = 0$ is not sufficient for a given linear or a metric map σ to be τ-negative for some orientation τ of X. For example, each automorphism of the star $K_{1,3}$ which cyclically permutes the set of its leaf vertices cannot be τ-negative for any orientation τ of $K_{1,3}$.

Theorem 3.4. *Let $\sigma : V(X) \to V(X)$ be a linear or a metric map on a tree X with $|V(X)| \geq 2$. Then there exists an orientation τ of X such that σ is τ-negative if and only if $p(X, \sigma) = 0$ and each cycle in $\Gamma(X, \sigma)$ is either a loop or has an even length.*

Proof. If σ is a τ-negative map, then clearly $p(X, \sigma) = 0$. Further, suppose that σ is metric and let $e_1 \to \ldots \to e_m \to e_1$ be a cycle in $\Gamma(X, \sigma)$ with $m \geq 1$. If m is odd, then σ^m is also a τ-negative map. Since σ is metric, then each edge e_i is σ^m-negative. By Proposition 3.1, $m \leq n(X, \sigma^m) \leq 1$ implying $m = 1$.

Now let σ be linear. We use induction on $|V(X)| \geq 2$. The induction basis clearly holds. If, additionally, σ is a metric map, then we are done. Thus, let σ be a linear non-metric map. This implies the existence of an edge $uv \in E(X)$ with zero outdegree in $\Gamma(X, \sigma)$. Consider the tree X' with the vertex set $V(X') = V(X) \backslash \{v\}$ and the edge set $E(X') = E(X - \{v\}) \cup \{uw : w \in N_X(v)\}$. One can think that X' is obtained from X by "contracting" the edge uv into a single vertex. Furthermore, consider the map

$$\sigma'(w) = \begin{cases} \sigma(w), & \text{if } \sigma(w) \neq v, \\ u, & \text{if } \sigma(w) = v \end{cases}$$

for all $w \in V(X')$. It is easy to see that σ' is linear on X' and also $\Gamma(X', \sigma') \simeq \Gamma(X, \sigma) - \{uv\}$. Since uv has zero outdegree in $\Gamma(X, \sigma)$ it does not lie on any cycle in $\Gamma(X, \sigma)$. By induction assumption, each non-loop cycle in $\Gamma(X', \sigma')$ has an even length and thus the same holds also for $\Gamma(X, \sigma)$. Therefore, the necessity of the condition is proved.

Now we prove the sufficiency of this condition. At first, assume that σ is an automorphism of X. Then by Proposition 2.3, the Markov graph $\Gamma = \Gamma(X, \sigma)$ is a disjoint union of cycles. Hence, Γ can be viewed as a permutation of the edge set $E(X)$. Let $E(X) = \bigsqcup_{i=1}^{k} \mathrm{orb}_\Gamma(e_i)$ and $m_i = |\mathrm{orb}_\Gamma(e_i)|$, where $e_i = u_i v_i \in E(X)$ (here $\mathrm{orb}_f(x) = \{x, f(x), \ldots, f^n(x), \ldots\}$ denotes the *orbit* of an element x under the self-map f). Put $\tau(e_i) = v_i$ for all $1 \leq i \leq k$ and

$$\tau(\Gamma^j(e_i)) = \begin{cases} \sigma^j(v_i), & \text{if } j \text{ is even}, \\ \sigma^j(u_i), & \text{if } j \text{ is odd}. \end{cases}$$

for all $1 \leq j \leq m_i - 1$, $1 \leq i \leq k$. Since for each $1 \leq i \leq k$ it holds $m_i = 1$ (and $p(X, \sigma) = 0$) or m_i is even, we can conclude that σ is a τ-negative map.

Further, we use induction on $|V(X)| \geq 2$. The case $|V(X)| = 2$ is obvious. Therefore, let $|V(X)| \geq 3$ and σ be a metric map. If $d^-_{\Gamma(X,\sigma)}(e) \geq 1$ for all edges $e \in E(X)$, then the equality

$$|A(\Gamma(X, \sigma))| = \sum_{e \in E(X)} d^+_{\Gamma(X,\sigma)}(e) = \sum_{e \in E(X)} d^-_{\Gamma(X,\sigma)}(e)$$

implies that $d^+_{\Gamma(X,\sigma)}(e) = d^-_{\Gamma(X,\sigma)}(e) = 1$ for all $e \in E(X)$. In this case, σ is an automorphism of X (see Proposition 2.3) and we are done. Otherwise, suppose there exists an edge $uv \in E(X)$ with zero indegree in $\Gamma(X, \sigma)$. Similarly to the proof of

Theorem 3.3, one can assume that $u \in L(X)$. Consider the tree $X' = X - \{u\}$ and the map $\sigma' = \mathrm{pr}_{V(X)\setminus\{u\}} \circ \sigma$. Clearly, σ' is a metric map on X'. By induction assumption, there exists an orientation τ' of X' such that σ' is τ'-negative. If $\sigma(u) = \sigma(v)$, or $\sigma(u) = v$ and $\sigma(v) = u$, then put

$$\tau(e) = \begin{cases} \tau'(e), & \text{if } e \neq uv, \\ v, & \text{if } e = uv \end{cases}$$

for all $e \in E(X)$. Otherwise, let $\sigma(u) \neq \sigma(v)$, and $\sigma(u) \neq v$ or $\sigma(v) \neq u$. If $\sigma(u) = u$ and $\sigma(v) = v$, then the edge uv is σ-positive which contradicts $p(X, \sigma) = 0$. Thus, we can conclude that $\sigma(u)\sigma(v) \in E(X)\setminus\{uv\}$. Further, $\sigma^{-1}(\tau'^{-1}(\sigma(u)\sigma(v))) \cap \{u, v\}$ is a singleton set. In this case we put

$$\tau(e) = \begin{cases} \tau'(e), & \text{if } e \neq uv, \\ w, & \text{if } e = uv \text{ and } \sigma^{-1}(\tau'^{-1}(\sigma(u)\sigma(v))) \cap \{u, v\} = \{w\} \end{cases}$$

for all $e \in E(X)$. It is clear that σ is a τ-negative map.

Finally, let σ be a linear non-metric map and $uv \in E(X)$ be an edge with zero outdegree in $\Gamma(X, \sigma)$. Again, consider the tree X' with the vertex set $V(X') = V(X)\setminus\{v\}$ and the edge set

$$E(X') = E(X - \{v\}) \cup \{uw : w \in N_X(v)\}$$

along with the map

$$\sigma'(w) = \begin{cases} \sigma(w), & \text{if } \sigma(w) \neq v, \\ u, & \text{if } \sigma(w) = v \end{cases}$$

for all $w \in V(X')$. Then σ' is linear on X'. By induction assumption, there is an orientation τ' of X' with σ' being τ'-negative map. If $d^-_{\Gamma(X,\sigma)}(uv) = 0$, then put

$$\tau(e) = \begin{cases} \tau'(e), & \text{if } e \text{ is not incident to } v, \\ \tau'(uw), & \text{if } e = vw \text{ and } w \neq u, \\ v, \text{ if } e = uv \end{cases}$$

for all $e \in E(X)$. Otherwise, $d^-_{\Gamma(X,\sigma)}(uv) = 1$ and thus $N^-_{\Gamma(X,\sigma)}(uv) = \{xy\}$ for some edge $xy \in E(X)$. In this case, put

$$\tau(e) = \begin{cases} \tau'(e), & \text{if } e \text{ is not incident to } v, \\ \tau'(uw), & \text{if } e = vw \text{ and } w \neq u, \\ \mathrm{pr}_{uv}(\sigma(\tau^{-1}(xy))), & \text{if } e = uv \end{cases}$$

for all $e \in E(X)$. Again, σ is a τ-negative map. □

4. DIRECTED PATHS AND CONVERSES TO PARTIAL FUNCTIONAL DIGRAPHS AS M-GRAPHS

It easily can be seen that any partial functional digraph is an M-graph (see [4]). Moreover, there always exists its realization (X, σ) with X being a star. In particular, a directed path is realizable on a star. In [6] it was proved that directed paths are realizable only on spiders.

Theorem 4.1 ([6]). *If $\Gamma(X, \sigma)$ is a path, then X is a spider. Moreover, if X is non-trivial, then its center is a unique fixed point of σ.*

We will call a spider $\mathrm{Sp}(a_1, \ldots, a_m)$ *balanced* if $|a_i - a_j| \le 1$ for all $1 \le i, j \le m$.

Theorem 4.2. *For a tree X there exists its map $\sigma : V(X) \to V(X)$ with $\Gamma(X, \sigma)$ being a path if and only if X is a balanced spider.*

Proof. First, we prove the necessity of this condition. Let $\Gamma(X, \sigma)$ be a path. Clearly, σ is a linear metric map on X (see Corollary 1.2). By Theorem 4.1, X is a spider. If X is a path with n vertices, then $X \simeq \mathrm{Sp}(n-1)$ is a balanced spider. Therefore, assume that X is a non-trivial spider with $u_0 \in V(X)$ being its center. Also, let $uv \in E(X)$ be the unique edge with zero indegree in $\Gamma(X, \sigma)$. By Proposition 2.2, uv is a leaf edge in X. Without loss of generality, let $u \in L(X)$.

Claim 1. For every leaf vertex $x \in L(X)$ there exists a number $k \ge 0$ such that $\sigma^k(u) = x$.

Let $x \in L(X)$ be an arbitrary leaf vertex and $xy \in E(X)$ be the corresponding leaf edge. Since $\Gamma(X, \sigma)$ is a path, there is a walk $uv = e_0 \to \ldots \to e_k = xy$ in $\Gamma(X, \sigma)$. In light of Theorem 2.1 this means the existence of an arc $uv \to xy$ in $\Gamma(X, \sigma^k)$. If $\sigma^k(u) \ne x$, then $\sigma^k(v) = x$ and $\sigma^k(u) = y$. We have $v \in [u_0, u]_X$ but $\sigma^k(v) = x \notin [\sigma(u_0), \sigma(u)]_X = [u_0, y]_X$ as $\operatorname{fix} \sigma = \{u_0\}$ (see Theorem 4.1). A contradiction with the linearity of σ^k. Therefore, $\sigma^k(u) = x$.

Claim 2. For every $k \ge 0$ we have

$$d_X(u_0, \sigma^k(u)) - 1 \le d_X(u_0, \sigma^{k+1}(u)) \le d_X(u_0, \sigma^k(u)).$$

Since σ is a metric map, then

$$d_X(u_0, \sigma^{k+1}(u)) = d_X(\sigma^{k+1}(u_0), \sigma^{k+1}(u)) \le d_X(\sigma^k(u_0), \sigma^k(u)) = d_X(u_0, \sigma^k(u)).$$

Further, to the contrary, assume that

$$d_X(u_0, \sigma^k(u)) - 2 \ge d_X(u_0, \sigma^{k+1}(u))$$

for some $k \ge 0$. Since σ is a linear map, we have

$$\sigma([u_0, \sigma^k(u)]_X) = \sigma([\sigma^k(u_0), \sigma^k(u)]_X) \subset [\sigma^{k+1}(u_0), \sigma^{k+1}(u)]_X = [u_0, \sigma^{k+1}(u)]_X.$$

This means that there exists a vertex $x \in [u_0, \sigma^{k+1}(u)]_X$ such that $|\sigma^{-1}(x)| \ge 3$ or there is a pair of different vertices $x_1, x_2 \in [u_0, \sigma^{k+1}(u)]_X$ with $|\sigma^{-1}(x_1)| \ge 2$ and

$|\sigma^{-1}(x_2)| \geq 2$. However, every linear map is *monotone* (meaning that the pre-image of every vertex is connected) implying that in both cases X would contain a pair of different edges each having zero outdegree in $\Gamma(X, \sigma)$. The obtained contradiction proves the second claim.

Now, if there are two different numbers $k_{1,2} \geq 0$ with

$$d_X(u_0, \sigma^{k_i}(u)) - 1 = d_X(u_0, \sigma^{k_i+1}(u))$$

for $i = 1, 2$, then a similar argument yields that X contains a pair of different edges with zero outdegrees in $\Gamma(X, \sigma)$. Therefore, there exists at most one number $k \geq 0$ with $d_X(u_0, \sigma^k(u)) - 1 = d_X(u_0, \sigma^{k+1}(u))$. Using Claim 1, we can conclude that $|d_X(u_0, x) - d_X(u_0, y)| \leq 1$ for all $x, y \in L(X)$ implying that X is a balanced spider.

To prove the sufficiency of this condition assume that X is a balanced spider. If $X = \{u_1 - \ldots - u_n\}$ is a path, then for the map

$$\sigma(u_i) = \begin{cases} u_{i+1}, & \text{if } 1 \leq i \leq n-1, \\ u_n, & \text{if } i = n \end{cases}$$

for $1 \leq i \leq n$, the Markov graph $\Gamma(X, \sigma)$ is a path. Thus assume X is a non-trivial spider centered at $u_0 \in V(X)$. Since X is balanced, there exists a number $a \geq 1$ such that $d_X(u_0, v) \in \{a, a+1\}$ for all $v \in L(X)$. Let $L(X) = \{v_1, \ldots, v_m\}$, $m \geq 3$. Without loss of generality, we can assume that there is $1 \leq k \leq m$ with

$$d_X(u_0, v_i) = \begin{cases} a, & \text{if } 1 \leq i \leq k, \\ a+1, & \text{if } k+1 \leq i \leq m \end{cases}$$

for all $1 \leq i \leq m$.

Further, for each $1 \leq i \leq m$ and $0 \leq j \leq d_X(u_0, v_i)$ denote by x_i^j the unique vertex from the interval $[u_0, v_i]_X$ with $d_X(u_0, x_i^j) = j$ (for example, $x_i^0 = u_0$ for all $1 \leq i \leq m$). If $k = m$, then put

$$\sigma(x) = \begin{cases} x_{i+1}^j, & \text{if } x = x_i^j \text{ and } 1 \leq i \leq m-1,\ j \neq 0, \\ x_1^{j-1}, & \text{if } x = x_m^j,\ j \neq 0, \\ u_0, & \text{if } x = u_0 \end{cases}$$

for all $x \in V(X)$. If $k \neq m$, then put

$$\sigma(x) = \begin{cases} x_{i+1}^j, & \text{if } x = x_i^j \text{ and } 1 \leq i \leq k-1,\ j \neq 0, \\ x_{i+1}^{j-1}, & \text{if } x = x_k^j,\ j \neq 0, \\ x_1^j, & \text{if } x = x_m^j,\ j \neq 0, \\ u_0, & \text{if } x = u_0 \end{cases}$$

for all $x \in V(X)$. In both cases the Markov graph $\Gamma(X, \sigma)$ is a path. $\square$

Example 4.3. Consider the tree X with

$$V(X) = \{0, \ldots, 9\} \quad \text{and} \quad E(X) = \{04, 15, 26, 37, 48, 59, 69, 79, 89\}.$$

Clearly, $X \simeq \mathrm{Sp}(3, 2, 2, 2)$ is a balanced spider. Put $\sigma(i) = i + 1$ for all $0 \le i \le 8$ and $\sigma(9) = 9$ (see Figure 1). Then the corresponding Markov graph

$$\Gamma(X, \sigma) = \{04 \to 15 \to 26 \to 37 \to 48 \to 59 \to 69 \to 79 \to 89\}$$

is a path with 9 vertices.

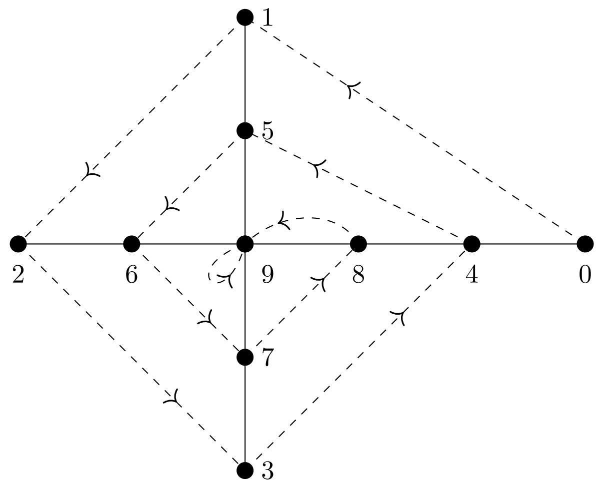

Fig. 1. Spider X and its map σ from Example 4.3

Spiders of a general type can be characterized as trees which admit maps of the very special type. Namely, a map on a tree is called *anti-expansive* if its Markov graph does not contain loops. Further, a map f on a graph G is a *neighborhood map* provided $f(u) \in N_G[u]$ for all $u \in V(G)$.

Proposition 4.4. *A tree is a spider if and only if it admits an anti-expansive linear neighborhood map.*

Proof. Let X be a spider. If X is a path, then fix any its vertex $u_0 \in V(X)$. Otherwise, assume u_0 is the center of X. The map σ_τ, where $X(\tau)$ is an in-tree centered at u_0, is clearly an anti-expansive neighborhood map. Moreover, $\Gamma(X, \sigma_\tau)$ is a disjoint union of paths implying that σ_τ is linear as well.

Conversely, let σ be an anti-expansive linear neighborhood map on a non-spider tree X. Then X contains two different vertices $u, v \in V(X)$ with $d_X(u) \ge 3$ and $d_X(v) \ge 3$. Since σ is anti-expansive, it has a unique fixed point. Let $\operatorname{fix}\sigma = \{u_0\}$. Without loss of generality, we can assume $u_0 \ne v$. Fix a vertex $x \in [u_0, v]_X$ with $vx \in E(X)$. Since $d_X(v) \ge 3$, there are two different vertices $y_1, y_2 \in N_X(v) \backslash [u_0, v]_X$.

Since σ is an anti-expansive neighborhood map, we can conclude that $\sigma(y_1) = \sigma(y_2) = v$ and $\sigma(v) = x$. This means that $vy_1 \to vx$ and $vy_2 \to vx$ in $\Gamma(X, \sigma)$ contradicting the linearity of σ (see Theorem 1.1). $\square$

Now we prove that not only partial functional digraphs but also their converses are M-graphs as well.

Theorem 4.5. *Let Γ^{co} be a partial functional digraph. Then Γ is an M-graph.*

Proof. Assume that Γ is weakly connected. Let us show that in this case Γ is an M-graph and there is its realization (X, σ) with $\operatorname{fix} \sigma \neq \emptyset$. We use induction on $|V(\Gamma)|$. If $V_0^+(\Gamma) = \emptyset$, then Γ is a cycle. Then Γ is realizable on a star with its center being the unique fixed point of the corresponding map.

Thus, let $V_0^+(\Gamma) \neq \emptyset$. Put $F = N_\Gamma^-$. Since Γ^{co} is a partial functional, F correctly defines a function from the set $V(\Gamma)\backslash V_0^-(\Gamma)$ into $V(\Gamma)$. Note that $|V_0^-(\Gamma)| \leq 1$ since Γ is weakly connected. If Γ is a path, then trivially Γ is an M-graph. Otherwise, we have a correctly defined function $f : V_0^+(\Gamma) \to \mathbb{N}$, where $f(u) = \min\{k \in \mathbb{N} : d_\Gamma^+(F^k(u)) \geq 2\}$.

Fix a vertex $u_0 \in V_0^+(\Gamma)$ with $f(u_0) = \min\{f(u) : u \in V_0^+(\Gamma)\}$ and consider the digraph $\Gamma' = \Gamma - \{u_0, \ldots, F^{f(u_0)-1}(u_0)\}$. Clearly, Γ' is also weakly connected and its converse digraph $(\Gamma')^{co}$ is a partial functional. By induction assumption, $\Gamma' \simeq \Gamma(X', \sigma')$ for some pair (X', σ') with $\operatorname{fix} \sigma' \neq \emptyset$. Let $\varphi' : V(\Gamma') \to E(X')$ be the corresponding isomorphism and $\varphi'(F^{f(u_0)}(u_0)) = x_0y_0$.

Since $d_\Gamma^+(F^{f(u_0)}(u_0)) \geq 2$, then

$$N_\Gamma^+(F^{f(u_0)}(u_0))\backslash\{F^{f(u_0)-1}(u_0)\} \neq \emptyset.$$

Fix a vertex $v_1 \in N_\Gamma^+(F^{f(u_0)}(u_0))\backslash\{F^{f(u_0)-1}(u_0)\}$ such that the edge $\varphi'(v_1)$ is incident to $\sigma'(x_0)$ in X'. Since $f(u_0) = \min\{f(u) : u \in V_0^+(\Gamma)\}$, then for every $2 \leq k \leq f(u_0)$ there exists a vertex $v_k \in N_\Gamma^+(v_{k-1})$ such that the corresponding edge $\varphi'(v_k)$ is incident to $(\sigma')^k(x_0)$. Finally, let $\varphi'(v_k) = x_ky_k$ for all $1 \leq k \leq f(u_0)$. Consider the new tree X with the vertex set

$$V(X) = V(X') \sqcup \{u_0, \ldots, F^{f(u_0)-1}(u_0)\}$$

and the edge set

$$\begin{aligned} E(X) = {} & (E(X')\backslash\{\varphi'(v_1), \ldots, \varphi'(v_{f(u_0)})\}) \\ & \cup \{x_kF^{f(u_0)-k}(u_0), F^{f(u_0)-k}(u_0)y_k : 1 \leq k \leq f(u_0)\}. \end{aligned}$$

One can think that X is obtained from X' by the subdivision of each edge $\varphi'(v_k)$ with the new vertex $F^{f(u_0)-k}(u_0)$. Put

$$\sigma(x) = \begin{cases} \sigma'(x), & \text{if } x \in V(X'), \\ F^{f(u_0)-k-1}(u_0), & \text{if } x = F^{f(u_0)-k}(u_0),\ 1 \leq k \leq f(u_0) - 1, \\ \sigma'^{f(u_0)+1}(x_0), & \text{if } x = u_0 \end{cases}$$

for all $x \in V(X)$. Now define the function $\varphi : V(\Gamma) \to E(X)$ as follows:

$$\varphi(v) = \begin{cases} \varphi'(v), & \text{if } v \in V(\Gamma)\backslash(\{u_0, \dots, F^{f(u_0)-1}(u_0)\} \cup \{v_1, \dots, v_{f(u_0)}\}), \\ x_k F^{f(u_0)-k}(u_0), & \text{if } v = F^{f(u_0)-k}(u_0),\ 1 \le k \le f(u_0), \\ F^{f(u_0)-k}(u_0) y_k, & \text{if } v = v_k,\ 1 \le k \le f(u_0) \end{cases}$$

for all $v \in V(\Gamma)$. Then φ is an isomorphism between Γ and $\Gamma(X, \sigma)$. Also, by construction, we have $\operatorname{fix} \sigma \neq \emptyset$. Since every digraph is a disjoint union of its weak components, then using Remark 2.6, one can conclude that each converse to a partial functional digraph is an M-graph. $\square$

Example 4.6. Consider the digraph Γ depicted on Figure 2. It is clear that Γ^{co} is a partial functional. For the spider X with

$$V(X) = \{0, \dots, 9\}, \quad E(X) = \{01, 02, 03, 04, 18, 25, 36, 67, 79\}$$

and its map

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 5 & 7 & 4 & 8 & 9 & 4 & 4 & 5 & 4 \end{pmatrix}$$

we have $\Gamma \simeq \Gamma(X, \sigma)$ (note that $\operatorname{fix} \sigma = \{0\}$). The explicit isomorphism is given by the map

$$f = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 \\ 18 & 04 & 10 & 25 & 79 & 03 & 02 & 36 & 67 \end{pmatrix}.$$

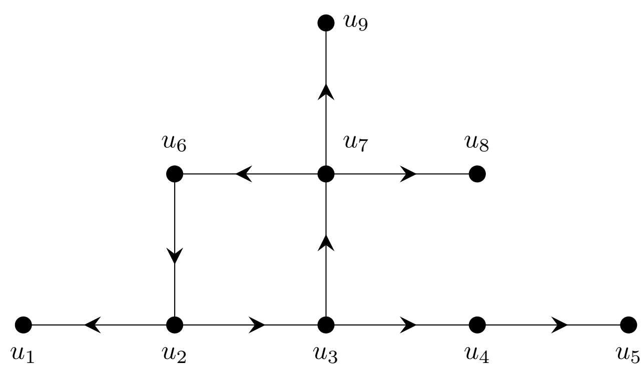

Fig. 2. Digraph Γ from Example 4.6

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