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Pylyavska O., Shatohina Ju.

THE NON-REGULAR 2-GROUPS SATISFYING THE CONDITION: EACH CYCLIC SUBGROUP IS CONTAINED IN THE CENTER OR HAS A TRIVIAL INTERSECTION WITH IT

Y. Berkovich formulated the following problem: «Suppose that p-group G satisfies the following condition: if C is a cyclic subgroup of G then either $C \leq Z(G)$ or $C \cap Z(G) = \{1\}$. Classify all such groups.» The authors have proved that each regular p-group satisfies this condition if and only if it is either abelian or a group of exponent p.

Here we regard the non-regular 2-groups. Some necessary conditions for non-abelian 2-group in which every cyclic subgroup is contained in the center or has a trivial intersection with it are pointed. The minimal example of non-regular 2-group with 2 generators satisfying this condition is constructed.

1. Introduction

Let G be a non-trivial finite p-group, $Z(G)$ – the center of G.

In his book [1] Y. Berkovich formulated the following problem. «Suppose that p-group G satisfies the following condition: if C is a cyclic subgroup of G then either $C \leq Z(G)$ or $C \cap Z(G) = \{1\}$. Classify all such groups».

The p-group which satisfies this condition we will call the ZC-group.

The authors ([2]) have proved that a regular p-group is ZC-group if and only if it is either the group of exponent p or an abelian group. The p-group G is called to be *regular* if for each $g,h \in G$ we have

$$g^p h^p = (gh)^p \prod_i s_i^p,$$

where s_i is an element from the commutator subgroup $\langle g,h \rangle$ of the group generated by g, h ([3],[4]).

It is easy to see that each non-abelian 2-group is non-regular.

In the first part of the paper we consider the question when the non-abelian ZC-group is the direct product of some ZC-groups M and N. We prove that this claim holds if and only if $\exp Z(M) = \exp Z(N) = p$;

In the second part of the paper we prove three theorems, which give the necessary conditions for 2-group to be the ZC-group.

In the third part we regard the 2-groups of class 3 and prove some necessary conditions for them to be the ZC-group.

In the final part of this paper we use these statements to construct the minimal example of non-regular 2-group with two generators, which is a ZC-group.

2. The factorization of the ZC-group as a direct product of ZC-groups

We want to prove the next theorem:

Theorem 1. ***Let M, N are the ZC-groups. The non-abelian group $G = M \times N$ is ZC-group if and only if*** $\exp Z(M) = \exp Z(N) = p$.

We need the next lemma for proving.

Lemma 1. ***Let non-abelian p-group G be ZC-group. Then Z(G) is an elementary abelian group.***

Proof. Let G be a non-abelian ZC-group. We want to show that G has an element g of order p, which does not belong to the center, $g \notin Z(G)$. Really, consider the factor-group $F = G/Z(G)$. Let $Z(F)$ be a center of F, $\bar{g} \in Z(F)$ – element of order p. For each preimage g of $\bar{g}$ we have $g^p \in Z(G)$ but $g \notin Z(G)$. G is a ZC-group, so $g^p = 1$.

Suppose that $\exp Z(G) > p$.

Let $g \in G$ be an element of exponent p, $g \notin Z(G)$. Choose the element $z_1 \in Z(G)$ such as $\exp z_1 = p^m > p$. The element $z_1 g$ does not belong to the center and has an exponent which is equal to the exponent of element z_1. The cyclic subgroup $C = \langle z_1 g \rangle$ has the non-trivial subgroup C^p, which is generat-

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ed by the element $(z_1g)^p = z_1^p g^p = z_1^p \neq 1$ and is contained in the center of G. The contradiction with the condition that G is a ZC-group leads to $\exp Z(G) = p$.

The Lemma is proved.

Proof of Theorem 1. Let $G = M \times N$ be a non-abelian group ZC-group, where M, N are ZC-groups. If $M(N)$ is non-abelian, then from Lemma 1 we have $\exp Z(G) = p$ ($\exp Z(N) = p$). Suppose M is abelian. Then M is contained in the center $Z(G)$ of group G. According to Lemma 1 we have $\exp Z(G) = \exp M = p$.

On the other hand, suppose that M, N are ZC-groups and $\exp Z(M) = \exp Z(N) = p$. Let C be a cyclic subgroup of G, $C = \langle g \rangle$ and let $C \not\subset Z(G)$. G is the direct product of M, N, so $g = g_1g_2$, where $g_1 \in M, g_2 \in N$. We may assume, without a loss of generality, that $g_1 \notin Z(G_1)$. Let $|g_1| = p^k, |g_2| = p^m$. If $k \geq m$ then $|g| = |g_1| = p^k$ and for each $i \leq k$ holds $g^{p^i} \notin Z(G)$, so C have a trivial intersection with $Z(G)$. If $k < m$ then $|g_2| > p$, according Theorem 1 $g_2 \notin Z(G_2)$. Similarly, we obtain $g^{p^i} \notin Z(G)$. Thus G is a ZC-group. The proof of Theorem 1 is complete.

3. The sufficient conditions for 2-group to be a ZC-group

Here we will regard non-abelian 2-groups.

We will denote the second member of the upper central series by $Z_2(G)$ and the nilpotency class of G by $cl(G)$.

For a finite p-group G and each non-negative integer k, we define subgroups $\Omega_k(G) = \langle g \in G \mid g^{p^k} = 1 \rangle$ and $\mho_k(G) = \langle x^{p^k} \mid x \in G \rangle$.

Theorem 2. ***Let non-abelian 2-group G be ZC-group. Then***

1) $Z_2(G)$ ***is an elementary abelian group;***

2) ***each element*** $g \in G$ ***of order 2 belongs to the centralizer*** $Z_2(G)$ ***of in G.***

Proof. Assume the non-abelian 2-group G is a ZC-group.

1) Suppose there is $y \in Z_2(G)$, such as $y^2 \neq 1$. G is a ZC-group, so $y^2 \notin Z(G)$. Then, there exists the element $a \in G$, such as $[a, y^2] = z \neq 1$, where $z \in Z(G)$. For each element $y \in Z_2(G)$ and each $a \in G$ holds $[a, y] \in Z(G)$. On the other hand, $z = [a, y^2] = [a, y][a, y]^y = [a, y]^2$. So we obtain $\exp Z(G) > 2$. This contradiction with Lemma 1 proves that $\exp Z_2(G) = 2$, so $Z_2(G)$ is an elementary abelian 2-group.

2) Let $g \in G, g^2 = 1$ and $y \in Z_2(G)$. Suppose $[g, y] = z \neq 1$, where $z \in Z(G)$. Then from $y^2 = g^2 = 1$ we have $(gy)^2 = z \in Z(G)$. According to Lemma 1 $\exp Z(G) = 2$, so $gy \notin Z(G)$ and the cyclic subgroup which is generated by this element has non-trivial intersection with $Z(G)$. The contradiction with the condition that G is a ZC-group proves the Theorem.

Next corollaries obviously follow from Theorem 2.

Corollaries. ***Let 2-group G be a ZC-group. Then***

1) $cl(G) \geq 3$;

2) $\Omega_1(G)$ ***is contained in the centralizer of*** $Z_2(G)$.

We will denote $G_1 = G$. For each non-negative integer $k > 1$, we define the subgroups $G_k = [G, G_{k-1}]$.

Theorem 3. ***Let 2-group G be a ZC-group and the derived subgroup*** G_2 ***is an elementary abelian group. If*** $cl(G) = l$ ***then*** $\exp G \leq 2^{l-1}$.

Proof. It is easy to see that condition $\exp G = 2^k$ is equivalent to

$$\mho_k(G) = \langle x^{p^k} \mid x \in G \rangle = 1,$$

$$\mho_{k-1}(G) = \langle x^{p^{k-1}} \mid x \in G \rangle \neq 1.$$

Let $cl(G) = l \geq 3$. Assume $\exp G = 2^k$, where $k > l - 1$. There is an element $a \in G$, such as $a^{2^{k-1}} \neq 1$. According to Lemma 1 $a \notin Z(G)$. Since G is a ZC-group, then $a^{2^{k-1}} \notin Z(G)$. Thus, there exists the element $b \in G$ such as $[b, a^{2^{k-1}}] \neq 1$. Denote $d_i = a^{2^i}$, $c_1 = [b, a]$, $c_{i+1} = [b, d_i]$. It is easy to see $d_i = d_{i-1}^2$ $(i = 2, \ldots, k-1)$ and $c_j \neq 1$ $(j = 1, \ldots, k)$. Since G_2 is an elementary abelian group, then $c_{i+1} = [b, d_i] = [b, d_{i-1}^2] = [b, d_{i-1}]^2[b, d_{i-1}, d_{i-1}] = [c_i, d_{i-1}]$. In this way from $c_1 \in G_2$ we obtain $c_i \in G_{i+1}$. Therefore we have $G_{k+1} \neq 1$, where $k + 1 > l$. As this contradicts the assumption that $cl(G) = l$, the Theorem is proved.

4. ZC-groups with nilpotency class 3

Let 2-group G be a ZC-group, and $cl(G) = 3$.

Theorem 4. ***If 2-group G is a ZC-group and*** $cl(G) = 3$ ***then*** $\Phi(G)$ ***is an elementary abelian group.***

Proof. The condition $cl(G) = 3$ implies that G_3 is contained in the $Z(G)$ and the derived subgroup G_2 is contained in the $Z_2(G)$. Therefore G_2 is an elementary abelian group. Theorem 3 implies $\exp G \leq 2^2$. Since all groups of exponent 2 are abelian, then $\exp G = 2^2$. In this case $\mho_2(G) = 1$, $\mho_1(G) \neq 1$, $\mho_1(G) \subseteq \Omega_1(G)$. From Corollary 2 we have $\Omega_1(G) \subseteq C_G(Z_2(G))$ and so each element $\mho_1(G)$ of commutes with each element of G_2. Note that Frattini subgroup $\Phi(G)$ of an arbitrary p-group G is generated by the derived subgroup G_2 and $\mho_1(G)$, namely $\Phi(G) = G_2 \cdot \mho_1(G)$, wherefrom $\Phi(G)$ is elementary abelian. The Theorem is proved.

Theorem 5. ***If 2-group G is a ZC-group and*** $cl(G) = 3$ ***then*** $\Phi(G) = \mho_1(G)$.

Proof. Note that the assumption of the theorem implies that $G_2 \subseteq Z_2(G)$. We want to show that every commutator $c \in G_2$ belongs to $\mho_1(G)$. Let $c = [b,a] \neq 1$. If $b^2 = 1$, then, by the Theorem 2, 2) the element b belongs to the centralizer of $Z_2(G)$ in G. Thus $(ab)^2 = a^2c$ and c belongs to $\mho_1(G)$. If $a^2 = d \neq 1, b^2 = f \neq 1$ then $(ab)^2 = a^2b^2c[c,b]$. The commutator $[c, b]$, where $c^2 = 1$, belongs to $\mho_1(G)$ as well as given above. Summing up, we obtain $G_2 \subseteq \mho_1(G)$, and thus $\Phi(G) = \mho_1(G)$. The Theorem is proved.

Corollary. ***If 2-group G is a ZC-group and*** $cl(G) = 3$ ***then*** $\mho_1(G) = A \times G_2$, ***where A is an elementary abelian group.***

Lemma 2. ***If 2-group G is a ZC-group and*** $cl(G) = 3$ ***then for each*** $g \in G$ ***and each*** $f \in \Phi(G)$ ***holds*** $[g, f] \in G_3$.

Proof. According to Theorem 5, for each $f \in \Phi(G)$ holds $f \in \mho_1(G)$ and consequantly there exists $a \in G$ such as $f = a^2$. According to Theorem 4 $\Phi(G)$ is an elementary abelian group, then for each $g \in G$ we have $[g, f] = [g, a^2] = [g, a]^2[g, a, a] = [g, a, a] \in G_3$. Lemma 2 is proved.

5. The construction of ZC-group

The aim of this part of the paper is to construct the minimal example of ZC-group.

Suppose that the following conditions hold:

1. a 2-group G is ZC-group;
2. $cl(G) = 3$;
3. G has two generators: $G = \langle a,b \rangle$.

According to the Theorem 3 we have $\exp(G) = 2^2$.

It is easy to see that the conditions 2), 3) give the minimal value for the number of generators, for the class of nilpotency of G and for the exponent of G.

Proposition 1. ***Suppose 2-group G is a ZC-group with 2 generators and*** $cl(G) = 3$. ***Then for each*** $g \in G \setminus \Phi(G)$ ***holds*** $g^2 \neq 1$ ***and*** $g^2 \notin Z(G)$.

Proof. Really, suppose that there exists $b \in G \setminus \Phi(G)$ such as $b^2 = 1$. Consider the subgroup $B = \langle b, \Phi(G) \rangle$. If B is abelian, then $\exp(B) = 2$. Since $\exp(G) > 2$ then there is an element $a \in G \setminus B$, such as $a^2 = d \neq 1$. Here $d \in \Phi(G) \subset B$, so $[d,b] = 1$. This being the case that G has two generators, we may consider $G = \langle a,b \rangle$. The element d commutes with a and b, so $g \in Z(G)$. We obtain the contradiction to the claim that G is a ZC-group.

If B is not abelian, then there exists an element $f \in \Phi(G)$, such as $[b, f] = z \neq 1$ and $z \in Z(G)$. From $b^2 = f^2 = 1$ follows that $(fb)^2 = z \in Z(G)$. Since $fb \notin Z(G)$, it contradicts the claim that G is a ZC-group. The Proposition is proved.

Proposition 2. ***Suppose 2-group G is a ZC-group with 2 generators,*** $G = \langle a,b \rangle$, ***and*** $cl(G) = 3$. ***Then*** $\mho_1(G) = H \times G_2$, ***where*** $H = \langle a^2 \rangle_2 \times \langle b^2 \rangle_2$.

Proof. We want to show for each $a,b \in G$, where $G = \langle a,b \rangle$, holds $a^2 = d \neq 1, b^2 = f \neq 1$ and $d \notin \langle f, G_2 \rangle$.

Really, suppose $d = fg$, where $g \in G_2$. From $G = \langle a,b \rangle$ we have $G_2 = \langle [b,a] \rangle \bmod G_3$. Denote $c = [b, a] \neq 1$. So $d = f^{\varepsilon_1}c^{\varepsilon_2}z$, where $z \in Z(G)$, $\varepsilon_1, \varepsilon_2 = 0,1$. If $\varepsilon_2 = 0$ then from $1 = [a,d] = [a, fz] = [a, f]$ we have $f \in Z(G)$. It contradicts to the claim that G is a ZC-group.

If $\varepsilon_1 = \varepsilon_2 = 1$ then

$$(ab)^2 = a^2b^2c[c,b] = fcz \cdot f \cdot c[c,b],$$

where $[c,b] \in Z(G)$. According to Lemma 1 $Z(G)$ is an elementary abelian group, so $(ab)^2 = z[c,b] \in Z(G)$. By Proposition 1 $(ab)^2 \neq 1$. We have the contradiction again.

If $\varepsilon_1 = 0$, $\varepsilon_2 = 1$ then $d = cz$ and $[c,a] = 1$. Thus $[b,d] = [b,a]^2[b,a,a] = c^2[c,a] = 1$. From $G = \langle a,b \rangle$ we have $d \in Z(G)$. It contradicts to the claim that G is a ZC-group.

Summing up we can conclude, that $d \notin \langle f, G_2 \rangle$ and that $\mho_1(G) = H \times G_2$, where $H = \langle d, f \rangle$. Since $d = a^2, f = b^2 \in \Phi(G)$ and $\Phi(G)$ is elementary abelian, then $H = \langle a^2 \rangle_2 \times \langle b^2 \rangle_2$. Proposition 2 is proved.

Proposition 3. ***Suppose 2-group G is a ZC-group with 2 generators and*** $cl(G) = 3$. ***Then*** $|G_3| > 2$.

Proof. The assumption that G has two generators, $G = \langle a,b \rangle$, implies that $G_2 = \langle [a,b] \rangle \bmod G_3$. Denote $[a,b] = c, c \in Z_2(G) \setminus Z(G)$. $\Phi(G)$ is elementary abelian, so

$$G_3 = \langle [c,a],[c,b],[h,a],[h,b] \mid h \in \Phi(G) \rangle.$$

Without a loss of generality we may assume that $[c,a] \neq 1, [c,a] = z_1 \in Z(G)$.

We want to show that $[b,c] = z_2 \notin \langle z_1 \rangle$. Really, $[b,c] = 1$ if then $[a, f] = [a, b^2] = [a,b]^2[a,b,b] = c^2[c,b] = 1$, where $f = b^2$. Therefore $f \in Z(G)$. It contradicts to the claim that G is a ZC-group.

If $[b,c] \neq 1, [b,c] \in \langle z_1 \rangle$ then for $b' = ab$ holds $[b',c] = 1$ and similarly we obtain $b'^2 \in Z(G)$. The contradictions obtained prove that $[b,c] = z_2 \notin \langle z_1 \rangle$, and, in this way, $|G_3| > 2$. The proof of Proposition 3 is completed.

Now we want to construct the 2-group of minimal order which is a ZC-group.

Suppose that G has two generators, $cl(G) = 3$, $|G_3| = 4$ and $G_3 = Z(G)$. Thus we may assume $G = \langle a,b,c,d,f,z_1,z_2 \rangle$, where $c = [b,a] \neq 1$, $d = a^2$, $h = b^2$, $z_1 = [c, a]$, $z_2 = [c, b]$, $z_1, z_2 \in Z(G)$. The last relations give $[a, f] = [a,b^2] = [a,b]^2[a,b,b]$.

Since $\Phi(G)$ is an elementary abelian group, $\Phi(G) = \langle c,d,f,z_1,z_2 \rangle$, then $[a,f]=[c,b]=z_2$. Analogously, $[b,d]=[c,a]=z_1$. So, we have the following relations of group G:

$$G=\langle a,b,c,d,f,z_1,z_2 \mid a^2=d, b^2=f,$$
$$c^2=d^2=f^2=z_1^2=z_2^2=1, [b,a]=c,$$
$$[c,a]=z_1, [d,a]=1, [f,a]=z_2, [z_i,a]=1,$$
$$[c,b]=z_2, [d,b]=z_1, [f,b]=[z_i,b]=1,$$
$$[d,c]=[f,c]=[z_i,c]=[f,d]=[z_i,d]=1,$$
$$[z_i,f]=[z_1,z_2]=1, (i=1,2)\rangle$$

Verify that this group G is a ceach $g\in G$ has the following presentation: $g=a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}f^{\lambda}z_1^{\mu}z_2^{\nu}$, where $\alpha,\beta,\gamma,\delta,\lambda,\mu,\nu=0,1$. Consider the cyclic subgroup $C=\langle g\rangle$. If $\alpha=\beta=\gamma=\delta=\lambda=0$ then $g\in Z(G)$ and C is contained in the center $Z(G)$.

If $\alpha=\beta=0$ then $g\in\Phi(G)$ and $g^2=1$, so C has a trivial intersection with center $Z(G)$.

If $\alpha=0,\beta=1$ then $g\notin\Phi(G)$, $g^2=b^2c^{\gamma}[c^{\gamma},b]\times d^{\delta}[d^{\delta},b]c^{\gamma}d^{\delta}f^{2\lambda}=fz$, where $z\in Z(G)$, $f\notin Z(G)$. Thus C has a trivial intersection with center $Z(G)$.

If $\alpha=1,\beta=0$ – as given above – C has a trivial intersection with center $Z(G)$.

If $\alpha=1,\beta=1$ then $g\notin\Phi(G)$, analogously $g^2=dfz$, where $z\in Z(G)$ by Proposition 2 $df\notin Z(G)$. Thus C has a trivial intersection with center $Z(G)$.

Summing up, we can easily see that G is a ZC-group.

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О. Пилявська, Ю. Шатохіна

НЕРЕГУЛЯРНІ 2-ГРУПИ, ЩО ЗАДОВОЛЬНЯЮТЬ УМОВУ: КОЖНА ЦИКЛІЧНА ПІДГРУПА МІСТИТЬСЯ В ЦЕНТРІ АБО МАЄ З НИМ ТРИВІАЛЬНИЙ ПЕРЕТИН

Автори висловлюють подяку професору З. Янку та професору В. Чепулічу, які запропонували розглянути наступну проблему, поставлену Я. Берковичем: «Нехай p-група G задовольняє умову: Якщо Z є циклічною підгрупою групи G, то $Z \le Z(G)$ або $Z \cap Z(G) = \{1\}$. Класифікувати всі такі групи». Групи, що задовольняють цю умову, назвемо ZC-групами. Авторами було доведено, що регулярні ZC-групи вичерпуються абелевими групами та групами експоненти p.

У даній роботі ми розглядаємо нерегулярні 2-групи. Довено деякі необхідні умови того, що неабелева 2-група є ZC-групою. Побудовано мінімальний приклад нерегулярної 2-групи з двома твірними, яка є ZC-групою.