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THE NONSTANDARD q-DEFORMATION OF ENVELOPING ALGEBRA $U(so_n)$ RELATED TO QUANTUM GRAVITY

We *describe properties of the nonstandard q-deformation* $U'_q(so_n)$ *of the universal enveloping algebra* $U(so_n)$ *of the Lie algebra* so_n *which is related to 2+1 quantum gravity on Riemannian surfaces. This algebra does not coincide with the Drinfeld-Jimbo quantum algebra* $U_q(so_n)$. *Many unsolved problems are formulated.*

1. **The** g-deformed algebra $U'_q(so_n)$

We consider a g-deformation $U'_q(so_n)$ of the universal enveloping algebra $U(so_n)$ of the Lie algebra so_n which does not coincide with the Drinfeld-Jimbo quantum algebra $U_q(so_n)$. The algebra $U'_q(so_n)$ is constructed without using the Cartan subalgebra and roots. This algebra has no root elements.

Existence of the g-deformation $U'_q(so_n)$ is explained by the following reason. The Lie algebra so_n of the rotation group *SO(n)* has two different structures related to two bases of the algebra so_n. The first structure is related to the basis consisting of a basis of the Cartan subalgebra and root elements. A g-deformation of this structure leads to the Drinfeld-Jimbo quantum algebra $U_q(so_n)$ which contains root elements (see, for example, [1]). The second structure is related to the basis of so_n consisting of skew-symmetric matrices $I_{ij}, i > j$. The m a t r $I_{ij}, i > j$, are defined as $I_{ij} = E_{ij} - E_{ji}$, where E_{ij} is the matrix with entries $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$. The universal enveloping algebra $U(so_n)$ is generated by a part of the basis elements $I_{ij}, i > j$, namely, by the elements $I_{21}, I_{32}, \cdots, I_{n,n-1}$. It is directly verified that these elements satisfy the relations

$$I^2_{i,i-1}I_{i+1,i} - 2I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I^2_{i,i-1} = -I_{i+1,i},$$

$$I_{i,i-1}I^2_{i+1,i} - 2I_{i+1,i}I_{i,i-1}I_{i+1,i} + I^2_{i+1,i}I_{i,i-1} = -I_{i,i-1},$$

$$I_{i,i-1}I_{j,j-1} - I_{j,j-1}I_{i,i-1} = 0 \quad \text{for} \quad |i-j| > 1.$$

The following Serre type theorem is a starting point for obtaining the algebra $U'_q(so_n)$.

Theorem 1. *The universal enveloping algebra* $U(so_n)$ *is isomorphic to the complex associative algebra (with a unit element) generated by elements* $I_{21}, I_{32}, \cdots, I_{n,n-1}$ *satisfying the above relations.*

We make the q-deformation of these relations by $2 \to [2] := (q^2 - q^{-2})/(q - q^{-1}) = q + q^{-1}$. As a result, we obtain the complex associative algebra (with a unit) generated by elements $I_{21}, I_{32}, \cdots, I_{n,n-1}$ satisfying the relations

$$I^2_{i,i-1}I_{i+1,i} - (q+q^{-1})I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I^2_{i,i-1} = -I_{i+1,i}, \quad (1)$$

$$I_{i,i-1}I^2_{i+1,i} - (q+q^{-1})I_{i+1,i}I_{i,i-1}I_{i+1,i} + I^2_{i+1,i}I_{i,i-1} = -I_{i,i-1}, \quad (2)$$

$$I_{i,i-1}I_{j,j-1} - I_{j,j-1}I_{i,i-1} = 0 \quad \text{for} \quad |i-j| > 1. \quad (3)$$

This algebra was introduced by us in [2] and is denoted *by* $U'_q(so_n)$.

The main motivation for studying this algebra is that the algebra of observables in 2+1 quantum gravity is isomorphic to the algebra $U'_q(so_n)$ or to its quotient algebra [3,4]. A q-analogue of the theory of harmonic polynomials (g-harmonic polynomials) on quantum vector space is also constructed by using the algebra $U'_q(so_n)$ (see [5]). The q-Laplace operator Δ_q, by means of which g-harmonic polynomials are defined, is invariant with respect to this algebra [6].

In $U'_q(so_n)$ we can determine [7] elements analogous to the matrices $I_{ij}, i > j$, of the Lie algebra so_n. In order to give them we use the notation $I_{k,k-1} \equiv I^+_{k,k-1} \equiv I^-_{k,k-1}$. Then for $\kappa > l+1$ we define recursively the elements

$$I^\pm_{kl} := [I_{l+1,l}, I_{k,l+1}]_{q^{\pm 1}} \equiv q^{\pm 1/2} I_{l+1,l}I_{k,l+1} - q^{\mp 1/2} I_{k,l+1}I_{l+1,l}.$$

(Note that similar sets of elements of $U'_q(so_n)$ are also introduced in [6]). The elements $I^+_{kl}, k > l,$ satisfy the commutation relations

$$[I^+{}_{ln}, I^+{}_{kl}]_q = I^+{}_{kn}, \quad [I^+{}_{kl}, I^+{}_{kn}]_q = I^+{}_{ln}, \quad [I^+{}_{kn}, I^+{}_{ln}]_q = I^+{}_{kl} \quad \text{for} \quad k > l > n, \tag{3}$$

$$[I^+{}_{kl}, I^+{}_{nr}] = 0 \quad \text{for } k > l > n > r \text{ and } k > n > r > l, \tag{4}$$

$$[I^+{}_{kl}, I^+{}_{nr}]_q = (q - q^{-1})(I^+{}_{lr} I^+{}_{kn} - I^+{}_{kr} I^+{}_{nl}) \quad \text{for} \quad k > n > l > r. \tag{5}$$

For $I^-{}_{kl}$, $\kappa > l$, the commutation relations are obtained by replacing $I^+{}_{kl}$ by $I^-{}_{kl}$ and *q* by q^{-1} When $q = 1$, then these commutation relations coincide with the corresponding relations for the matrices I_{ij}, $i > j$, of the Lie algebra so_n.

The algebra $U_q(so_n)$ can be defined as the associative algebra generated by the elements $I^+{}_{ij}$, $i > j$, satisfying the defining relations (4)-(6). That is, the algebra $U'_q(so_n)$ can be defined by means of quadratic defining relations.

Using the diamond lemma, it is proved [8] the Poincare-Birkhoff-Witt theorem for the algebra $U'_q(so_n)$:

Theorem 2. *The elements*

$$I_{21}^{+\,m_{21}} I_{31}^{+\,m_{31}} I_{32}^{+\,m_{32}} \cdots I_{n1}^{+\,m_{n1}} \cdots I_{n,n-1}^{+\,m_{n,n-1}}, \quad m_{ij} = 0, 1, 2, \cdots,$$

form a basis of the algebra $U'_q(so_n)$. This assertion is true if $I^+{}_{ij}$ are replaced by the corresponding elements $I^-{}_{ij}$.

2. Center of $U'_q(so_n)$

In order to describe central elements of the algebra $U'(so_n)$ we form the elements

$$J^{\pm}_{k_1,k_2,\cdots,k_{2r}} = q^{\mp\frac{r(r-1)}{2}} \sum_{s \in S_{2r}}{}' \varepsilon_{q^{\pm 1}}(s) \times I^{\pm}_{k_{s(2)},k_{s(1)}} I^{\pm}_{k_{s(4)},k_{s(3)}} \cdots I^{\pm}_{k_{s(2r)},k_{s(2r-1)}}, \tag{7}$$

of the algebra $U'(so_n)$ (see [9]), where $1 \le k_1 < k_2 < \cdots < k_{2r} \le n$ and summation runs over all permutations *s* of indices $k_1, k_2, \cdots, k_{2r}$ such that

$$k_{s(2)} > k_{s(1)}, \; k_{s(4)} > k_{s(3)}, \; \cdots,$$

$$k_{s(2r)} > k_{s(2r-1)}, \; k_{s(2)} < k_{s(4)} < \cdots < k_{s(2r)}.$$

The symbol $\varepsilon_{q^{\pm 1}}(s) \equiv (-q^{\pm 1})^{\ell(s)}$ stands for the *q*-analogue of Levi-Chivita antisymmetric tensor, $\ell(s)$ means the length of permutation *s* (the definition of a length see in standard handbooks on symmetric groups). Note that in the limit $q \to 1$ both sets in (7) reduce to the set of components of rank $2r$ antisymmetric tensor operator of the Lie algebra so_n.

Theorem 3. *[6, 9]. The elements*

$$C_n^{(2r)} = \sum_{1 \le k_1 < k_2 < \ldots < k_{2r} \le n} \tau J^+{}_{k_1,k_2,\ldots,k_{2r}} J^-{}_{k_1,k_2,\cdots,k_{2r}} \tag{8}$$

where $\tau = q^{k_1+k_2+\ldots+k_{2r}-r(n+1)}$ $r = 1, 2, \cdots$, {n/2} *({a} means the integral part of a), are Casimir elements of $U'_q(so_n)$, that is, they belong to the center of this algebra. If η is even, then $C_n^{(n)+} \equiv J^+{}_{1,2,\cdots,n}$ and $C_n^{(n)-} \equiv J^-{}_{1,2,\cdots,n}$ also belong to the center of $U'_q(so_n)$.*

If *q* is a root of unity, then (as in the case of Drinfeld-Jimbo quantum algebras) there exist additional central elements of $U'_q(so_n)$ which are given by the following theorem:

Theorem 4. *[10, 11]. Let $q^k = 1$ for $k \in \mathbb{N}$ and $q^j \ne 1$ for $0 < j < k$. Then the elements*

$$C^{(k)}(I^+{}_{rl}) = \sum_{j=0}^{\{k/2\}} \frac{(-1)^j (k-j-1)!}{j!(k-2j)!} \times \frac{1}{(q-q^{-1})^{2j}} I^+{}_{rl}{}^{k-2j}, \quad r > l, \tag{9}$$

where $\{k/2\}$ is the integral part of k/2, belong to the center of $U'_q(so_n)$.

Conjecture 1. For *q* not a root of unity the set of central elements $C_n^{(2r)}$, $r = 1, 2, \cdots, \{(n-1)/2\}$, and the element $C_n^{(n)+}$ (if *n* is even) generates the center of the algebra $U_q(so_n)$. If $q = 1$ (that is, in the case of the enveloping algebra $U(so_n)$), then this conjecture is true, that is, these elements generate the center of $U'_q(so_n)$.

Conjecture 2. For *q* a root of unity, the central elements of Theorems 3 and 4 generate the center of the algebra $U'_q(so_n)$. The central elements of Theorem 3 algebraically depend on those of Theorem 4.

3. The embedding $U'_q(so_n)$ -> $U_q(sl_n)$

An important property of the algebra $U'_q(so_n)$ is that it can be embedded into the Drinfeld-Jimbo quantum algebra $U_q(sl_n)$ (see [12]). This quantum algebra is generated by the elements E_i, F_i, $K_i^{\pm 1} = q^{\pm H_i}$, $i = 1, 2, \cdots, n-1$, satisfying the relations $K_i K_j = K_j K_i$, $K_i K_i^{-1} = K_i^{-1} K_i = 1$ and

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$E_i^2E_{i\pm1}-(q+q^{-1})E_iE_{i\pm1}E_i+E_{i\pm1}E_i^2=0,$$
$$F_i^2F_{i\pm1}-(q+q^{-1})F_iF_{i\pm1}F_i+F_{i\pm1}F_i^2=0,$$
$$[E_i,E_j]=0,\ \ [F_i,F_j]=0\quad \text{for}\quad |i-j|>1,$$

where a_{ij} are elements of the Cartan matrix of the Lie algebra sl_n.

Let us introduce the elements

$$\tilde{I}_{j,j-1}=F_{j-1}-qq^{-H_{j-1}}E_{j-1},\quad j=2,3,\cdots,n,$$

of $U_q(sl_n)$. It is proved in [12] that there exists the algebra homomorphism $\varphi: U'_q(so_n)\to U_q(sl_n)$ uniquely determined by the relations $\varphi(I_{i+1,i})=\tilde{I}_{i+1,i}$, $i=1,2,\cdots,n-1$. The following theorem states that this homomorphism is in fact an isomorphism.

Theorem 5. *The homomorphism* $\varphi: U'_q(so_n)\to U_q(sl_n)$ *determined by* $\varphi(I_{i+1,i})=\tilde{I}_{i+1,i}$, $i=1,2,\cdots,n-1$, *is an isomorphism of* $U'_q(so_n)$ *to* $U_q(sl_n)$.

In [6] the authors of that paper state that this homomorphism is an isomorphism and say that it can be proved by means of the Diamond Lemma. However, we could not restore their proof and found another one in [8]. Theorem 5 has the following important corollary, proved in [8]:

Corollary. *Finite dimensional irreducible representations of* $U'_q(so_n)$ *separate elements of this algebra, that is, for any* $a\in U'_q(so_n)$> *there exists a finite dimensional irreducible representation* T *of* $U'_q(so_n)$ *such that* $T(a)\neq 0$.

Note that this corollary is true for q not a root of unity as well as for q a root of unity. It is important for the representation theoiy of the algebra $U'_q(so_n)$.

Problem. We conjecture that the algebra $U'(so_n)$ is connected with some extension of the Drinfeld-Jimbo quantum algebra $U_q(so_n)$. This conjecture is proved in [13] for the case η = 3. It is shown there that there is an isomorphism $\varphi: U'_q(so_3)\to \hat{U}_q(sl_2)$, where $\hat{U}_q(sl_2)$ is an extension of the quantum algebra $U_q(sl_2)$.

In order to determine the isomorphism $\varphi: U'_q(so_3)\to \hat{U}_q(sl_2)$ we note that the quantum algebra $U_q(sl_2)$ is generated by elements q^H, q^{-H}, E, F satisfying the relations

$$q^Hq^{-H}=q^{-H}q^H=1,$$
$$q^HEq^{-H}=qE,\quad q^HFq^{-H}=q^{-1}F, \tag{10}$$
$$[E,F]:=EF-FE=\frac{q^{2H}-q^{-2H}}{q-q^{-1}}. \tag{11}$$

In order to relate the algebras $U'_q(so_3)$ and $U_q(sl_2)$ we need to extend $U_q(sl_2)$ by the elements $(q^kq^H+q^{-k}q^{-H})^{-1}$. We denote by $\hat{U}_q(sl_2)$ the associative algebra with unit element generated by q^H, q^{-H}, E, F, $(q^kq^H+q^{-k}q^{-H})^{-1}$, $k\in\mathbf{Z}$, satisfying the defining relations (10) and (11) of the algebra $U_q(sl_2)$ and the following natural relations:

$$(q^kq^H+q^{-k}q^{-H})^{-1}(q^kq^H+q^{-k}q^{-H})=$$
$$=(q^kq^H+q^{-k}q^{-H})(q^kq^H+q^{-k}q^{-H})^{-1}=1,$$
$$q^{\pm H}(q^kq^H+q^{-k}q^{-H})^{-1}=(q^kq^H+q^{-k}q^{-H})^{-1}q^{\pm H},$$
$$(q^kq^H+q^{-k}q^{-H})^{-1}E=E(q^{k+1}q^H+q^{-k-1}q^{-H})^{-1},$$
$$(q^kq^H+q^{-k}q^{-H})^{-1}F=F(q^{k-1}q^H+q^{-k+1}q^{-H})^{-1}.$$

There exists a unique algebra homomorphism ψ : $U'_q(so_3)\to\hat{U}_q(sl_2)$ such that

$$\psi(I_{21})=\frac{\mathrm{i}}{q-q^{-1}}(q^H-q^{-H}),$$
$$\psi(I_{32})=(E-F)(q^H+q^{-H})^{-1},$$

where $\mathrm{i}:=\sqrt{-1}$. It is proved in [8] that for q not a root of unity the homomorphism $\psi: U'_q(so_3)\to\hat{U}_q(sl_2)$ is injective.

Conjecture. The homomorphism $\psi: U'_q(so_3)\to\hat{U}_q(sl_2)$ is injective when q is a root of unity.

4. Finite dimensional classical type representations of $U'_q(so_n)$

If q is not a root of unity, the algebra $U'_q(so_n)$ has two types of irreducible finite dimensional representations:

(a) representations of the classical type (at $q\to 1$ they give the corresponding finite dimensional irreducible representations of the Lie algebra so_n);

(b) representations of the nonclassical type (they do not admit the limit $q\to 1$).

Let us describe (in the framework of a q-analogue of Gel'fand-Tsetlin formalism) irreducible finite dimensional representations of the algebras $U'_q(so_n)$, $\eta\geq 3$, which are g-deformations of the finite dimensional irreducible representations of the Lie algebra so_n. As in the classical case, they are given by sets m_n consisting of $\{n/2\}$ numbers $m_{1,n}, m_{2,n},\dots, m_{\{n/2\},n}$ (here $\{n/2\}$ denotes integral part of n/2) which are all integral or all half-integral and satisfy the dominance conditions

$$m_{1,2p+1}\geq m_{2,2p+1}\geq\dots\geq m_{p,2p+1}\geq 0, \tag{12}$$
$$m_{1,2p}\geq m_{2,2p}\geq\dots\geq m_{p-1,2p}\geq |m_{p,2p}| \tag{13}$$

for η = 2p + 1 and n = 2p, respectively. These representations are denoted by $T_{\mathbf{m}_n}$. For a basis in a representation space we take the q-analogue of Gel'fand-Tsetlin basis which is obtained by successive reduction of the representation $T_{\mathbf{m}_n}$ to the subalgebras $U'_q(\mathrm{so}_{n-1}),\ U'_q(\mathrm{so}_{n-2}),\ \cdots,\ U'_q(\mathrm{so}_3),\ U'_q(\mathrm{so}_2) := U(\mathrm{so}_2)$. As in the classical case, its elements are labelled by Gel'fand-Tsetlin tableaux

$$\{\xi_n\} \equiv \left\{ \begin{array}{c} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \dots \\ \mathbf{m}_2 \end{array} \right\}, \qquad (14)$$

where the components of $\mathbf{m}_n$ and $\mathbf{m}_{n-1}$ satisfy the usual (classical) "betweenness" conditions. The basis element defined by tableau $\{\xi_n\}$ is denoted as $|\{\xi_n\}\rangle$ or simply as $|\xi_n\rangle$.

It is convenient to introduce the so-called l-coordinates

$$\begin{aligned} l_{j,2p+1} &= m_{j,2p+1} + p - j + 1, \\ l_{j,2p} &= m_{j,2p} + p - j, \end{aligned} \qquad (15)$$

for the numbers $m_{i,k}$. The operator $T_{\mathbf{m}_n}(I_{j+1,j})$ of the representation $T_{\mathbf{m}_n}$ of $U'_q(so_n)$ acts upon Gel'fand-Tsetlin basis elements, labelled by (14), by the formula

$$T_{\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p} \frac{A^j_{2p}(\xi_n)}{q^{l_{j,2p}} + q^{-l_{j,2p}}} |(\xi_n)^{+j}_{2p}\rangle - $$
$$ - \sum_{j=1}^{p} \frac{A^j_{2p}((\xi_n)^{-j}_{2p})}{q^{l_{j,2p}} + q^{-l_{j,2p}}} |(\xi_n)^{-j}_{2p}\rangle, \qquad (16)$$

$$T_{\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \mathrm{i}\, C_{2p-1}(\xi_n)|\xi_n\rangle +$$
$$+ \sum_{j=1}^{p-1} \frac{B^j_{2p-1}(\xi_n)}{[2l_{j,2p-1} - 1][l_{j,2p-1}]} |(\xi_n)^{+j}_{2p-1}\rangle -$$
$$- \sum_{j=1}^{p-1} \frac{B^j_{2p-1}((\xi_n)^{-j}_{2p-1})}{[2l_{j,2p-1} - 1][l_{j,2p-1} - 1]} |(\xi_n)^{-j}_{2p-1}\rangle \qquad (17)$$

In these formulas, $(\xi_n)^{\pm j}_k$ means the tableau (14) in which j-th component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k} \pm 1$. The coefficients A^j_{2p}, B^j_{2p-1}, C_{2p-1} in (16) and (17) are given by the expressions

$$A^j_{2p}(\xi_n) = \left(\frac{\prod_{i=1}^{p} [l_{i,2p+1} + l_{j,2p}][l_{i,2p+1} - l_{j,2p} - 1]}{\prod_{i\neq j}^{p} [l_{i,2p} + l_{j,2p}][l_{i,2p} - l_{j,2p}]} \times \right.$$
$$\left. \times \frac{\prod_{i=1}^{p-1} [l_{i,2p-1} + l_{j,2p}][l_{i,2p-1} - l_{j,2p} - 1]}{\prod_{i\neq j}^{p} [l_{i,2p} + l_{j,2p} + 1][l_{i,2p} - l_{j,2p} - 1]} \right)^{\frac{1}{2}} \qquad (18)$$

$$B^j_{2p-1}(\xi_n) = \left(\frac{\prod_{i=1}^{p} [l_{i,2p} + l_{j,2p-1}][l_{i,2p} - l_{j,2p-1}]}{\prod_{i\neq j}^{p-1} [l_{i,2p-1} + l_{j,2p-1}][l_{i,2p-1} - l_{j,2p-1}]} \times \right.$$
$$\left. \times \frac{\prod_{i=1}^{p-1} [l_{i,2p-2} + l_{j,2p-1}][l_{i,2p-2} - l_{j,2p-1}]}{\prod_{i\neq j}^{p-1} [l_{i,2p-1} + l_{j,2p-1} - 1][l_{i,2p-1} - l_{j,2p-1} - 1]} \right)^{\frac{1}{2}} \qquad (19)$$

$$C_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p} [l_{s,2p}] \prod_{s=1}^{p-1} [l_{s,2p-2}]}{\prod_{s=1}^{p-1} [l_{s,2p-1}][l_{s,2p-1} - 1]}, \qquad (20)$$

where numbers in square brackets mean q-numbers defined by

$$[a] := (q^a - q^{-a})/(q - q^{-1}).$$

A proof of the fact that formulas (16)-(20) indeed determine a representation of U'_q (so„) is given in [14].

Theorem 6. *The representations $T_{\mathbf{m}_n}$ are irreducible and pairwise nonequivalent.*

Conjecture. The Casimir elements of Theorem 3 separate irreducible representations of the classical type, that is, for any irreducible representations $T_{\mathbf{m}_n}$ and $T_{\mathbf{m}'_n}$ there exists a central element C of Theorem 3 such that $T_{\mathbf{m}_n}(C) \neq T_{\mathbf{m}'_n}(C)$.

5. Finite dimensional nonciassical type representations of $U'_q(so_n)$

Irreducible finite dimensional nonciassical type representations are given by sets $\epsilon := (\epsilon_2, \epsilon_3, \cdots, \epsilon_n)$, $\epsilon_i = \pm 1$, and by sets $\mathbf{m}_n$ consisting of $\{n/2\}$ **half-integral** numbers $m_{1,n}, m_{2,n}, \cdots, m_{\{n/2\},n}$ (here $\{n/2\}$ denotes integral part of n/2) satisfying the dominance conditions

$$m_{1,2p+1} \geq m_{2,2p+1} \geq \cdots \geq m_{p,2p+1} \geq 1/2,$$
$$m_{1,2p} \geq m_{2,2p} \geq \cdots \geq m_{p-1,2p} \geq m_{p,2p} \geq 1/2.$$

These representations are denoted by $T_{\epsilon,\mathbf{m}_n}$

For a basis in the representation space we use the analogue of the basis of the previous section. Its elements are labelled by tableaux (14), where the components of $\mathbf{m}_k$ and $\mathbf{m}_{k-1}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \dots$$
$$\dots \geq m_{p,2p+1} \geq m_{p,2p} \geq 1/2,$$
$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \dots$$
$$\dots \geq m_{p-1,2p-1} \geq m_{p,2p}.$$

The basis element defined by tableau $\{\xi_n\}$ is denoted as $|\xi_n\rangle$

As in the previous section, it is convenient to introduce by formula (15) the l-coordinates. The operator $T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\epsilon,\mathbf{m}_n}$

of $U'_q(\mathrm{so}_n)$ acts upon our basis elements by the formulas

$$T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle =$$

$$= \delta_{m_{p,2p},1/2} \frac{\epsilon_{2p+1}}{q^{1/2}-q^{-1/2}} D_{2p}(\xi_n)|\xi_n\rangle +$$

$$+ \sum_{j=1}^{p} \frac{A^j_{2p}(\xi_n)}{q^{l_{j,2p}}-q^{-l_{j,2p}}} |(\xi_n)^{+j}_{2p}\rangle -$$

$$- \sum_{j=1}^{p} \frac{A^j_{2p}((\xi_n)^{-j}_{2p})}{q^{l_{j,2p}}-q^{-l_{j,2p}}} |(\xi_n)^{-j}_{2p}\rangle,$$

where the summation in the last sum must be from 1 to p -1 if $m_{p,2p} = 1/2$, and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$T_{\epsilon,\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \epsilon_{2p}\hat{C}_{2p-1}(\xi_n)|\xi_n\rangle +$$

$$+ \sum_{j=1}^{p-1} \frac{B^j_{2p-1}(\xi_n)}{[2l_{j,2p-1}-1][l_{j,2p-1}]_+} |(\xi_n)^{+j}_{2p-1}\rangle -$$

$$- \sum_{j=1}^{p-1} \frac{B^j_{2p-1}((\xi_n)^{-j}_{2p-1})}{[2l_{j,2p-1}-1][l_{j,2p-1}-1]_+} |(\xi_n)^{-j}_{2p-1}\rangle,$$

where $[a]_+ = (q^a+q^{-a})/(q-q^{-1})$. In these formulas, $(\xi_n)^{\pm j}_k$ means the tableau $\{\xi_n\}$ in which *j-th* component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k}\pm 1$. Matrix elements A^j_{2p} and B^j_{2p-1} are given by the same formulas as in (16) and (17) (that is, by the formulas (18) and (19)) and

$$\hat{C}_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}]_+ \prod_{s=1}^{p-1}[l_{s,2p-2}]_+}{\prod_{s=1}^{p-1}[l_{s,2p-1}]_+[l_{s,2p-1}-1]_+},$$

$$D_{2p}(\xi_n) = \frac{\prod_{i=1}^{p}[l_{i,2p+1}-\frac{1}{2}]\prod_{i=1}^{p-1}[l_{i,2p-1}-\frac{1}{2}]}{\prod_{i=1}^{p-1}[l_{i,2p}+\frac{1}{2}][l_{i,2p}-\frac{1}{2}]}.$$

For the operators $T_{\epsilon,\mathbf{m}_n}(I_{3,2})$ and $T_{\epsilon,\mathbf{m}_n}(I_{2,1})$ we have

$$T_{\epsilon,\mathbf{m}_n}(I_{3,2})|\xi_n\rangle = \frac{1}{q^{m_{1,2}}-q^{-m_{1,2}}}(([l_{1,3}+$$

$$+m_{1,2}][l_{1,3}-m_{1,2}-1])^{1/2}|(\xi_n)^{+1}_2\rangle -$$

$$-([l_{1,3}+m_{1,2}-1][l_{1,3}-m_{1,2}])^{1/2}|(\xi_n)^{-1}_2\rangle)$$

if $m_{1,2}\neq \frac{1}{2}$,

$$T_{\epsilon,\mathbf{m}_n}(I_{3,2})|\xi_n\rangle = \frac{1}{q^{1/2}-q^{-1/2}}(\epsilon_3[l_{1,3}-1/2]|\xi_n\rangle +$$

$$+([l_{1,3}+1/2][l_{1,3}-3/2])^{1/2}|(\xi_n)^{+1}_2\rangle)$$

if $m_{1,2}=\frac{1}{2}$, and $T_{\epsilon,\mathbf{m}_n}(I_{2,1})|\xi_n\rangle = \epsilon_2[m_{1,2}]_+|\xi_n\rangle$.

The above formulas for the operators $T_{\epsilon,\mathbf{m}_n}(I_{k,k-1})$ are given in [15].

Theorem 7. *The representations $T_{\epsilon,\mathbf{m}_n}$ are irreducible. The representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\epsilon',\mathbf{m}'_n}$ are pairwise nonequivalent for $(\epsilon,\mathbf{m}_n)\neq(\epsilon',\mathbf{m}'_n)$. For any admissable $(\epsilon,\mathbf{m}_n)$ and $\mathbf{m}'_n$ the representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\mathbf{m}'_n}$ are pairwise nonequivalent.*

Conjecture. If q is not a root of unity, then every irreducible finite dimensional representation of $U'_q(so_n)$ is equivalent to one of the representations $T_{\mathbf{m}_n}$ of the classical type or to one of the representations $T_{\epsilon,\mathbf{m}_n}$ of the nonclassical type. This conjecture is proved for the algebra $U'_q(so_3)$ (see [11]).

Theorem 8. *If q is not a root of unity, then a restriction of any irreducible representation of $U_q(sl_n)$ to the subalgebra $U'_q(so_n)$ is completely reducible and contains only irreducible representations of the classical type.*

The assertion on complete reducibility is proved in [6]. The second part of Theorem 8 is proved in [16].

Problems: 1. It is necessary to show that finite dimensional representations of the algebra $U'_q(so_n)$ at q not a root of unity are completely reducible. This assertion is proved for $n = 3$ in [17].

2. Note that central elements of the algebra $U'_q(so_n)$ from Theorem 3 do not separate irreducible representations of the nonclassical type, that is, there exist nonequivalent irreducible representations of the nonclassical type on which all central elements take the same eigenvalues. Namely, it can be shown that on representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\epsilon',\mathbf{m}_n}$ (with the same $\mathbf{m}_n$) Casimir elements take the same eigenvalues. It is necessary to show that central elements separate representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\epsilon,\mathbf{m}'_n}$ with different $\mathbf{m}_n$ and $\mathbf{m}'_n$

3. Using the embedding $\psi : U'_q(\mathrm{so}_3) \to \hat{U}_q(\mathrm{sl}_2)$ it is shown in [13] how to define tensor products of irreducible representations of $U'_q(so_3)$. Using the embedding $U'_q(so_n)$ -> $U_q(\mathrm{sl}_n)$ it is shown in [16] how to multiply the classical type representations $T_{\mathbf{m}'_n}$ and $T_{(1,0,\cdots,0)}$ of $U'(\mathrm{so}_n)$ at q not a root of unity. (The decomposition of such tensor product is such as in the classical case.) It is necessary to show how to multiply any irreducible representations of $U'_q(so_n)$ for q not a root of unity.

6. Irreducible representations of $U'_q(\mathrm{so}_n)$ for q a root of unity

It is well-known that a Drinfeld-Jimbo quantum algebra $U_q(g)$ for q a root of unity is a finite dimensional vector space over the center of $U_q(g)$. The same assertion is true (see [8]) for the algebra $U'_q(\mathrm{so}_n)$.

Using this assertion and the Poincare-Birkhoff-Witt theorem for $U'_q(so_n)$ it is easily proved the following theorem:

Theorem 9. *If q is a root of unity, then any irreducible representation of $U'_q(so_n)$ is finite dimensional.*

Irreducible representations of the algebra $U'_q(so_n)$ for q a root of unity are described in [8]. For construction of these irreducible representations of $U'_q(so_n)$, it is used the method of D. Arnaudon and A. Chakrabarti (used by them for construction of irreducible representations of the quantum algebra $U_q(sl_n)$ when q is a root of unity). If $q^p = 1$ and p is an odd integer, then there exists the series of irreducible representations of $U_q(so_n)$ which act on p^N-dimensional vector space (where N is the number of positive roots of the Lie algebra so_n) and are given by $r = \dim so_n$ complex parameters. These representations are irreducible for generic values of these parameters. These representations constitute the main class of irreducible representations of $U'_q(so_n)$. For some special values of the representation parameters in C^r the representations are reducible. These reducible representations give many other classes of (degenerate) irreducible representations which are given by less number of parameters or by parameters, values of which cover subsets of $\mathbb{C}^r$ of Lebesgue measure 0. As in the case of irreducible representations of the quantum algebra $U_q(sl_n)$, it is difficult to enumerate all irreducible representations of these classes. However, the most important classes of these degenerate representations can be constructed. In particular, in [8] we give 2^{n-l} classes of these representations, which are an analogue of the nonclassical type irreducible representations of $U'_q(so_n)$ for q not a root of unity.

Note that the problem of classification of irreducible representations of the algebra $U'_q(so_n)$ at q a root of unity is very complicated problem. (The same is true for the Drinfeld-Jimbo quantum algebras.) This problem is complicated even for the case of the algebra $U'_q(so_3)$ (see [12] for details).

Problems: 1. Let q be a root of unity: $q^k - 1$, It is necessary to find explicit expressions for the values $T(C^{(k)}(I_{i,j}))$, (i,j) = (2,1), (3,2), (3,1), for irreducible representations T of the algebra $U'_q(so_3)$. These values are known for some of irreducible representations. Namely, then for the irreducible representations $R_l^{(1)}$ from section 9 of [13] we have

$$R_l^{(1)}(C^{(k)}(I_{21})) = R_l^{(1)}(C^{(k)}(I_{32})) =$$
$$= R_l^{(1)}(C^{(k)}(I_{31})) = \frac{2}{n}(q-q^{-1})^{-k}\cos\frac{\pi k}{2},$$

and for the irreducible representations $R_{ab\lambda}$ from [13] $R_{ab\lambda}(C^{(k)}(I_{21})) = \frac{2}{n}(q-q^{-1})^{-k}\cos k\sigma$, where $\lambda = e^{-i\sigma}$

2. For q a root of unity, irreducible representations of the algebra $U'_q(so_3)$ map the set of Casimir elements $C^{(k)}(I_{21}), C^{(k)}(I_{32}), C^{(k)}(I_{31})$ to $\mathbb{C}^3$ We conjecture that the image of this map coincides with $\mathbb{C}^3$

3. It is conjectured that there exists a closed domain $D \subset \mathbb{C}^3$ of Lebesgue measure 0 such that the mapping $T : (C^{(k)}(I_{21}), C^{(k)}(I_{32}), C^{(k)}(I_{31})) \to \mathbb{C}^3$ is a one-to-one map for $\mathbb{C}^3 \setminus D$

7. Restriction of representations of $U_q(sl_n)$ to $U'_q(so_n)$

In this section we assume that q is not a root of unity. The algebra $U'_q(so_n)$ is a subalgebra of the quantum algebra $U_q(sl_n)$. Therefore, we may restrict irreducible finite dimensional representations of the algebra $U_q(sl_n)$ to the subalgebra $U'_q(so_n)$. Generally speaking, such a restriction leads to reducible representations of the subalgebra. It was proved in [17] that each irreducible finite dimensional representation of $U_q(sl_n)$ under restriction to $U'_q(so_n)$ decomposes into a direct sum of irreducible representations of this subalgebra. N. Iorgov has proved (will be published) that such a decomposition contains only irreducible representations of the classical type. However, explicit formula for the decomposition is known only for the restriction $U_q(sl_3)$ -> $U'_q(so_3)$.

Irreducible finite dimensional representations of $U_q(sl_3)$ are given by three integers $\ell = (l_1, l_2, l_3)$ such that $l_1 \geq l_2 \geq l_3$. We denote such the representation by R_ℓ. Irreducible finite dimensional classical type representations of $U'_q(so_3)$ are denoted by T_k, where k is a nonnegative integral or half-integral number.

In order to find which irreducible representations of $U'(so_3)$ are contained in the decomposition of $R_\ell \downarrow_{U'_q(so_3)}$ we split in [18] the spectrum Spec $R_\ell(I_{21})$ of the representation operator $R_\ell(I_{21})$ into spectra of operators $T_k(I_{21})$ of irreducible representations T_k of $U'_q(so_3)$. (It is proved in [18] that such splitting is unique.) As a result, we have that

$$R_\ell \downarrow_{U'_q(so_3)} = \sum_s{}' \sum_{k=s}^{s+l_2-l_3} T_k$$

if $l_1 - l_2$ is odd and

$$R_\ell \downarrow_{U'_q(so_3)} = \sum_s{}' \sum_{k=s}^{s+l_2-l_3} T_k \oplus \sum_r{}' T_r$$

if $l_1 - l_2$ is even, where $\sum_s'$ means the summation over the values $l_1 - l_2, l_1 - l_2 - 2, l_1 - l_2 - 4, \cdots, 1$ (or 2) and the last sum $\sum'_r$ is over the values $l_2 - l_3, l_2 - l_3 - 2, l_2 - l_3$ - 4, ··· ,0 (or 1). Note that these decompositions coincide with the corresponding decompositions for the reduction *SU(3)* -> S0(3).

8. Applications

There are the following main applications of the algebra $U'_q(so_n)$ and its irreducible representations:

1. The theory of orthogonal polynomials and special functions (especially, the theory of *q*-orthogonal polynomials and basic hypergeometric functions). This direction is not good worked out. Some ideas of such applications can be found in [19].

2. The algebra $U'_q(So_n)$ (espesially its particular case $U'_q(SO_3)$) is related to the algebra of observables in 2+1 quantum gravity on the Riemmanian surfaces (see the papers [3], [4], [20], [21]).

3. A g-anaiogue of the Riemannian symmetric space *SU(n)/SO(n)* is constructed by means of the algebra $U'_q(so_n)$. This construction is fulfilled in the paper [12].

4. A g-analogue of the theory of harmonic polynomials (g-harmonic polynomials on quantum vector space $\mathbb{R}^n_q$) is constructed by using the algebra $U'_q(so_n)$. In particular, a g-analogue of different separations of variables for the g-Laplace operator is given by means of this algebra and its subalgebras. This theory is contained in the papers [6] and [22].

5. The algebra $U_q(so_n)$ also appear in the theoiy of links in the algebraic topology (see [23]).

1. A. *Klimyk* and *K. Schmiidgen:* Quantum Groups and Their Representations, Springer, Berlin, 1997.
2. A. *M. Gaviilik and A. U. Klimyk:* Lett. Math. Phys. **21** (1991) 215.
3. *J. Nelson, T. Regge, and F. Zertuche:* Nucl. Phys. B339 (1990) 227.
4. *J. Nelson, and T. Regge:* Commun. Math. Phys. **155** (1993) 561.
5. *N. Z. Iorgov and A. U. Klimyk:* J. Math. Phys., 42 (2001) 2315.
6. *M. Noumi, T. Umeda, and M. Wakagama:* Compos. Math. 104 (1996)227.
7. A. *M. Gavnlik and N. Z. Iorgov:* Ukr. J. Phys. 43 (1998) 453.
S. *N. Z. Iorgov and A. U. Klimyk:* Methods of Funct. Anal. Topol. 6, N 3 (2000) 56.
9. A. M. *Gavnlik and N. Z. Iorgov:* Heavy Ion Phys. 11, N 1-2 (2000) 29.
10. M. *Havlicek, A. U. Klimyk, and S. Posta:* Czech. J. Phys. 50 (2000) 79.
11. M. *Havlicek and S. Posta:* J. Math. Phys. 42 (2001) 472.
12. M. *Noumi:* Adv. Math. **123** (1996) 16.
13. M. *Havlicek, A. U. Klimyk, and S. Posta:* J. Math. Phys. 40 (1999) 2135.
14. A. M. *Gaviilik and N. Z. Iorgov:* Methods of Funct. Anal. Topol. 3, N4(1997) 51.
15. *N. Z. Iorgov and A. U. Klimyk: Czech.* J. Phys. 50 (2000) 85.
16. *N. Z. Iorgov:* J. Phys. A: Math. Gen., to be published.
17. *N. Z. Iorgov:* Methods of Funct. Anal. Topol. 5, N 2 (1999) 22.
IS. *I. I. Kachurik and A. U. Klimyk: J.* Phys. A: Math. Gen. 34 (2001) 1.
19. A. *U. Klimyk and I. I. Kachuiik:* Commun. Math. Phys. **175** (1996) 89.
20. A. M. *Gaviilik:* Proc. Inst. Math. NAS Ukraine 30 (2000), 304.
21. L. Ciekhov and *V. Fock:* Czech. J. Phys. 50 (2000) 1201 .
22. *N. Z, Iorgov and* A. *U. Klimyk:* J. Math. Phys. 42, N 3 (2001).
23. *D. Bullock and J. H. Przytycki:* Multiplicative structure of Kauffman bracket skein module quantization, e-print: math.QA/9902117.

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НЕСТАНДАРТНА q-ДЕФОРМАЦІЯ ОГОРТУЮЧОЇ АЛГЕБРИ U(so_n), ЗВ'ЯЗАНА З КВАНТОВОЮ ГРАВІТАЦІЄЮ

Описуються властивості нестандартної q-деформації U' (so_n) універсальної огортуючої алгебри U(so_n) алгебри JIi so_n, зв'язаної з 2+1 квантовою гравітацією на ріманових поверхнях. Ця алгебра не співпадає з квантовою алгеброю Дрінфельда-Джімбо $U_q(so_n)$. Формулюється багато нерозв'язаних проблем.