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RISK MODELLING APPROACHES FOR STUDENT-LIKE MODELS WITH FRACTAL ACTIVITY TIME

The paper focuses on value at risk ($V@R$) measuring for Student-like models of markets with fractal activity time (FAT). The fractal activity time models were introduced by Heyde to try to encompass the empirically found characteristics of real data and elaborated on for Variance Gamma, normal inverse Gaussian and skewed Student distributions. But problem of evaluating an value at risk for this model was not researched. It is worth to mention that if we use normal or symmetric Student's models than $V@R$ can be computed using standard statistical packages. For calculating $V@R$ for Student-like models we need Monte Carlo method and the iterative scheme for simulating N scenarios of stock prices. We model stock prices as a diffusion processes with the fractal activity time and for modeling increments of fractal activity time we use another diffusion process, which has a given marginal inverse gamma distribution.

The aim of the paper is to perform and compare $V@R$ Monte Carlo approach and Markowitz approach for Student-like models in terms of portfolio risk. For this purpose we propose procedure of calculating $V@R$ for two types of investor portfolios. The first one is uniform portfolio, where d assets are equally distributed. The second is optimal Markowitz portfolio, for which variance of return is the smallest out of all other portfolios with the same mean return.

The programmed model which was built using R-statistics can be used as to the simulations for any asset and for construct optimal portfolios for any given amount of assets and then can be used for understanding how this optimal portfolio behaves compared to other portfolios for Student-like models of markets with fractal activity time.

Also we present numerical results for evaluating $V@R$ for both types of investor portfolio. We show that optimal Markovitz portfolio demonstrates in the most of cases the smallest possible Value at Risk comparing with other portfolios. Thus, for making investor decisions under uncertainty we recommend to apply portfolio optimization and value at risk approach jointly.

Keywords: Value at Risk, Student distribution, Monte-Carlo method, Fractal Activity Time model, Optimal portfolio.

Introduction

Making decisions under uncertainty is a complex and important problem, which one can face in different spheres, particularly in the area of investments, where participants strive to gain the desired level of income and protect themselves against losses. For the control of potential losses was proposed the using of the value at risk ($V@R$) monetary risk measure by the regulations Basel I and Basel II.

For a given portfolio, time horizon T, and probability p, the $V@R$ of level p can be defined informally as the maximum possible loss during that time after we exclude all worse outcomes whose combined probability is at most p. More formally, $V@R$ is defined such that the probability of a loss greater than $V@R$ is (at most) p while the probability of a loss less than $V@R$ is (at least) $1 - p$. Common parameters for standard $V@R$ are 1% and 5% probabilities and one day and two-week horizons, although other combinations are in use.

$V@R$ can be estimated either parametrically (for example, variance-covariance $V@R$ in Markowitz) or nonparametrically (for examples, historical simulation $V@R$ or root-finding algoritms $V@R$ in Ivanov). A McKinsey report published in May 2012 estimated that 85% of large banks were using historical simulation. The other 15% used Monte Carlo methods. For the using of Monte Carlo method we need to make some assumptions about models which we choose for the risky factors.

The paper focuses on value at risk measuring for Student-like models of markets with fractal activity time. The fractal activity time model was introduced by Heyde (1999) to try to encompass the empirically found characteristics of real data and elaborated on for Variance Gamma, normal inverse Gaussian and skewed Student distributions [1; 2; 8]. If we use normal or symmetric Student's models than $V@R$ can be computed using standard statistical packages. For calculating $V@R$ for model proposed in [2; 8] we need Monte Carlo method and the iterative scheme for simulating N scenarios of stock prices.

In this paper we consider the simulations for each asset and for some investor portfolios built out of them: uniform portfolio as the one where N assets are equally distributed and optimal Markowitz portfolio as the one, variance of which return is the smallest out of all other portfolios with the same mean return. Notice, that comparison of $V@R$ and Markowitz efficient frontier approaches were discussed in [3], where for applying Monte Carlo method was chosen GBM.

The paper is organized as follows. In Section 2 we remain formal definition of probability functional V@R for risk measuring and consider the main steps in a basic Monte Carlo approach to V@R estimating.

In Section 3 we discuss the assumptions about market data, which are necessary for performing of first step of Monte Carlo method. For this aim we describe time-changed processes for Student-like models with depends. This section is based on the papers [2], [8], [9], [10], [11] where models of the generalized diffusion process with *"market"* time are presented and discussed.

In Section 4 we use the iterative scheme for simulating N scenarios of stock prices for our model, which was proposed in [6]. We model market time increments as a diffusion processes with a given marginal inverse gamma distribution.

In Section 5 we propose procedure of calculating $V@R$ for different types of investor portfolio.

In Section 6 we present numerical results for evaluating $V@R$ for both types of investor portfolio.

Monte-Carlo Method for evaluating Value at Risk

Let S is a vector of risk factors, Δt is $V@R$ horizon (one day or two weeks), ΔS is a change of risk factors over Δt, Y is a loss in portfolio value resulting from change ΔS over Δt The loss Y is the difference between the current value of the portfolio and the portfolio value at the end of the $V@R$ horizon Δt if the risk factors move from S to $S+\Delta S$.

Reducing the variance of random variable (return on asset/portfolio) leads to minimization of losses, however, it also leads to minimization of income. Therefore we need to introduce such a metric, which would encounter only bad effects of risks, and which will not encounter the positive properties of risks (e.g unexpected income).

Probability functionals have been objects of many theoretical and empirical investigations of risk measuring. For background on probability (risk) functionals see Pflug [5] (2005), for instance. For a seminal work on axiomatic definitions for risk functionals see Artzner et al (1999).

Let us recall the definition of probability (risk) functional $V@R$ of level α.

Let (Ω, F, P) be the probability space and suppose for $p \in [1, +\infty)$ a linear space $L(p)$ of real valued random variables

$$Y : \Omega \to R^1$$

such that $E(|Y|^p) < \infty$ is defined on it.

Definition. The value-at-risk of level

$$\alpha, 0 < \alpha \leq 1$$

for random variable $Y \in L(p)$ is a probability functional, defined as α-quantile of the profit (loss) function

$$V@R_\alpha(Y) = G^{-1}(\alpha) = inf\{\ y \in R : G(Y) \geq \alpha\}\ , \quad (1)$$

where G is the distribution function of $Y \in L(p)$, G^{-1} is the quantile function of α, $0 < \alpha \leq 1$.

In general, even though the distribution function G may fail to possess a left or right inverse, the quantile function G^{-1} behaves as an "almost sure left inverse" for the distribution function, in the sense that

$$G^{-1}(G(Y)) = Y \quad (2)$$

almost surely.

Often it is recommended (for examples by regulators Basel I and Basel II) to denote $V@R$ as the low quantile with minus sign [7]:

$$V@R_\alpha(Y) = -G^{-1}(\alpha) \quad (3)$$

For evaluating $V@R$ there are some methods. $V@R$ can be estimated either parametrically (for example, variance-covariance $V@R$) or nonparametrically (for examples, historical simulation $V@R$ or resampled $V@R$). A McKinsey report published in May 2012 estimated that 85% of large banks were using historical simulation and the other 15% used Monte Carlo methods. The main steps in a basic Monte Carlo approach to estimating loss probabilities are as follows:

1. Generate N scenarios by sampling changes in risk factors $\Delta S(1)$,..., $\Delta S(N)$ over horizon Δt.

2. Revalue portfolio at end of horizon Δt in scenarios

$$S + \Delta S(1), ..., S + \Delta S(N);$$

determine losses $Y(1), .., Y(N)$ by subtracting revaluation in each scenario from current portfolio value; build the empirical distribution function $G(Y)$.

3. Find a quantile (3) of given probability α.

The first step requires some assumptions about market data. These assumptions are being considered in the next section.

Student-like models with fractal activity time

Let us consider the market, which consists of a risk- free bond with price B_t and risky stocks with price S_t. Price of the bond evolves according to formula for continuous rates. The price of the underlying traded asset S_t is a strong solution of the following stochastic differential equation [2]:

$$dS_t = \mu S_t dt + (\theta + \sigma^2/2) S_t dT_t + \sigma S_t dW_{T_t}, \; t \geq 0, \tag{4}$$

where T_t, $t \geq 0$, is a random time change or fractal activity time, that is positive non-decreasing process such that $T_0 = 0$, and $W_t, t \geq 0$, is a standard Brownian motion independent of the process T_t. The meaning of the coefficients before dt, dT, and dW_t, you can find in [11]. This model differs from the previous one in that the Brownian motion does not depend on the usual calendar time, but on some random process T_t, otherwise, from market time. Market time is a positive non-descending stochastic process with stationary increase that are subordinated to the gamma-inverse distribution. The idea of using "market" time is intuitively correct, because the change in stock prices occurs randomly, rather than at certain points in time.

The fractal activity time model was introduced by Heyde (1999) to try to encompass the empirically found characteristics of real data and elaborated on for Variance Gamma, normal inverse Gaussian and skewed Student distributions [1; 2]. In paper [2; 8], we considered two constructions of activity time. The first construction is based on reciprocal gamma diffusion type processes and leads to stationary returns with exact Student marginal distribution. The second construction uses a superposition of two reciprocal gamma diffusion type processes and leads to Student-like marginal distribution.

The increments over unit time are

$$\tau_t = T_t - T_{t-1}, t = 1, 2, \ldots$$

and the returns are given by

$$X_t = \log\left(\frac{S_t}{S_{t-1}}\right) \sim \mu + \theta\tau_t + \sigma\tau_t^{\frac{1}{2}} W_1, \tag{5}$$

where $\sim$ denotes equality in distribution.

If increments $\tau_t \sim R\Gamma\left(\frac{\nu}{2}, \frac{\delta^2}{2}\right)$, with PDF

$$f_{R\Gamma}(x) = \frac{\left(\frac{\delta^2}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} x^{-\frac{\nu}{2}-1} e^{-\frac{\delta^2}{2x}}, \quad x > 0 \tag{6}$$

then assuming $\theta = 0$ and $\sigma = 1$, the log returns X_t is stationary process with marginal Student $T(\mu, \delta, \nu)$ distribution

$$f_{St}(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\delta\sqrt{\pi}\Gamma(\frac{\nu}{2})} \frac{1}{[1 + (\frac{x-\mu}{\delta})^2]^{\frac{\nu+1}{2}}}, \quad \in R, \tag{7}$$

where $\mu \in R$ is a location parameter, $\delta > 0$ is a scaling parameter, $\nu > 0$ is a tail index.

If $\theta \neq 0$ and $\sigma \neq 0$, then returns are skewed Student distributed.

After choosing model for risk factors we need simulating (generating) N scenarios for this model over time horizon.

Simulating for Student-like models with fractal activity time

For simulating N scenarios for Student-like models with fractal activity time over time horizon we proposed the following iterative scheme [6], which follows from (4):

$$x_{k+1} = x_k + \mu x_k \triangle t + (\theta + \frac{\delta^2}{2}) x_k \tau_k + \sigma\sqrt{\tau_k}\varepsilon_k \tag{8}$$

where μ, σ and θ are constants, which can be found from historical data; ε - white noise with normal standard distribution, and τ is a stationary process of active time, with inverse gamma distribution, which was modeled earlier (see [2], [8], [9], [10]).

Now we need to construct a iterative scheme for stochastic diffusion process τ_t with a given marginal gamma-inverse density:

$$f = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}. \tag{9}$$

where $\alpha = \frac{\delta^2}{2}$; $\beta = \frac{v}{2}$.

Using Bibby's article[4] and our paper [10] the process τ_t can be determined by the equation:

$$d\tau = -\theta\left(\tau - \frac{v}{\delta^2 - 2}\right) dt + \sqrt{\frac{4\theta}{\delta^2 - 2}\tau^2} dW, \tag{10}$$

where θ - coefficient of the autocorrelation function.

From (10) we can easily build an iterative scheme:

$$\tau_{k+1} = \tau_k - \theta\left(x - \frac{v}{\delta^2 - 2}\right) \triangle t + \sqrt{\frac{4\theta}{\delta^2 - 2}\tau^2 \triangle t}\varepsilon_k \tag{11}$$

Thus, for simulating we use iterative scheme (8), where τ_t can be generated by (11).

Procedure of finding $V@R$ for some kinds of investor portfolios

Now we discuss procedure of finding $V@R$ for two types of portfolios $x = (x_1, ..., x_d)$, where x_i denotes the fraction of asset i in this collection. The first one is uniform portfolio, where d assets are equally distributed. The second is optimal portfolio, where variance of which return is the smallest out of all other portfolios with the same mean return. It seems reasonable that the portfolio with the same average return, but lower risks of loses should be "better" for profit-seeking investor.

Therefore according Markovitz approach, the portfolio will be called optimal if it has the smallest possible variance under fixed expected value of its returns. Optimal portfolios can be analyzed in a mean-variance framework if the returns on the assets are jointly elliptically distributed, including the special case in which they are jointly normally distributed.

Under mean-variance analysis, it can be shown that every minimum-variance portfolio given a particular expected return (that is, every efficient portfolio) can be formed as a combination of any two efficient portfolios.

Formally, given $x = (x_1, x_2, ...x_d)$ is the distribution of our funds among assets 1 to d we need:

$$\sigma_x^2 \to min \tag{12}$$

subject to:

$$\mu_x = \sum_{i=1}^{d} \mu_i x_i \; = r, \tag{13}$$

$$\sum_{i=1}^{d} x_i = 1. \tag{14}$$

This problem (12)-(14) of solving conditional extremum problem can be reduced to unconditional extremum problem using Lagrange multiplier method and Lagrangian function is written in the form:

$$L(x_1, \ldots, x_d, u, v) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{ij}^2 x_i x_j - $$
$$-v \times \left[\sum_{i=1}^{d} \mu_i x_i - r\right] - u \times \left[\sum_{i=1}^{d} x_i - 1\right], \tag{15}$$

where σ_{ij}^2 is covariance of assets i and j, μ_i is a mean return on asset i, r - targeted mean return of a portfolio.

Thus the procedure includes the finding covariance matrix σ_{ij}^2 of assets i and j, building the optimal portfolio, applying iterative scheme for generating N scenarios by sampling changes in risk factors $\Delta S(1)$,..., $\Delta S(N)$ over horizon Δt, determining losses $Y(1), .., Y(N)$ by subtracting revaluation in each scenario from current portfolio value; building the empirical distribution function $G(Y)$ and finding a quantile for given probability α.

Numerical results

In this section, numerical results between historical simulation and Monte-Carlo simulation for investor optimal portfolio and the portfolio with uniform distribution of stocks are demonstrated.

We took three assets for illustrating our approaches:
Asset 1 - Facebook shares
Assets 2- Boeing shares
Asset 3 - Goldman Sacks shares
and considered two different portfolios: uniform, where

$$x = (1/3, 1/3, 1/3),$$

and optimal

$$x = (0.413, 0.225, 0.362),$$

which was built using Lagrange multiplier method. We obtained the following values of Value at Risk during monthly time interval and confidence level $p = 0.95$, which are presented in Table 1.

Table 1. Numerical $V@R$ comparisons

	Historical Simulation	Normal Assumption	Monte-Carlo FAT Simulation
Facebook	-8.374	-9.054	-12.042
Boing	-10.698	-10.652	-13.373
Goldman	-10.461	-10.692	-12.420
Opt.Port.	-8.082	-7.314	-8.940
Uniform Port.	-8.111	-7.410	-9.467

Overall, the results show that the optimal portfolio demonstrates the smallest possible Value at Risk under different probabilistic scenarios.

Conclusions

In this paper we propose procedure for $V@R$ evaluating by Monte Carlo method for Student-like models with fractal activity time. We use this method for some investor portfolios of risky assets. We show that optimal Markovitz portfolio demonstrates in the most of cases the smallest possible Value at Risk comparing with other portfolios. Thus, for making investor decisions under uncertainty we recommend to apply portfolio optimization and value at risk approach jointly.

However we have perspectives for further research. It is interesting to find necessary and /or sufficient conditions which provide smallest possible Value at Risk for optimal portfolio comparing with other investor portfolios. The other problem is to include some derivatives in investor portfolio. Finally it is useful to apply the expected shortfall risk measure.

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ПІДХОДИ ДО МОДЕЛЮВАННЯ РИЗИКУ ДЛЯ СТЬЮДЕНТ-ПОДІБНИХ МОДЕЛЕЙ ІЗ ФРАКТАЛЬНИМ АКТИВНИМ ЧАСОМ

Статтю присвячено проблемі вимірювання ризику (V@R) для Стьюдент моделей ринків з фрактальним активним часом, (FAT). Моделі ринків з фрактальним активним часом були введені Хейде, щоб спробувати охопити емпірично знайдені характеристики реальних даних і покращити наявні моделі. Ці моделі вже було досліджено для Variance Gamma розподілу, normal inverse Gaussian розподілу і skewed Student розподілу. Проте проблеми вимірювання ризику в цих моделях не було досліджено. Варто зауважити, якщо ми використовуємо моделі з нормальним розподілом або з симетричним розподілом Стьюдента, то V@R можна обчислити за допомогою стандартних статистичних пакетів. Для розрахунку V@R для моделей із скособоченим розподілом Стьюдента, нам знадобиться метод Монте-Карло та ітераційна схема для моделювання N сценаріїв цін акцій. Ми моделюємо ціни акцій як процеси дифузії з фрактальним активним часом, а для моделювання приростів процесу цього нового часу ми використовуємо інший процес дифузії, який має заданий граничний зворотний гамма-розподіл. Мета роботи полягає у застосуванні та порівнянні методу Монте-Карло для обчислення міри ризику V@R та підходу Марковіца для моделей типу Стьюдента, у термінах портфельного ризику. Для цього ми пропонуємо процедуру розрахунку V@R для двох типів портфелів інвесторів. Перший – однорідний портфель, де активи на d розподілені порівну. Другий – оптимальний портфель Марковіца, для якого дисперсія прибутковості є найменшою з усіх інших портфелів з такою ж середньою прибутковістю. Запрограмована модель, яка була побудована з використанням R-статистики, може бути використана для моделювання для будь-якого активу та для побудови оптимальних портфелів для будь-якої заданої кількості активів. Також цю модель можна використати, щоб зрозуміти, як цей оптимальний портфель поводиться порівняно з іншими портфелями для моделей типу Стьюдента на ринках з фрактальним часом активності. Також ми наводимо числові результати для оцінки V@R для обох типів портфеля інвестора. Показано, що оптимальний портфель

Марковіца демонструє в більшості випадків найменшу можливу міру ризику порівняно з іншими портфелями. Таким чином, для прийняття рішень інвесторами в умовах невизначеності ми рекомендуємо спільно застосовувати оптимізацію портфеля та підхід вимірювання ризику.

Ключові слова: міра ризику, розподіл Стьюдента, Монте-Карло метод, модель з активним фрактальним часом, оптимальний портфель.

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