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Contents

Abstract

Let $\mathrm{Bin}(X)$ be a collection of all groupoids on some non-empty set X. Define the operation $\Box : \mathrm{Bin}^2(X) \to \mathrm{Bin}(X)$ so that $x(\circ\Box *)y = (x \circ y) * (y \circ x)$ for all $x, y \in X$ and $(X, \circ), (X, *) \in \mathrm{Bin}(X)$. Let $\circ_{lz}$ denote *left-zero* operation $(\forall x, y \in X : x \circ_{lz} y = x)$ on X. Then, $(X, \circ_{lz})$ is an identity of $(\mathrm{Bin}(X), \Box)$. Similarly, define *right-zero* $\circ_{rz} \in \mathrm{Bin}(X)$ $(\forall x, y \in X : x \circ_{rz} y = y)$.

We consider the center of $(\mathrm{Bin}(X), \Box)$ and represent its elements as graphs. Furthermore, we investigate distributivity from the left in $\mathrm{Bin}(X)$ and its interaction with $\Box$-product. We show that the only operation that is left-distributive over all possible $\circ \in \mathrm{Bin}(X)$ is $\circ_{rz} \in \mathrm{Bin}(X)$ and that any $\circ \in \mathrm{Bin}(X)$ is left-distributive over $\circ_{lz}, \circ_{rz} \in \mathrm{Bin}(X)$.

Keywords: *Groupoid,* $\mathrm{Bin}(X)$*, left-distributivity, graph of groupoid.*

Introduction

The study of algebraic structures is of particular interest in mathematics and its applications. The theory of groups, rings, and fields is widely used in the natural and computer sciences. An overview of structures equipped with one binary operation was given by R. H. Bruck in [1]. Here, groupoids, as one of the most natural yet simplest examples of the interplay between a set and a binary operation, were considered.

In [2], H.S. Kim and J. Neggers defined a collection $\mathrm{Bin}(X)$ of all groupoids on some fixed set X. They also proposed an operation to multiply entities of this collection, and since the initial set is fixed, this product may be treated as a product of binary operations themselves. In turn, the collection and the operation appeared to form a monoid.

H. F. Fayoumi published a study [3] on commutativity in the discussed monoid. She considered the notion of a center of the algebraic structure (a class of elements of X commutative with any other its element). It yielded to the series of rather intriguing requirements to belongings of center, which are referred to as locally-zero. Fayoumi continued the research in [4], where she devised several approaches to factorization of elements of $\mathrm{Bin}(X)$. These were also reinforced with the variety of examples of such a factorization in well-known groupoids.

Later, S.S. Ahn, H.S. Kim and J. Neggers pointed out in [5] that locally-zero groupoids could quite naturally define an undirected graph. Not only did they formalize the concept, but also regarded several notions of graph and groupoid theories, that turned out to complement one another.

This paper is devoted to studying the structure of $\mathrm{Bin}(X)$. In particular, we study the notion of $\mathrm{Bin}(X)$ and develop the vision of distributivity from the left in terms of it. Additionally, a sequence of useful results is provided for the left-distributive relation. We consider the center of $\mathrm{Bin}(X)$ along with its entities' graph representation. We belabor the point by proposing the way to construct a graph of the product of locally-zero operations in terms of set-theoretical symmetric difference. Further, we study in [6] associativity and invertibility in the center of $\mathrm{Bin}(X)$.

1 Definitions

1.1 Binary operations and algebraic structures

Definition 1.1. Let X be some non-empty set. A *binary operation* is a mapping $\varphi : X \times X \to X$ that assigns to each pair $(x, y) \in X \times X$ some element $z \in X$.

Example 1.2. Let $X = \mathbb{R}$. One could consider mappings φ, ψ such that:

$$\forall x, y \in \mathbb{R} : \varphi(x, y) = x \cdot y;$$
$$\forall x, y \in \mathbb{R} : \psi(x, y) = x + y.$$

It is easy to see that both φ and ψ are consistent with the definition for binary operations. In most cases, we shall omit notation like above and simply write $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Definition 1.3. A pair $(X, \circ)$, with $X \neq \emptyset$ being some set and $\circ$ - some binary operation, is referred to as *groupoid* (*magma, binary system*) whenever

$$\forall x, y \in X : x \circ y \in X.$$

In this case, set X is said to be *closed under* $\circ$.

Remark 1.4. It might seem that any set equipped with some binary operation is closed under it. For the sake of contradiction, consider $(\mathbb{N}, -)$, with the operation $- : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. Now, for $(2, 3) \in \mathbb{N}$ we obtain:

$$2 - 3 = -1 \notin \mathbb{N},$$

and so, $(\mathbb{N}, -)$ is not a groupoid.

Definition 1.5. A binary operation $\circ$ is *associative over* X if

$$\forall x, y, z \in X : x \circ (y \circ z) = (x \circ y) \circ z.$$

Definition 1.6. A groupoid $(X, \circ)$ is called a *semigroup* if $\circ$ is associative.

Example 1.7. Consider multiplication and subtraction on $\mathbb{R}$.

- $(\mathbb{R}, \cdot)$ is a semigroup;

- $(\mathbb{R}, -)$ is a groupoid, but not a semigroup.

Definition 1.8. A semigroup $(X, \circ)$ is a *monoid* if there exists an *identity* (*neutral element*) $e \in X$, i.e.

$$\exists e \in X : \forall x \in X : e \circ x = x \circ e = x.$$

Example 1.9. A pair $(\mathbb{R}, \cdot)$, where $\cdot$ denotes the multiplication of real numbers, is a monoid with identity being 1. That is:

$$\forall r \in \mathbb{R} : 1 \cdot r = r \cdot 1 = r.$$

Definition 1.10. Let $(X, \circ)$ be a groupoid. A *center* of groupoid is defined by:

$$Z_\circ(X) = \{\, x \in X \mid \forall y \in X : x \circ y = y \circ x \,\}.$$

Remark 1.11. The above definition is valid for semigroups, monoids and further algebraic structures.

Definition 1.12. Let $(X, \circ)$ be a groupoid. The element $x \in X$ is called an *idempotent* with respect to $\circ$ if it holds that:

$$x \circ x = x.$$

Example 1.13. Consider $(\mathbb{N}, \cdot)$ and $(\mathbb{N}, +)$. Here, $1 \in \mathbb{N}$ is idempotent with respect to multiplication, whilst not with respect to addition.

Definition 1.14. The operation $\circ$ is an *idempotent operation* over X if all elements of X are idempotent with respect to $\circ$, i.e.:

$$\forall x \in X : x \circ x = x.$$

Definition 1.15. Let $\circ, \bullet$ be some binary operations defined on some set X. We say that $\circ$ is *left-distributive* over $\bullet$ if:

$$\forall x, y, z \in X : x \circ (y \bullet z) = (x \circ y) \bullet (x \circ z).$$

Whenever it is the case, we write $\circ \hookrightarrow \bullet$.

Definition 1.16. Let $(X, \circ_{lz})$ be a groupoid. The operation $\circ_{lz}$ is said to be *left-zero operation* if:

$$\forall x, y \in X : x \circ_{lz} y = x.$$

Definition 1.17. Let $(X, \circ_{rz})$ be a groupoid. The operation $\circ_{rz}$ is said to be *right-zero operation* if:

$$\forall x, y \in X : x \circ_{rz} y = y.$$

Observation 1.18. *Any non-empty set X equipped with $\circ_{lz}$ ($\circ_{rz}$) is a semigroup.*

Proof. For $\circ_{lz}$, consider an associativity equation. That is, for all $x, y, z \in X$:

$$\begin{gathered} x \circ_{lz} (y \circ_{lz} z) = (x \circ_{lz} y) \circ_{lz} z \\ x \circ_{lz} y = x \circ_{lz} z \\ x = x, \end{gathered}$$

and, thus, associativity follows. Similarly, for $\circ_{rz}$:

$$\begin{gathered} x \circ_{rz} (y \circ_{rz} z) = (x \circ_{rz} y) \circ_{rz} z \\ x \circ_{rz} z = y \circ_{rz} z \\ z = z, \end{gathered}$$

which was to be shown. □

We say that $(X, \circ_{lz})$ is a *left-zero semigroup* (*leftoid*), $(X, \circ_{rz})$ - *right-zero semigroup* (*rightoid*) over X.

Definition 1.19. An operation † defined on set X is a *constant* (at $x_0 \in X$) operation if and only if:

$$\forall x, y \in X : x \dagger y = x_0.$$

1.2 Set theory

Definition 1.20. A set X is *well-ordered* if any $S \subseteq X$ with $S \neq \emptyset$ has a least element.

Definition 1.21. A *total order on X* is a binary relation $\leq \subseteq X \times X$ which satisfies:

- *reflexivity:* $\forall x \in X : x \leq x$
- *antisymmetry:* $\forall x, y \in X : (x \leq y \wedge y \leq x) \Rightarrow x = y$

- *transitivity:* $\forall x, y, z \in X : (x \leq y \leq z) \Rightarrow x \leq z$
- *weak connectivity:* $\forall x, y \in X : x \leq y \vee y \leq x$

Theorem 1.22. *Any well-ordered set has a total order.*

Theorem 1.23. (Zermelo). *Any non-empty set can be well-ordered.*

1.3 Graph theory

Definition 1.24. An (*undirected*) *graph* G is a pair (V, E) whereby:

(1) $V = V(G)$ - a set of *vertices*;

(2) $E = E(G) \subseteq V^{(2)} = \{\, \{u, v\} \mid u, v \in V \,\}$ - a set of *edges*.

Definition 1.25. Two vertices $u, v \in V(G)$ are said to be *adjacent* if there exists $e \in E(G)$ such that $e = \{u, v\}$.

Definition 1.26. An undirected graph is *simple* if it contains neither loops nor parallel edges.

Definition 1.27. An *empty* graph L_n is a graph with $|V(L_n)| = n$ and $E(L_n) = \emptyset$.

Definition 1.28. A *complete* graph K_n is a graph such that any two of its n vertices are adjacent.

Observation 1.29. *On n vertices, $2^{C_n^2}$ simple graphs could be generated.*

Definition 1.30. Let $G = (V, E)$ be a graph. A *complement* $\overline{G}$ is defined by:

$$V(\overline{G}) = V(G);$$
$$E(\overline{G}) = \{\, \{u, v\} \mid \{u, v\} \notin E(G) \,\}.$$

Definition 1.31. Let G, H be graphs on the same vertex set. An (*edge*) *symmetric difference* $G \triangle H$ is a graph defined by:

$$V(G \triangle H) = V(G) = V(H);$$
$$E(G \triangle H) = (E(G) \cup E(H)) \setminus (E(G) \cap E(H))$$

Definition 1.32. A *directed graph* D is a pair (V, A) whereby:

(1) $V = V(D)$ - a set of *vertices*;

(2) $A = A(D) \subseteq V \times V = \{\, (u, v) \mid u, v \in V \,\}$ - a set of *arcs*.

Definition 1.33. Let $v \in V(D)$ be a vertex of digraph D. Then the sets

$$N_D^+[v] = \{\, u \in V \mid (v, u) \in A \,\},$$

$$N_D^-[v] = \{\, w \in V \mid (w, v) \in A \,\}$$

are called *outgoing-neighbourhood* and *ingoing-neighbourhood*, respectively.

2 The monoid of all binary operations Bin(X)

Definition 2.1. Let $X \neq \emptyset$ be a set. We denote by $\mathrm{Bin}(X)$ the collection of all groupoids (magmas, binary systems) on X, i.e.

$$\mathrm{Bin}(X) = \{ (X, *) \mid \forall x, y \in X : x * y \in X \}.$$

Now let $(X, \circ), (X, *) \in \mathrm{Bin}(X)$. We define an operation $\Box$ as follows:

$$\forall x, y \in X : x \Box y = (x \circ y) * (y \circ x).$$

We write $(X, \circ)\Box(X, *)$ in this case. However, it is even more convenient to write $\circ, * \in \mathrm{Bin}(X)$ and so $\circ\Box*$, since the set X is predefined.

Theorem 2.2. $(\mathrm{Bin}(X), \Box)$ *forms a semigroup, i.e. the operation* $\Box$ *is associative.*

Proof. Let $*, \circ, \bullet \in \mathrm{Bin}(X)$. We prove that:

$$*\Box(\bullet\Box\circ) = (*\Box\bullet)\Box \circ .$$

Let us start with the left-hand side. We denote $\diamond = \bullet\Box\circ$. Then:

$$\forall x, y \in X : *\Box\diamond = (x * y) \diamond (y * x) = ((x * y) \bullet (y * x)) \circ ((y * x) \bullet (x * y)).$$

Similarly, on the right-hand side, with, say, $\star = *\Box\bullet$, we prove:

$$\forall x, y \in X : \star\Box\circ = (x \star y) \circ (y \star x) = ((x * y) \bullet (y * x)) \circ ((y * x) \bullet (x * y)),$$

and, thus, $\Box$ is associative. $\Box$

Theorem 2.3. *Left-zero semigroup* $\circ_{lz} \in \mathrm{Bin}(X)$ *is an identity element of* $(\mathrm{Bin}(X), \Box)$*, i.e.* $(\mathrm{Bin}(X), \Box)$ *is a monoid.*

Proof. For all $* \in \mathrm{Bin}(X)$, we show that:

$$\circ_{lz}\Box* = *; \tag{1}$$

$$*\Box\circ_{lz} = *. \tag{2}$$

For (1), we obtain:

$$\forall x, y \in X : \circ_{lz}\Box* = ((x \circ_{lz} y) * (y \circ_{lz} x)) = x * y,$$

which tells that applying (1) is the same as applying plain $*$. Correspondingly, for (2) we conclude:

$$\forall x, y \in X : *\Box\circ_{lz} = ((x * y) \circ_{lz} (y * x)) = x * y,$$

which proves (2) from the same considerations. Hence, $\circ_{lz}$ is an identity. $\Box$

3 Left distributivity relation on $\mathbf{Bin}(X)$

Let $\circ, \bullet \in \mathrm{Bin}(X)$. Recall that if $\circ$ is left-distributive over $\bullet$, which is

$$\forall x, y, z \in X : x \circ (y \bullet z) = (x \circ y) \bullet (x \circ z),$$

we write $\circ \hookrightarrow \bullet$.

Definition 3.1. Let $\circ \in \mathrm{Bin}(X)$. Its outgoing-neighbourhood $N^+[\circ]$ and ingoing-neighbourhood $N^-[\circ]$ are defined as follows:

$$N^+[\circ] = \{ * \mid \circ \hookrightarrow * \};$$

$$N^-[\circ] = \{ \bullet \mid \bullet \hookrightarrow \circ \}.$$

In the following, we show that for any $\circ \in \mathrm{Bin}(X)$, its outgoing neighbourhood is closed under $\Box$.

Theorem 3.2. *Let $\circ, \bullet_1, \bullet_2 \in \mathrm{Bin}(X)$ be operations with $\circ \hookrightarrow \bullet_1, \circ \hookrightarrow \bullet_2$. Then, $\circ \hookrightarrow \bullet_1 \Box \bullet_2$.*

Proof. Firstly, denote $\oslash = \bullet_1 \Box \bullet_2$, meaning:

$$y \oslash z = (y \bullet_1 z) \bullet_2 (z \bullet_1 y)$$

We want to show:

$$\forall x, y, z \in X : x \circ (y \oslash z) = (x \circ y) \oslash (x \circ z),$$

Deriving left-hand side, we obtain:

$$x \circ (y \oslash z) = x \circ ((y \bullet_1 z) \bullet_2 (z \bullet_1 y)).$$

Use the fact that $\circ \hookrightarrow \bullet_2$:

$$x \circ ((y \bullet_1 z) \bullet_2 (z \bullet_1 y)) = (x \circ (y \bullet_1 z)) \bullet_2 (x \circ (z \bullet_1 y)).$$

Similarly, use $\circ \hookrightarrow \bullet_1$:

$$(x \circ (y \bullet_1 z)) \bullet_2 (x \circ (z \bullet_1 y)) = ((x \circ y) \bullet_1 (x \circ z)) \bullet_2 ((x \circ z) \bullet_1 (x \circ y)) = (x \circ y) \oslash (x \circ z),$$

which is exactly what we wanted to show. $\Box$

Fore some fixed operation, there might occur a necessity to describe sets of operations for which left-distibutivity holds.

Example 3.3. Consider $\mathrm{Bin}(\mathbb{R})$ as well as standard on $\mathbb{R}$ operations of addition $+ : \mathbb{R}^2 \to \mathbb{R}$, multiplication $* : \mathbb{R}^2 \to \mathbb{R}$ and division $/ : \mathbb{R}^2 \to \mathbb{R}$. One could conclude:

(1) $* \in N^-[+]$;

(2) $+ \notin N^-[*]$;

(3) $+ \notin N^+[/]$.

Let us describe neighbourhoods of some specific operations.

Proposition 3.4. *Let $\dagger \in \mathrm{Bin}(X)$ be a constant at $x_0 \in X$ operation. Then:*

(1) $N^+[\dagger] = \{ \circ \mid x_0 \in X$ is an idempotent with respect to $\circ \}$;

(2) $N^-[\dagger] = \{ * \mid x_0 \in X$ is a right-zero with respect to $* \}$.

Proof. For (1), consider some $\circ \in \mathrm{Bin}(X)$ such that $\dagger \hookrightarrow \circ$, i.e., for all $x, y, z \in X$:

$$\begin{aligned} x \dagger (y \circ z) &= (x \dagger y) \circ (x \dagger z) \\ x_0 &= x_0 \circ x_0, \end{aligned}$$

And, thus, x_0 is an idempotent for $\circ$.

Analogously, for (2), consider $* \in \mathrm{Bin}(X)$ with $* \hookrightarrow \dagger$. Then, for all $x, y, z \in X$:

$$\begin{aligned} x * (y \dagger z) &= (x * y) \dagger (x * z) \\ x * x_0 &= x_0, \end{aligned}$$

Which leads us to a conclusion, that x_0 is a right-zero for $*$. □

We take now left- and right-zero semigroups into account.

Proposition 3.5. *Let $\circ_{rz} \in \mathrm{Bin}(X)$ be a right-zero semigroup. Then:*

$$N^+[\circ_{rz}] = N^-[\circ_{rz}] = \mathrm{Bin}(X).$$

Proof. Let $\circ_{rz} \hookrightarrow *$ for some $* \in \mathrm{Bin}(X)$. For all $x, y, z \in X$, it follows:

$$\begin{gathered} x \circ_{rz} (y * z) = (x \circ_{rz} y) * (x \circ_{rz} z) \\ y * z = y * z. \end{gathered}$$

Therefore, $\circ_{rz}$ is left-distributive over any operation.

Similarly, one could consider $\bullet \hookrightarrow \circ_{rz}$ for some $\bullet \in \mathrm{Bin}(X)$, which, for all $x, y, z \in X$, leads to:

$$\begin{gathered} x \bullet (y \circ_{rz} z) = (x \bullet y) \circ_{rz} (x \bullet z) \\ x \bullet z = x \bullet z. \end{gathered}$$

Hence, any operation is left-distributive over the right-zero operation. □

Proposition 3.6. *Let $\circ_{lz} \in \mathrm{Bin}(X)$ be a left-zero semigroup. Then:*

(1) $N^+[\circ_{lz}] = \{ * \mid * \in \mathrm{Bin}(X)$ *is an idempotent operation*$\}$;

(2) $N^-[\circ_{lz}] = \mathrm{Bin}(X)$.

Proof. For (1), let $* \in \mathrm{Bin}(X)$ with $\circ_{lz} \hookrightarrow *$. For any $x, y, z \in X$, it is easy to see that:

$$\begin{gathered} x \circ_{lz} (y * z) = (x \circ_{lz} y) * (x \circ_{lz} z) \\ x = x * x. \end{gathered}$$

As $x \in X$ was chosen arbitrarily, we could say that $\circ_{lz}$ is left-distributive over idempotent operations.

As for (2), again, for $\bullet \in \mathrm{Bin}(X)$ with $\bullet \hookrightarrow \circ_{lz}$ and for all $x, y, z \in X$, we conclude:

$$\begin{gathered} x \bullet (y \circ_{lz} z) = (x \bullet y) \circ_{lz} (x \bullet z) \\ x \bullet y = x \bullet y, \end{gathered}$$

which implies that any operation is left-distributive over the leftoid. □

By means of the last theorems, we can formulate the following results.

Theorem 3.7. *For all $\circ_i \in \mathrm{Bin}(X), i \in I$, it holds:*

$$\{\circ_{rz}, \circ_{lz}\} \;=\; \bigcap_{\circ_i \,\in\, Bin(X)} N^+[\circ_i].$$

Proof. To prove the equality of two sets we have to prove the inclusion from both sides.

"$\subseteq$" The inclusion from right-hand side to the left-hand side is fairly obvious. It follows immediately from **Proposition 3.5** together with **Proposition 3.6**.(2).

"$\supseteq$" Suppose that there exists $\bullet \in \bigcap_{\circ_i \in \mathrm{Bin}(X)} N^+[\circ_i]$ such that it holds $\bullet \notin \{\circ_{rz}, \circ_{lz}\}$. Now, that means that we could find $x, y, u, w \in X$ such that:

$$\begin{cases} x \bullet y \neq x \\ u \bullet w \neq w \end{cases}$$

Consider a binary operation $* \in \mathrm{Bin}(X)$ satisfying the following conditions:

(1) $x * w = y$

(2) $x * u = x$

(3) $\forall z \in X : z * (u \bullet w) = z$

Remembering it must hold that $* \hookrightarrow \bullet$, we conclude, on the one hand

$$x * (u \bullet w) \overset{(3)}{=} x,$$

and, on the other hand,

$$x * (u \bullet w) = (x * u) \bullet (x * w) \overset{(1),(2)}{=} x \bullet y,$$

leading us to

$$x = x \bullet y,$$

which is a contradiction. Therefore, $\bullet \notin N^+[*]$. □

Theorem 3.8. *For all* $\circ_i \in \mathrm{Bin}(X), i \in I$, *it holds:*

$$\{\circ_{rz}\} \;=\; \bigcap_{\circ_i \in Bin(X)} N^-[\circ_i].$$

Proof. "$\subseteq$" The inclusion from right-hand side to the left-hand side follows from **Proposition 3.5**.

"$\supseteq$" We assume, there is $* \in \bigcap_{\circ_i \in \mathrm{Bin}(X)} N^-[\circ_i]$ with $* \neq \circ_{rz}$. That means there exist $x, y \in X$ such that:

$$x * y \neq y.$$

Now consider $\diamond \in \mathrm{Bin}(X)$ with:

$$x \diamond x = (x * x) \diamond (x * x) = y.$$

According to our assumption it must hold that $* \hookrightarrow \diamond$, giving:

$$\begin{aligned} x * (x \diamond x) &= (x * x) \diamond (x * x) \\ x * y &= y. \end{aligned}$$

That means $* \notin N^-[\diamond]$ and, therefore, $* \notin \bigcap_{\circ_i \in \mathrm{Bin}(X)} N^-[\circ_i]$. $\square$

4 Center of Bin(X)

Getting back to the monoid $(\mathrm{Bin}(X), \Box)$, one might want to describe the commutative property with respect to $\Box$. That is, we want to find the operations for which:

$$\circ \Box \bullet = \bullet \Box \circ .$$

Or, in terms of arbitrary pairs $x, y \in X$:

$$(x \circ y) \bullet (y \circ x) = (x \bullet y) \circ (y \bullet x),$$

which is obviously not always the case. Whenever it is the case, we may think of the following structure.

Definition 4.1. The center of $\mathrm{Bin}(X)$ is denoted $\mathrm{ZBin}(X)$ and defined by:

$$\mathrm{ZBin}(X) = \{ \bullet \in \mathrm{Bin}(X) \mid \forall \circ \in \mathrm{Bin}(X) : \bullet \Box \circ = \circ \Box \bullet \}.$$

Let us now discuss a content of $\mathrm{ZBin}(X)$.

Proposition 4.2. *Both* $\circ_{lz}, \circ_{rz} \in \mathrm{Bin}(X)$ *are in* $\mathrm{ZBin}(X)$.

Proof. First, consider $\circ_{lz} \in \mathrm{Bin}(X)$. We want to show:

$$\forall * \in \mathrm{Bin}(X) : \circ_{lz} \Box * = * \Box \circ_{lz}$$

Considering the above product for all $x, y \in X$:

$$\begin{gathered} (x \circ_{lz} y) * (y \circ_{lz} x) = (x * y) \circ_{lz} (y * x) \\ x * y = x * y \end{gathered}$$

which implies:

$$\forall * \in \mathrm{Bin}(X) : \circ_{lz} \Box * = * \Box \circ_{lz} = *$$

Similarly, for $\circ_{rz}$ and for all $x, y \in X$ we obtain:

$$\begin{gathered} (x \circ_{rz} y) * (y \circ_{rz} x) = (x * y) \circ_{rz} (y * x) \\ y * x = y * x \end{gathered}$$

implying again:

$$\forall * \in \mathrm{Bin}(X) : \circ_{rz} \Box * = * \Box \circ_{rz} = *,$$

and the result follows. $\Box$

Proposition 4.3. *If* $\bullet \in \mathrm{ZBin}(X)$*, then* $\bullet$ *is an idempotent operation, i.e*

$$\forall x \in X : x \bullet x = x.$$

Proof. Suppose that there exists $x \in X$ for which $x \bullet x \neq x$. Now let $* \in \mathrm{Bin}(X)$ be an operation such that:

$$\forall a, b \in X : a * b = x.$$

Now, from $\bullet \in \mathrm{ZBin}(X)$:

$$\begin{gathered}(a * b) \bullet (b * a) = (a \bullet b) * (b \bullet a) \\ x \bullet x = x.\end{gathered}$$

Which is a contradiction. Observing that the same procedure could be done for any $x \in X$, the proposition follows. □

Proposition 4.4. *If* $\bullet \in \mathrm{ZBin}(X)$*, then, for all* $x, y \in X$ *with* $x \neq y$*,* $\{x, y\} = \{x \bullet y, y \bullet x\}$*.*

Proof. Recall that by Zermelo's Theorem any set can be well-ordered, which implies the existence of total order on the set.

Denote by $(X, \leq)$ a set X together with its total order. Thus, we could now construct the following operations $\underline{\circ}\,, \overline{\circ} \in \mathrm{Bin}(X)$:

$$\forall x, y \in X : x \,\underline{\circ}\, y = \min\{x, y\};$$

$$\forall x, y \in X : x \,\overline{\circ}\, y = \max\{x, y\}.$$

Now consider the products $\underline{\circ} \,\square\, \bullet$, $\overline{\circ} \,\square\, \bullet$:

$$(x \,\underline{\circ}\, y) \bullet (y \,\underline{\circ}\, x) = \begin{cases} x \bullet x, & \text{if } x \leq y \\ y \bullet y, & \text{otherwise} \end{cases} = \begin{cases} x, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases} = \min\{x, y\};$$

$$(x \,\overline{\circ}\, y) \bullet (y \,\overline{\circ}\, x) = \begin{cases} y \bullet y, & \text{if } x \leq y \\ x \bullet x, & \text{otherwise} \end{cases} = \begin{cases} y, & \text{if } x \leq y \\ x, & \text{otherwise.} \end{cases} = \max\{x, y\}.$$

As well as $\bullet \,\square\, \underline{\circ}\,, \bullet \,\square\, \overline{\circ}\,$:

$$(x \bullet y) \,\underline{\circ}\, (y \bullet x) = \min\{x \bullet y, y \bullet x\};$$

$$(x \bullet y) \,\overline{\circ}\, (y \bullet x) = \max\{x \bullet y, y \bullet x\}.$$

Note that by the initial assumption $\bullet \in \mathrm{ZBin}(X)$ and therefore the above products should be equal, namely:

$$\min\{x, y\} = \min\{x \bullet y, y \bullet x\};$$
$$\max\{x, y\} = \max\{x \bullet y, y \bullet x\}.$$

which could be shortly rewritten as $\{x, y\} = \{x \bullet y, y \bullet x\}$. $\square$

Proposition 4.5. *If $\bullet \in \mathrm{ZBin}(X)$, then any distinct $x, y \in X$ form either left-zero semigroup or right-zero semigroup with respect to $\bullet$, i.e.*

$$\forall \bullet \in ZBin(X) : \forall x, y \in X : x \neq y : \forall a, b \in \{x, y\} : a \bullet b = \begin{cases} a, & \text{if } \bullet = \circ_{lz} \\ b, & \text{if } \bullet = \circ_{rz} \end{cases}$$

Proof. Consider $\bullet \in \mathrm{ZBin}(X)$. Now, choose $x, y \in X$ such that $x \neq y$. Note that, by **Proposition 4.3**, we shall have:

$$x \bullet x = x$$

$$y \bullet y = y.$$

Keeping that in mind, four different cases for $\bullet$ on $\{x, y\}$ are possible:

$$\begin{array}{c|cc} \bullet_1 & x & y \\ \hline x & x & x \\ y & y & y \end{array} \qquad \begin{array}{c|cc} \bullet_2 & x & y \\ \hline x & x & y \\ y & x & y \end{array} \qquad \begin{array}{c|cc} \bullet_3 & x & y \\ \hline x & x & y \\ y & y & y \end{array} \qquad \begin{array}{c|cc} \bullet_4 & x & y \\ \hline x & x & x \\ y & x & y \end{array}$$

Observe that $\bullet_1 = \circ_{lz}$ and $\bullet_2 = \circ_{rz}$ on $\{x, y\}$. We want to show that $\bullet_3, \bullet_4$ could not be the case for $\bullet \in \mathrm{ZBin}(X)$. Consider the following:

$$\bullet_3 \Box \bullet_4 = (x \bullet_3 y) \bullet_4 (y \bullet_3 x) = y \bullet_4 y = y$$

$$\bullet_4 \Box \bullet_3 = (x \bullet_4 y) \bullet_3 (y \bullet_4 x) = x \bullet_3 x = x$$

$$y \neq x,$$

and hence, $\bullet$ is not $\bullet_3$ or $\bullet_4$, so it is either $\circ_{lz}$ or $\circ_{rz}$. It is easy to see that both $\circ_{lz}$ and $\circ_{rz}$ on $\{x, y\}$ do not violate commutative property with respect to any operation and, therefore, the desired follows. $\square$

We could show the converse.

Proposition 4.6. *If, for two distinct $x, y \in X$ and some $\bullet \in \mathrm{Bin}(X)$, $(\{x, y\}, \bullet)$ forms either left-zero or right-zero semigroup, then $\bullet \in \mathrm{ZBin}(X)$.*

Proof. For some $\triangle \in \mathrm{Bin}(X)$, consider the products:

$$(x \bullet y) \triangle (y \bullet x)$$

$$(x \triangle y) \bullet (y \triangle x)$$

In case $(\{x, y\}, \bullet)$ is a left-zero semigroup, we obtain:

$$(x \bullet y) \triangle (y \bullet x) = x \triangle y$$
$$(x \triangle y) \bullet (y \triangle x) = x \triangle y$$

Analogously, if $(\{x, y\}, \bullet)$ is a right-zero semigroup:

$$(x \bullet y) \triangle (y \bullet x) = y \triangle x$$
$$(x \triangle y) \bullet (y \triangle x) = y \triangle x$$

Which implies $\bullet \in \mathrm{ZBin}(X)$ in both cases. □

Combining the above properties, we construct the member of $\mathrm{ZBin}(X)$.

Definition 4.7. A groupoid $\bullet \in \mathrm{Bin}(X)$ is said to be *locally-zero* if:

(1) $\forall x \in X : x \bullet x = x$.

(2) $\forall x, y \in X : x \neq y : (\{x, y\}, \bullet)$ - left-zero or right-zero semigroup.

Corollary 4.8. *A groupoid $\bullet \in \mathrm{ZBin}(X)$ if and only if it is locally-zero.*

Proof. Follows immediately from **Propositions 4.3 - 4.5**. □

Definition 4.9. Let $\circ \in \mathrm{Bin}(X)$ and let $x, y, z \in X$. Then $(x, y, z)_X$ is the *associative under $\circ$ triplet* if, for any permutation of x, y, z the associative equality holds. That is:

$$\forall a, b, c \in \{x, y, z\} : a \circ (b \circ c) = (a \circ b) \circ c.$$

If we require associativity for locally-zero, we obtain the following result.

Theorem 4.10. *Let $\bullet \in \mathrm{ZBin}(X)$ be a semigroup. Then $\bullet \in \{\circ_{lz}, \circ_{rz}\}$, i.e. it is either left- or right-zero semigroup on X.*

Proof. Recall that $\bullet \in \mathrm{Bin}(X)$ is a semigroup if it is associative on X. Fix now the triplet $(x, y, z)_X$ with $x \neq y \neq z$. Also recall that by **Proposition 4.5**, we got that the operation $\bullet \in \mathrm{ZBin}(X)$ forms either left- or right-zero semigroup on $\{x, y\}$, $\{x, z\}$, $\{y, z\}$. Let us show that the triplet $(x, y, z)_X$ is associative if and only if $\bullet$ is left- or right-zero semigroup on $\{x, y\}$, $\{x, z\}$, $\{y, z\}$ simultaneously. We consider other cases:

1. On $\{x, y\}$, $\bullet = \circ_{lz}$; on $\{x, z\}$, $\bullet = \circ_{lz}$; on $\{y, z\}$, $\bullet = \circ_{rz}$:
$$y \bullet (x \bullet z) = (y \bullet x) \bullet z$$
$$y \neq z;$$

2. On $\{x, y\}$, $\bullet = \circ_{lz}$; on $\{x, z\}$, $\bullet = \circ_{rz}$; on $\{y, z\}$, $\bullet = \circ_{rz}$:
$$y \bullet (z \bullet x) = (y \bullet z) \bullet x$$
$$y \neq x;$$

3. On $\{x, y\}$, $\bullet = \circ_{rz}$; on $\{x, z\}$, $\bullet = \circ_{rz}$; on $\{y, z\}$, $\bullet = \circ_{lz}$:
$$z \bullet (x \bullet y) = (z \bullet x) \bullet y$$
$$z \neq y;$$

4. On $\{x, y\}$, $\bullet = \circ_{rz}$; on $\{x, z\}$, $\bullet = \circ_{lz}$; on $\{y, z\}$, $\bullet = \circ_{lz}$:
$$x \bullet (z \bullet y) = (x \bullet z) \bullet y$$
$$x \neq y;$$

5. On $\{x, y\}$, $\bullet = \circ_{rz}$; on $\{x, z\}$, $\bullet = \circ_{lz}$; on $\{y, z\}$, $\bullet = \circ_{rz}$:
$$x \bullet (y \bullet z) = (x \bullet y) \bullet z$$
$$x \neq z;$$

6. On $\{x, y\}$, $\bullet = \circ_{lz}$; on $\{x, z\}$, $\bullet = \circ_{rz}$; on $\{y, z\}$, $\bullet = \circ_{lz}$:
$$z \bullet (y \bullet x) = (z \bullet y) \bullet x$$
$$z \neq x,$$

which was to be shown. Since the choice of $(x, y, z)_X$ was arbitrary, the theorem follows. $\square$

5 Graph representation of locally-zero operations

In the last section, we introduced the notion of locally-zero operation.

Observation 5.1. *Let X be a non-empty set. If $|X| = n, n \in \mathbb{N}$, then there are $2^{C_n^2}$ locally-zero operations on X.*

Therefore, we could naturally associate an (undirected) graph with locally-zero operation.

Definition 5.2. Let $\circ \in \mathrm{Bin}(X)$ be a locally-zero operation. A graph $G(\circ) = (V, E)$ is defined as follows:

$$\begin{aligned} &V(G(\circ)) = X; \\ &E(G(\circ)) = \{\, \{x, y\} \mid \circ = \circ_{lz} \text{ on } \{x, y\}\, \}. \end{aligned}$$

Remark 5.3. Note that, by **Proposition 4.3**, we got that any locally-zero operation is idempotent. Together with **Definition 5.2**, this implies consequently $\{x, x\} \in E(G(\circ))$ for all $x \in X$. We treat therefore loops on generated graphs as redundant.

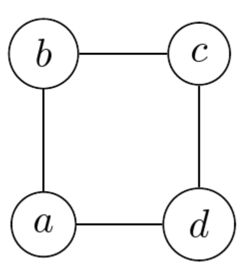
Example 5.4. Define $X = \{a, b, c, d\}$. Consider two locally-zero operations $\circ, \bullet \in \mathrm{Bin}(X)$ with the following Cayley tables:

$\circ$	a	b	c	d
a	a	a	c	a
b	b	b	b	d
c	a	c	c	c
d	d	b	d	d

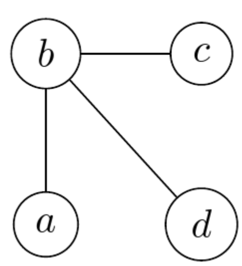
$\bullet$	a	b	c	d
a	a	a	c	d
b	b	b	b	b
c	a	c	c	d
d	a	d	c	d

Their graphs could be illustrated as follows:

$G(\circ)$:



$G(\bullet)$:



By **Proposition 4.2** and **Corollary 4.8**, left- and right-zero semigroups are also locally-zero.

Proposition 5.5. *Let X be a set with $|X| = n < \infty$. For $\circ_{lz} \in \mathrm{Bin}(X)$, the graph $G(\circ_{lz}) = K_n$.*

Proof. Fix $x, y \in X$. We have:

$$x \circ_{lz} y = x,$$

and so $\{x, y\} \in E(G(\circ_{lz}))$. Since the choice of x, y was arbitrary, any two vertices are thus adjacent and the proposition follows. □

Proposition 5.6. *Let X be a set with $|X| = n < \infty$. For $\circ_{rz} \in \mathrm{Bin}(X)$, the graph $G(\circ_{rz}) = L_n$.*

Proof. Fix again $x, y \in X$. We have:

$$x \circ_{rz} y = y \neq x,$$

and hence $\{x, y\} \notin E(\circ_{rz})$, implying that $E(\circ_{rz}) = \emptyset$. □

We consider now graphs generated by the $\Box$-product of locally-zero operations. If we fix two locally-zero operations $\circ, * \in \mathrm{Bin}(X)$, we note that $\circ\Box* = *\Box\circ$, as they are contained in $\mathrm{ZBin}(X)$. Hence, the graphs $G(\circ\Box*)$ and $G(*\Box\circ)$ are the same.

Proposition 5.7. *Let $\circ \in \mathrm{Bin}(X)$ be a locally-zero operation. Then the graph $G(\circ\Box\circ_{rz}) = \overline{G(\circ)}$.*

Proof. For the sake of simplicity, denote $\circ\Box\circ_{lz} = \oslash$. Now fix $x, y \in X$. We show now that if $\{x, y\} \in E(G(\circ))$, then $\{x, y\} \notin E(G(\oslash))$. We obtain:

$$x \oslash y = (x \circ y) \circ_{rz} (y \circ x) = y \circ x = y.$$

Similarly, it follows that $y \oslash x = x$ and thus $\{x, y\} \notin E(G(\oslash))$. Now if $\{x, y\} \notin E(G(\circ))$, it must hold that $\{x, y\} \in E(G(\oslash))$. Note that if $\{x, y\} \notin E(G(\circ))$, then $\circ$ is a right-zero semigroup on $\{x, y\}$ by **Proposition 4.5**. Therefore:

$$x \oslash y = (x \circ y) \circ_{rz} (y \circ x) = y \circ x = x,$$

as well as $y \oslash x = y$. It leads us to a conclusion that $\oslash = \circ_{lz}$ on $\{x, y\}$. Thus, $\{x, y\} \in E(G(\oslash))$ and the proposition follows. □

Remark 5.8. Note that by **Theorem 2.3** the equality $G(\circ\Box\circ_{lz}) = G(\circ)$ is fairly obvious for any $\circ \in \mathrm{ZBin}(X)$.

In turn, the above facts could be generalized for a pair of arbitrary locally-zero operations.

Theorem 5.9. *Let $\circ, * \in \mathrm{Bin}(X)$ be locally-zero operations. Then the graph $G(\circ\Box*) = \overline{G(\circ) \triangle G(*)}$.*

Proof. Denote $\circ\Box* = \oslash$. Consider the product:

$$\forall x, y \in X : x \oslash y = (x \circ y) * (y \circ x) = \begin{cases} x, & \text{if } \circ = \circ_{lz}, * = \circ_{lz} \\ x, & \text{if } \circ = \circ_{rz}, * = \circ_{rz} \\ y, & \text{if } \circ = \circ_{lz}, * = \circ_{rz} \\ y, & \text{if } \circ = \circ_{rz}, * = \circ_{lz}. \end{cases}$$

We conclude, on the one hand, $\{x, y\} \in E(G(\oslash))$ if it is in both $E(G(\circ))$ and $E(G(*))$. On the other hand, $\{x, y\} \in E(G(\oslash))$ if it is neither in $E(G(\circ))$ nor in $E(G(*))$. It could be rewritten as:

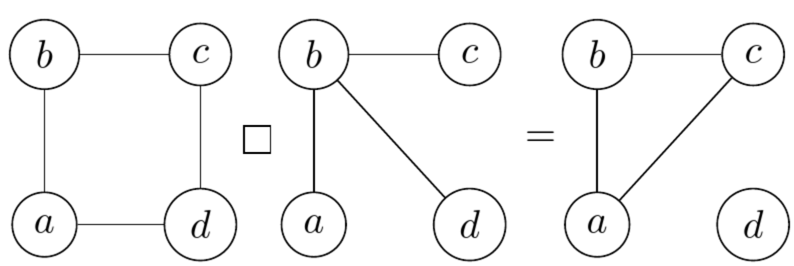
$$G(\oslash) = (G(\circ) \cap G(*)) \cup (\overline{G(\circ)} \cap \overline{G(*)}).$$

In compliance with **Definition 1.31**, one could easily identify a complement to a symmetric difference. So it follows:

$$G(\circ\Box*) = G(\oslash) = \overline{G(\circ) \triangle G(*)}.$$

$\Box$

Example 5.10. Recall the operations $\circ, \bullet \in \mathrm{Bin}(X)$ presented in **Example 5.4**. If we $\Box$-multiply them, we obtain the following:



Summary

Section 1 provides an overview of important facts in different areas of mathematical research. In **Subsection 1.1**, we recall the way to construct an algebraic structure, namely groupoid, semigroup, and monoid. In **Subsection 1.2**, Zermelo's Theorem along with its prerequisite set-theoretical framework are stated, as these find themselves essential further on in the study. A brief introduction to Graph Theory is available in **Subsection 1.3**.

We introduce $\mathrm{Bin}(X)$ and $\square$-product in **Section 2**. We show that they are a semigroup and a monoid together by proving associativity and finding an identity, respectively.

Distributivity from the left is considered in **Section 3**. We introduce the notion of classes of operations that are left-distributive over the fixed operation and vice versa. We refer to these classes as in- and out-neighbourhoods of an operation and define them for some notable operations. We describe furthermore the intersection of all in- and out-neighbourhoods for the elements of $\mathrm{Bin}(X)$.

In **Section 4**, we characterize the center of $\mathrm{Bin}(X)$ with respect to $\square$-product. The necessary conditions for an element of the center are proposed throughout a range of propositions. This includes idempotency, closure on any two-element subset of X, and more. The operation from the center is then called locally-zero.

We proceed with locally-zero operations in **Section 5**. One could associate an undirected simple graph with them. We find out which locally-zero operations generate complete and empty graphs of order coinciding with the cardinality of X. We propose a characterization of graphs generated by $\square$-product of locally-zero operations and describe these graphs for special locally-zeros.

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