Discrete Mathematics 346 (2023) 113614

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Unique eccentric point graphs and their eccentric digraphs

Artem Hak, Vladyslav Haponenko, Sergiy Kozerenko *, Andrii Serdiuk

Faculty of Computer Sciences, National University of Kyiv-Mohyla Academy, Skovorody str. 2, 04070 Kyiv, Ukraine

A R T I C L E I N F O

Article history:
Received 19 January 2023
Received in revised form 4 July 2023
Accepted 11 July 2023
Available online 25 July 2023

Keywords:
Unique eccentric point graph
Eccentric digraph
Eccentricity of a vertex
Block graph

A B S T R A C T

We study graph-theoretic properties of eccentric digraphs of unique eccentric point graphs (shortly, uep-graphs). The latter are the connected graphs in which every vertex has a unique eccentric vertex. In particular, we characterize uep-graphs and the corresponding eccentric digraphs in the following classes: self-centered graphs having the number of vertices twice as diameter, block graphs, and graphs with diameter three. Also, we obtain non-trivial properties of weak components in eccentric digraphs of uep-graphs with diameter four and pose several open questions in this direction.

1. Introduction

The eccentricity of a given vertex u in a connected graph G is the maximum distance from it to the other vertices in G. Any vertex v which attains this distance is called an eccentric vertex for u in G. Since the relation "being an eccentric vertex for" is not necessarily symmetric, it is naturally encoded by a directed graph on the same vertex set – the eccentric digraph $\mathrm{ED}(G)$ (see [1–3,11]).

One of the main problems in studying eccentric digraphs is to describe the structure of $\mathrm{ED}(G)$ for some concrete graph classes (this is referred to as "Open problem 1" in [1]). Moving in the opposite direction, the following question arises: if instead we pose some conditions on $\mathrm{ED}(G)$, can we extract some properties of the corresponding graphs G? In this paper, we address these two broad problems for the unique eccentric point graphs (or simply, uep-graphs). These are the connected graphs in which every vertex has a unique eccentric vertex [10]. It is clear that G is a uep-graph if and only if each vertex in $\mathrm{ED}(G)$ has an out-degree one (these are the so-called functional digraphs). The structure of (finite) functional digraphs is well-known [7, Theorem 16.5]: each of their weak components is an orientation of some unicyclic pseudograph H with the edges on the unique cycle C (which as well can be a loop at some vertex or a pair of parallel edges between two vertices) of H being oriented cyclically and other edges being oriented towards C. However, not every functional digraph is an eccentric digraph of some uep-graph. Moreover, describing eccentric digraphs of uep-graphs up to isomorphism seems to be a hard problem.

In this work, we study the structural properties of eccentric digraphs for general uep-graphs. In particular, we completely characterize uep-graphs and their respective eccentric digraphs in several graph classes.

The paper is organized as follows. In Section 2, we give main definitions and assemble all the preliminary results about uep-graphs from [10], which will be used throughout this paper. In Section 3.1, we present our results starting with basic properties of eccentric digraphs of uep-graphs (including the characterization of self-centered uep-graphs having the number of vertices twice as its diameter, see Proposition 3.5). Section 3.2 deals with uep block graphs. In particular, we extend the

* Corresponding author.
E-mail address: kozerenkosergiy@ukr.net (S. Kozerenko).

https://doi.org/10.1016/j.disc.2023.113614

characterization of uep trees from [10] to uep block graphs (Theorem 3.8) and, as a corollary, completely describe the structure of their eccentric digraphs (Proposition 3.9). In Section 3.3, we provide a characterization of uep-graphs with diameter three (Theorem 3.10) and describe their eccentric digraphs as well (Proposition 3.13). Section 3.4 contains results about the eccentric digraphs of uep-graphs with diameter four (Theorem 3.14). In Section 4, we pose several open questions concerning the structure of weak components in eccentric digraphs of uep-graphs.

We note that several results of this paper (namely, Proposition 3.1, Theorem 3.8 and Proposition 3.9) were announced at Xth All-Ukrainian Conference of Young Scientists in Physics and Mathematics [5].

2. Main definitions and preliminary results

2.1. Undirected graphs

An *undirected graph* or just a *graph* is an ordered pair $G=(V,E)$, where $V=V(G)$ is the set of its *vertices* and $E=E(G)$ is the set of its *edges* (which are some 2-element subsets of V). In this paper, all the considered graphs are finite. Also, for a pair of vertices $u,v\in V$ the edge $\{u,v\}$ will be shortly denoted as uv.

As usual, by K_n, $K_{m,n}$, and C_n we denote the n-vertex complete graph, the complete bipartite graph having parts of cardinalities m,n, and the n-cycle, respectively.

Two graphs G and H are called *isomorphic* if there is an *isomorphism* between them, i.e. a bijection $f:V(G)\to V(H)$ such that $uv\in E(G)$ if and only if $f(u)f(v)\in E(H)$. If G and H are isomorphic, then we write $G\simeq H$.

The *complement* of a graph G is the graph $\overline{G}$ having $V(\overline{G})=V(G)$ and $E(\overline{G})=\{uv:u\neq v \text{ and } uv\notin E(G)\}$. The *union* of graphs G,H is the graph $G\cup H$ with $V(G\cup H)=V(G)\sqcup V(H)$ and $E(G\cup H)=E(G)\cup E(H)$. For a graph G and a number $m\in\mathbb{N}$, we write mG for the union of m isomorphic copies of G. For a set of vertices $A\subset V(G)$, by $G[A]$ we denote the subgraph of G induced by A. Also, we put $G-A=G[V(G)\backslash A]$ and $G-u=G-\{u\}$ for any vertex $u\in V(G)$.

The *neighborhood* of a vertex u in a graph G is the set $N_G(u)=\{v\in V(G):uv\in E(G)\}$. The *closed neighborhood* of u in G is the set $N_G[u]=N_G(u)\cup\{u\}$. The degree of u is the number $d_G(u)=|N_G(u)|$. A vertex $u\in V(G)$ is called a *leaf vertex* provided $d_G(u)=1$.

A set of vertices $A\subset V(G)$ is *dominating* provided for every $u\in V(G)\backslash A$ there is $a\in A$ with $au\in E(G)$.

A graph is called *connected* if there is a path between each pair of its vertices (otherwise, it is *disconnected*). A *connected component* of a graph is its maximal connected subgraph. The vertex set of a connected graph G is equipped with the standard metric d_G, where $d_G(u,v)$ equals the length (i.e. the number of edges) of a shortest path between u and v in G. For a vertex $u\in V(G)$ and a set $A\subset V(G)$ in a connected graph G, we put $d_G(u,A)=\min\{d_G(u,a):a\in A\}$.

For a pair of vertices $u,v\in V(G)$ in a connected graph G, we define the *metric interval* between them as the set $[u,v]_G=\{x\in V(G):d_G(u,x)+d_G(x,v)=d_G(u,v)\}$.

The *eccentricity* of a vertex u in a connected graph G is the number $\mathrm{ecc}_G(u)=\max\{d_G(u,v):v\in V(G)\}$. A vertex $v\in V(G)$ is called an *eccentric vertex for* u in G provided $\mathrm{ecc}_G(u)=d_G(u,v)$. A vertex v is an *eccentric vertex* in G if it is an eccentric vertex for some $u\in V(G)$. The *radius of a graph* G is the value $\mathrm{rad}(G)=\min\{\mathrm{ecc}_G(u):u\in V(G)\}$ and the *diameter of* G is $\mathrm{diam}(G)=\max\{\mathrm{ecc}_G(u):u\in V(G)\}$. It is clear that $\mathrm{diam}(G)=\max\{d_G(u,v):u,v\in V(G)\}$. A pair of vertices is *diametral* in G provided $d_G(u,v)=\mathrm{diam}(G)$.

The *center* of a connected graph G is the set of its vertices whose eccentricities equal $\mathrm{rad}(G)$. The *periphery* of G is the set of vertices having their eccentricities equal $\mathrm{diam}(G)$. A graph G is called *self-centered* provided its center (equivalently, periphery) equals the whole vertex set $V(G)$.

A connected graph without cycles is called a *tree*. A *path* P_n is a tree with n vertices that has at most two leaf vertices. A *star* $K_{1,n-1}$ is a tree with n vertices that has at most one non-leaf vertex. A *bi-star* is a tree that has exactly two non-leaf vertices.

A vertex in a (finite) graph is called a *cut vertex* if its deletion increases the number of connected components. Hence, for a connected graph G, a vertex $u\in V(G)$ is a cut vertex in G if and only if $G-u$ is disconnected. A connected graph is called 2-*connected* provided it has no cut vertices. A *block* in a graph is its maximal 2-connected subgraph. A graph is called a *block graph* if every its block is a complete subgraph. For example, any tree is a block graph.

The next fundamental result about the center of a graph will be used in the characterization of uep block graphs (see Theorem 3.8).

Proposition 2.1. [6] *The center of a connected graph lies in a block.*

2.2. Directed graphs

A *directed graph* or, shortly, a *digraph* is an ordered pair $D=(V,A)$, where $V=V(D)$ is the set of its *vertices* and $A=A(D)\subset V\times V$ is the set of its *arcs*. The existence of an arc $(u,v)\in A(D)$ will be also denoted as $u\to v$ in D. An arc of the form (u,u) is called a *loop* at vertex u. The *out-degree* $d^+_D(u)$ of a vertex $u\in V(D)$ is the number of arcs of the form $u\to v$, $v\in V(D)$. Similarly, the *in-degree* $d^-_D(u)$ of u is the number of arcs of the form $v\to u$, $v\in V(D)$. The *out-neighborhood* of u is the set of vertices $N^+_D(u)=\{v\in V(D):u\to v\}$. And the *in-neighborhood* of u is the set $N^-_D(u)=\{v\in V(D):v\to u\}$.

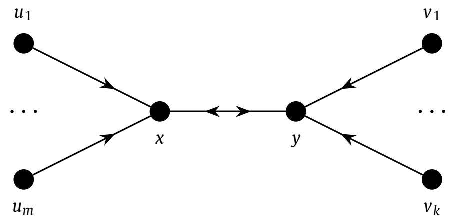

Fig. 1. The functional digraph $D_{m,k}$.

Clearly, $d^+_D(u) = |N^+_D(u)|$ and $d^-_D(u) = |N^-_D(u)|$ for all $u \in V(D)$. Two vertices $u, v \in V(D)$ are called *adjacent* provided $u \to v$ or $v \to u$ in D.

Two digraphs D_1 and D_2 are called *isomorphic* if there is an *isomorphism* between them, i.e. a bijection $f : V(D_1) \to V(D_2)$ such that $u \to v$ in D_1 if and only if $f(u) \to f(v)$ in D_2. The latter will be denoted by $D_1 \simeq D_2$.

A digraph D is called *weakly connected* provided the corresponding undirected graph (which is obtained from D by ignoring orientations, multiple edges and loops) is connected. A maximal weakly connected subgraph of D is called its *weak component*.

A *path* in a digraph D is the ordered set of vertices $u_1, \dots, u_m$ such that $u_i \to u_{i+1}$ for all $1 \le i \le m-1$. A path is called *simple* provided its vertices (and hence, arcs) are pairwise different. A simple path $u_1, \dots, u_m$ is called *induced* if the $u_i \to u_j$ in D implies $j = i+1$.

An *m-cycle* in a digraph D is an ordered set of m different vertices $u_1, \dots, u_m$, where $u_i \to u_{i+1}$ and $u_m \to u_1$ in D. It is clear that a 1-cycle is just a loop. A 2-cycle frequently will be denoted just as $u_1 \leftrightarrow u_2$.

A digraph D is called *functional* provided $d^+_D(u) = 1$ for every $u \in V(D)$. It is clear that functional digraphs having vertex set V are in one-to-one correspondence with functions of the form $f : V \to V$. To describe the structure of functional digraphs, we need one more definition. An *in-tree* is a digraph D obtained from an (undirected) tree X by orienting each edge in X towards some fixed vertex u (more formally, $V(T) = V(X)$ and $A(T) = \{(x, y) : xy \in E(X) \text{ and } y \in [x, u]_X\}$). The corresponding vertex u is the *root* of an in-tree T. Graph-theoretic structure of finite functional digraphs can be described pretty easily. Namely, every weak component D' of a functional digraph D contains a unique cycle C such that each weak component in $D' - A(C)$ is an in-tree T, and $V(C)$ contains the set of roots of these in-trees T.

Another type of functional digraphs, which will appear many times in this paper, is constructed as follows. For a pair of non-negative integers $m, k \in \mathbb{Z}_+$, we define the digraph $D_{m,k}$ to have the vertex set $V(D_{m,k}) = \{x, y, u_1, \dots, u_m, v_1, \dots, v_k\}$ and the arc set

$$A(D_{m,k}) = \{(x, y), (y, x)\} \cup \{(u_i, x), (v_j, y) : 1 \le i \le m, 1 \le j \le k\}$$

(see Fig. 1). For example, $D_{0,0}$ is just a 2-cycle.

Let G be a connected graph. The *eccentric digraph* [2] of G is the digraph $\mathrm{ED}(G)$ with $V(\mathrm{ED}(G)) = V(G)$ and

$$A(\mathrm{ED}(G)) = \{(u, v) : v \text{ is an eccentric vertex for } u \text{ in } G\}.$$

More on eccentric digraphs of general connected graphs (and even disconnected digraphs) as well as several of their classes can be found in [1,3].

2.3. Unique eccentric point graphs

A connected graph G is called a *unique eccentric point* or just a *uep-graph* provided its eccentric digraph $\mathrm{ED}(G)$ is functional. In other words, G is a uep-graph if each of its vertices has a unique eccentric vertex in G. It is clear that the only uep-graph G with $\mathrm{diam}(G) = 1$ is K_2. Uep-graphs with diameter two also can be easily characterized.

Theorem 2.2. [10] *A connected graph G is a uep-graph with* $\mathrm{diam}(G) = 2$ *if and only if* $\overline{G} \simeq mK_2$.

It was also proved in [10] that each uep-graph G with $\mathrm{diam}(G) = 3$ is either self-centered or *upper-diameter critical* (these are connected graphs with the property that the addition of any new edge decreases the diameter). In this paper, we generalize the latter result by giving a complete characterization of non-self-centered uep-graphs G with $\mathrm{diam}(G) = 3$ (see Theorem 3.10).

We note that the class of uep-graphs is a highly non-trivial one. For example, even the self-centered uep-graphs (which are also known as *even graphs* [4] or *diametral graphs* [9]) are very interesting in themselves. This class of uep-graphs contains several natural subclasses such as balanced, harmonic, and symmetric even graphs (again, see [4]). However, in some restricted graph classes uep-graphs can be nicely characterized.

Theorem 2.3. [10] *A tree T with $n \ge 2$ vertices is a uep-graph if and only if T has exactly two central and two peripheral vertices.*

The next lemma is a simple technical result that will be used extensively throughout this paper.

Lemma 2.4. [10] *Let $uv \in E(G)$ be an edge in a uep-graph G. If $\mathrm{ecc}_G(u) \neq \mathrm{ecc}_G(v)$, then u and v have the same eccentric vertex in G.*

The following characterization of self-centered uep-graphs was obtained also in [10].

Theorem 2.5. [10] *A uep-graph is self-centered if and only if each its vertex is eccentric.*

In the next section, we show that self-centered uep-graphs can be also characterized in terms of their eccentric digraphs (see Corollary 3.2).

It also can be easily shown that a uep-graph cannot have a diameter twice its radius. This simple result will be used in our characterization of uep block graphs (see Theorem 3.8).

Proposition 2.6. [10] *For any uep-graph G, it holds* $\mathrm{diam}(G) \leq 2\,\mathrm{rad}(G) - 1$.

3. Main results

3.1. Eccentric digraphs of general uep-graphs

As we know from the structure of functional digraphs, for a uep-graph G, each weak component in $\mathrm{ED}(G)$ consists of a unique cycle C and some in-trees directed to C. The following observation is the starting point in the study of eccentric digraphs of uep-graphs.

Proposition 3.1. *The eccentric digraph of a uep-graph with $n \geq 2$ vertices has cycles only of length two.*

Proof. Let G be a uep-graph with $n \geq 2$ vertices and $u_1 \to \cdots \to u_m \to u_1$ be a cycle in $\mathrm{ED}(G)$. Then $\mathrm{ecc}_G(u_1) = d_G(u_1, u_2) \leq \mathrm{ecc}_G(u_2) = d_G(u_2, u_3) \leq \cdots \leq \mathrm{ecc}_G(u_m) = d_G(u_m, u_1) \leq \mathrm{ecc}_G(u_1)$. Hence, $\mathrm{ecc}_G(u_1) = \cdots = \mathrm{ecc}_G(u_m)$. In particular, $\mathrm{ecc}_G(u_1) = d_G(u_1, u_2) = d_G(u_m, u_1)$ implying that $u_2 = u_m$. Thus $m \leq 2$. But since $n \geq 2$, we have $m = 2$. □

Proposition 3.1 asserts the following criterion for self-centered uep-graphs in terms of their eccentric digraphs.

Corollary 3.2. *Let G be a uep-graph with $n \geq 2$ vertices. Then G is self-centered if and only if each weak component in $\mathrm{ED}(G)$ is isomorphic to $D_{0,0}$.*

Proof. Necessity. In a self-centered graph, if a vertex u is an eccentric vertex of a vertex v, then v must also be an eccentric vertex of u. Now, if G is a self-centered uep-graph, then both vertices of every diametral pair in G must have out-degree 1 and in-degree 1 in $\mathrm{ED}(G)$. Thus, each weak component D' in $\mathrm{ED}(G)$ is a cycle. From Proposition 3.1 we obtain that each such D' is a 2-cycle. Hence, $D' \simeq D_{0,0}$.

Sufficiency. If each weak component in $\mathrm{ED}(G)$ is isomorphic to $D_{0,0}$, then clearly every vertex from G is eccentric. Thus, by Theorem 2.5, G is self-centered. □

In what follows, we will consider three types of weak components in eccentric digraphs. Namely, a weak component D' in $ED(G)$ for a uep-graph G is called

- *bald*, if $D' \simeq D_{0,0}$;
- *half-bald*, if exactly one of the two vertices on a 2-cycle in D' has in-degree one;
- *full*, if D' is neither bald nor half-bald.

Proposition 3.3. *If an eccentric digraph of a uep-graph with $n \geq 3$ vertices has a non-full weak component, then it has at least two weak components.*

Proof. Let G be a uep-graph with $n \geq 3$ vertices and x, y be a diametral pair in G. Since $n \geq 3$, G is not complete implying that $\mathrm{diam}(G) \geq 2$. Denote by D' the weak component in $\mathrm{ED}(G)$, which contains x, y. Now fix a vertex $u \in [x, y]_G \cap N_G(x)$. We have $\mathrm{ecc}_G(u) \geq d_G(u, y) = d_G(x, y) - 1 = \mathrm{diam}(G) - 1$. If $\mathrm{ecc}_G(u) = \mathrm{diam}(G)$, then $u \notin V(D')$ (as peripheral vertices lie on cycles in the eccentric digraph and there is exactly one 2-cycle per weak component in $\mathrm{ED}(G)$), hence $ED(G)$ has at least two weak components. Otherwise, $\mathrm{ecc}_G(u) = \mathrm{diam}(G) - 1$ and there is an arc $u \to y$ in $ED(G)$. Similarly, consider a vertex $v \in [x, y]_G \cap N_G(y)$. If $\mathrm{ecc}_G(v) = \mathrm{diam}(G)$, then $ED(G)$ has at least two weak components. If $\mathrm{ecc}_G(v) = \mathrm{diam}(G) - 1$, then $v \to x$ in $ED(G)$. In the latter case, D' is a full weak component in $ED(G)$. Therefore, if $ED(G)$ has a non-full weak component, then it has at least two weak components. □

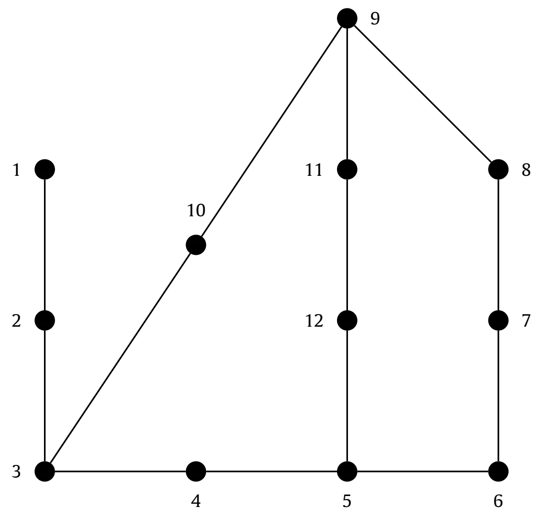

Fig. 2. Uep-graph G.

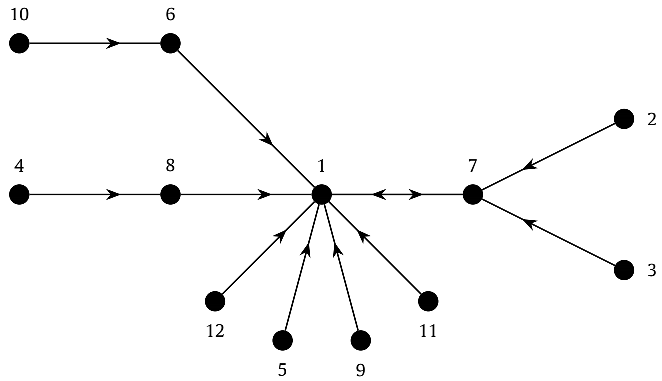

Fig. 3. The eccentric digraph $\mathrm{ED}(G)$ for the uep-graph G from Fig. 2.

Note that full weak components in the eccentric digraph of a uep-graph can contain induced paths of lengths larger than one. Indeed, Fig. 2 depicts a uep-graph G whose eccentric digraph $\mathrm{ED}(G)$ has induced paths of length two (see Fig. 3). However, the lengths of such induced paths are bounded in terms of the diameter of a graph.

Proposition 3.4. *Let G be a uep-graph with $\mathrm{diam}(G) \geq 4$. Then the length of a longest induced path in $\mathrm{ED}(G)$ is at most $\left\lfloor \frac{\mathrm{diam}(G)}{2} \right\rfloor - 1$.*

Proof. Let $u_0 \to u_1 \to \cdots \to u_m$ be any longest induced path in $\mathrm{ED}(G)$. Then $(u_{i+1}, u_i) \notin A(\mathrm{ED}(G))$ for all $0 \leq i \leq m-1$. Also, by Proposition 3.1, the last vertex u_m on the path lies on a 2-cycle in $\mathrm{ED}(G)$, say $u_m \leftrightarrow v$. Since G is a uep-graph, $\mathrm{ecc}_G(u_{i+1}) > \mathrm{ecc}_G(u_i)$ for all $0 \leq i \leq m-1$. Hence, $\mathrm{ecc}_G(u_0) \leq \mathrm{ecc}_G(u_m) - m$.

For $m = 1$ the statement clearly holds. Suppose that $m \geq 2$. In this case, we have $ecc_G(u_m) = d_G(u_m, v) \leq d_G(u_m, u_0) + d_G(u_0, v) \leq 2\,\mathrm{ecc}_G(u_0) - 2 \leq 2(\mathrm{ecc}_G(u_m) - m) - 2$ implying that $m \leq \left\lfloor \frac{\mathrm{ecc}_G(u_m)}{2} \right\rfloor - 1 \leq \left\lfloor \frac{\mathrm{diam}(G)}{2} \right\rfloor - 1$. $\square$

We also note that the bound from Proposition 3.4 is tight. Indeed, the graph G from Fig. 2 has $\mathrm{diam}(G) = 6$ and the lengths of two largest induced paths in $\mathrm{ED}(G)$ are equal $\left\lfloor \frac{\mathrm{diam}(G)}{2} \right\rfloor - 1 = 2$. Trivially, for uep-graphs G with $\mathrm{diam}(G) = 3$ the length of a longest induced path in $\mathrm{ED}(G)$ is at most one.

As we know from Corollary 3.2, a uep-graph G is self-centered if and only if each weak component in $\mathrm{ED}(G)$ is bald. It is clear that such a graph G has an even number of vertices. Moreover, we have the following result.

Proposition 3.5. *Every self-centered uep-graph G has at least $2\,\mathrm{diam}(G)$ vertices. Moreover, the equality $|V(G)| = 2\,\mathrm{diam}(G)$ holds if and only if $G \simeq K_2$ or $G \simeq C_m$ for an even $m \geq 4$.*

Proof. Fix an arbitrary diametral pair $x, y \in V(G)$ and some shortest path $x - u_1 - \cdots - u_{\mathrm{diam}(G)-1} - y$ between x and y in G. For every $1 \le i \le \mathrm{diam}(G)-1$ fix an eccentric vertex v_i for u_i in G. It is clear that, $v_i \ne x, y, u_j$ for all $1 \le j \le \mathrm{diam}(G)-1$ (as each vertex in a self-centered graph is peripheral). Hence, $|V(G)| \ge 2\,\mathrm{diam}(G)$.

Now let us prove the second statement of the proposition.

Sufficiency. If $G \simeq K_2$, then the assertion clearly holds. Similarly, if $G \simeq C_m$ for an even $m \ge 4$, then G is a symmetric even graph (and hence, a self-centered uep-graph) with $\mathrm{diam}(G) = \frac{m}{2}$.

Necessity. If $\mathrm{diam}(G) = 1$, then G is complete and hence, $G \simeq K_2$. Further, assume $d = \mathrm{diam}(G) \ge 2$. From Corollary 3.2 it follows that $\mathrm{ED}(G)$ has exactly d weak components each being bald. We want to show that every vertex in G has degree two. To the contrary, assume that there is a vertex $a_1 \in V(G)$ with $d_G(a_1) \ne 2$. Let $a_1 \leftrightarrow b_1$ be the corresponding weak component in $\mathrm{ED}(G)$. Fix a shortest path $a_1 - a_2 - \cdots - a_d - b_1$ between a_1 and b_1 in G. For any $1 \le i \le d$ by b_i denote the eccentric vertex for a_i in G (hence, $a_i \leftrightarrow b_i$, $1 \le i \le d$ are the weak components in $\mathrm{ED}(G)$).

If $d_G(a_1) = 1$, then $\mathrm{ecc}_G(a_1) = \mathrm{ecc}_G(a_2) + 1$ as $d \ge 2$. Hence, in this case G cannot be self-centered.

Now let $d_G(a_1) \ge 3$. In this case, fix two different neighbors $b_i, b_j \in N_G(a_1) \backslash \{a_2\}$. Let $i < j$. Then there is a path $b_i - a_1 - \cdots - a_i$ between b_i and a_i of length $1 + i - 1 = i < j \le d$, which is a contradiction.

Therefore, G is a (finite) connected graph with $d_G(u) = 2$ for all $u \in V(G)$. Hence, $G \simeq C_m$ for the even number $m = 2\,\mathrm{diam}(G)$. □

We note that there are 2 self-centered uep-graphs with 6 vertices (namely, $K_6 - 3K_2$ and C_6), and exactly 3 such graphs having 8 vertices (namely, $K_8 - 4K_2$, C_8, $K_{4,4} - 4K_2$). A computer search showed that the number of these graphs with 10 vertices is 24.

3.2. Uep block graphs

The similarity between block graphs and trees in the context of uep-graphs shows up directly in the criterion of uep trees from [10]. Moreover, it turns out that the statement of Theorem 2.3 can be extended to connected block graphs. To present this result, we need the next useful metric characterization of block graphs as well as one technical lemma that follows after.

Theorem 3.6. [8] *A connected graph G is a block graph if and only if its metric d_G satisfies the "4-point condition": for any $x, y, z, t \in V(G)$ it holds*

$$d_G(x,y) + d_G(z,t) \le \max\{d_G(x,z) + d_G(y,t), d_G(x,t) + d_G(y,z)\}.$$

We note that trees are precisely triangle-free graphs that satisfy the 4-point condition.

Lemma 3.7. *In a connected block graph each eccentric vertex is peripheral.*

Proof. Let G be a connected block graph and v be an eccentric vertex for some vertex u in G. Fix a diametral pair x, y in G and use Theorem 3.6 for the vertices u, v, x, y:

$$\begin{aligned}\mathrm{ecc}_G(u) + \mathrm{diam}(G) &= d_G(u,v) + d_G(x,y) \\ &\le \max\{d_G(u,x) + d_G(v,y), d_G(u,y) + d_G(v,x)\}.\end{aligned}$$

If $d_G(u,x) + d_G(v,y) \le d_G(u,y) + d_G(v,x)$, then

$$\mathrm{ecc}_G(u) + \mathrm{diam}(G) \le d_G(u,y) + d_G(v,x) \le \mathrm{ecc}_G(u) + \mathrm{diam}(G)$$

implying that $\mathrm{ecc}_G(u) = d_G(u,y)$ and $d_G(v,x) = \mathrm{diam}(G)$. The case $d_G(u,x) + d_G(v,y) \ge d_G(u,y) + d_G(v,x)$ is considered similarly (here $d_G(v,y) = \mathrm{diam}(G)$). Hence, v is a peripheral vertex in G. □

Now we are ready to present the main result of this subsection.

Theorem 3.8. *A connected block graph with $n \ge 2$ vertices is a uep-graph if and only if it has exactly two central and two peripheral vertices.*

Proof. Necessity. Let G be a uep block graph with $n \ge 2$ vertices. If $\mathrm{rad}(G) = 1$, then by Proposition 2.6, $\mathrm{diam}(G) \le 1$. In this case, G is a complete uep-graph, hence $G \simeq K_2$. Hence, let $\mathrm{rad}(G) \ge 2$. To show that G contains exactly two peripheral vertices, we assume that x, y and a, b are two different diametral pairs in G. Since G is a uep-graph, we have $\{x,y\} \cap \{a,b\} = \emptyset$. Using Theorem 3.6 for the vertices a, b, x, y, we obtain

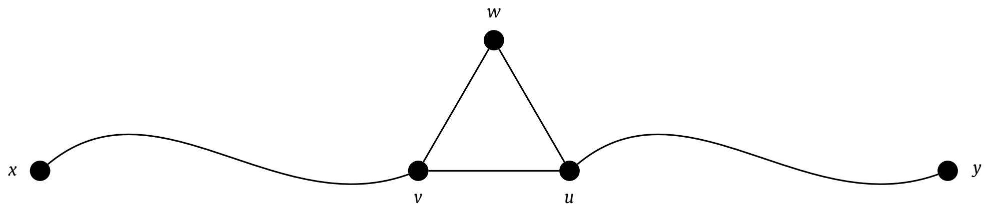

Fig. 4. Three central vertices u, v, w lie in a common block.

$$2\,\mathrm{diam}(G) = d_G(a,b) + d_G(x,y) \le \max\{d_G(a,x) + d_G(b,y), d_G(a,y) + d_G(b,x)\}.$$

If $d_G(a,x) + d_G(b,y) \ge d_G(a,y) + d_G(b,x)$, then $2\,\mathrm{diam}(G) \le d_G(a,x) + d_G(b,y)$. Hence, $d_G(a,x) = \mathrm{diam}(G)$. Therefore, a and y are two different eccentric vertices for x in G. Similarly, the inequality $d_G(a,x)+d_G(b,y) \le d_G(a,y)+d_G(b,x)$ would imply $d_G(b,x) = \mathrm{diam}(G)$. The obtained contradiction shows that G has exactly two peripheral vertices, say x, y. By Lemma 3.7, x and y are the only eccentric vertices in G.

Further, consider a central vertex $u \in V(G)$. Without loss of generality, let x be its eccentric vertex in G. Also, fix some vertex $v \in [u,x]_G \cap N_G(u)$. We have $d_G(x,u) = \mathrm{rad}(G)$ and $d_G(x,v) = \mathrm{rad}(G) - 1$. Hence, y is the eccentric vertex for v in G, implying $d_G(y,v) = \mathrm{ecc}_G(v) \ge \mathrm{rad}(G)$. On the other hand, $d_G(y,v) \le d_G(y,u) + 1 \le \mathrm{rad}(G) - 1 + 1 = \mathrm{rad}(G)$. Therefore, $d_G(y,v) = \mathrm{rad}(G)$. Similarly, $d_G(y,u) = \mathrm{rad}(G) - 1$. Thus, G contains at least two central vertices, namely u, v.

Assume that there exists another central vertex $w \in V(G)\backslash\{u,v\}$. Combining Proposition 2.1 with the definition of a block graph, we conclude that $wu, wv \in E(G)$ (see Fig. 4). Without loss of generality, suppose that x is the eccentric vertex for w in G. Then $d_G(w,x) = \mathrm{rad}(G)$ and $d_G(w,y) = \mathrm{rad}(G) - 1$ (as w is adjacent to the vertex v having $d_G(v,y) = \mathrm{rad}(G)$).

Fix a vertex $t \in [w,y]_G \cap N_G(w)$. We have $d_G(t,y) = \mathrm{rad}(G) - 2$. Clearly, $t \ne u, v$ as $d_G(u,y) = \mathrm{rad}(G) - 1$ and $d_G(v,y) = \mathrm{rad}(G)$. If $ut \in E(G)$, then the vertices u, v, w, t induce a 2-connected subgraph in G, implying that v and t lie in a common block in G. In this case, $vt \in E(G)$ and, therefore, $d_G(v,y) \le d_G(v,t) + d_G(t,y) = 1 + \mathrm{rad}(G) - 2 = \mathrm{rad}(G) - 1$, which is a contradiction. Thus, $d_G(u,t) = 2$ (as there is a path $u - w - t$ in G). Now we use Theorem 3.6 for the vertices u, t, w, y in order to obtain a contradiction:

$$\begin{aligned}1 + \mathrm{rad}(G) &= d_G(u,t) + d_G(w,y)\\ &\le \max\{d_G(u,w) + d_G(t,y), d_G(u,y) + d_G(t,w)\}\\ &= \max\{1 + \mathrm{rad}(G) - 2, \mathrm{rad}(G) - 1 + 1\} = \mathrm{rad}(G).\end{aligned}$$

This means that G contains exactly two central vertices.

Sufficiency. Let x, y be the two peripheral vertices and u be some central vertex in G. Using Lemma 3.7, we can assume that x is an eccentric vertex for u in G. As in the proof of **Necessity**, fix a vertex $v \in [u,x]_G \cap N_G(u)$. In a similar way, we can prove that v is a central vertex and y is the unique eccentric vertex for v in G. Further, we use Theorem 3.6 for the vertices x, u, y, v:

$$\begin{aligned}2\,\mathrm{rad}(G) &= d_G(x,u) + d_G(y,v) \le \max\{d_G(x,y) + d_G(u,v), d_G(x,v) + d_G(u,y)\}\\ &= \max\{\mathrm{diam}(G) + 1, \mathrm{rad}(G) - 1 + d_G(u,y)\}\\ &\le \max\{\mathrm{diam}(G) + 1, 2\,\mathrm{rad}(G) - 1\}.\end{aligned}$$

Hence, $2\,\mathrm{rad}(G) \le \mathrm{diam}(G) + 1$. Combining this inequality with Proposition 2.6, we obtain $\mathrm{diam}(G) = 2\,\mathrm{rad}(G) - 1$. Further, fixing the vertex $w \in [v,y]_G \cap N_G(v)$, we can prove that w is a central vertex in G implying that $w = u$.

Now let $z \in V(G)$ be an arbitrary vertex in G. By Lemma 3.7, x and y are the only candidates for the eccentric vertices of z in G. We want to prove that $d_G(z,x) \ne d_G(z,y)$ (thus proving that z has a unique eccentric vertex in G). To the contrary, suppose $d_G(z,x) = d_G(z,y)$. Let z_0 be such a vertex z with minimal distance $d_G(z,\{u,v\})$. It is clear that $d_G(z_0,\{u,v\}) \ge 1$ (as $d_G(z_0,\{u,v\}) = 0$ would imply $z_0 = u$ or $z_0 = v$, and this case is already covered). Fix a vertex $t \in N_G(z_0)$ with $d_G(t,\{u,v\}) = d_G(z_0,\{u,v\}) - 1$. The minimality of $d_G(z_0,\{u,v\})$ asserts $d_G(t,x) \ne d_G(t,y)$. Without loss of generality, we can assume that $d_G(t,x) > d_G(t,y)$. Use Theorem 3.6 for the vertices z, y, t, x:

$$\begin{aligned}d_G(z_0,y) + d_G(t,x) &\le \max\{d_G(z_0,t) + d_G(x,y), d_G(z_0,x) + d_G(t,y)\}\\ &= \max\{1 + \mathrm{diam}(G), d_G(z_0,y) + d_G(t,y)\}\\ &= \max\{2\,\mathrm{rad}(G), d_G(z_0,y) + d_G(t,y)\},\end{aligned}$$

implying that $d_G(z_0,y) + d_G(t,x) \le 2\,\mathrm{rad}(G)$. Hence, $d_G(z_0,y) \le 2\,\mathrm{rad}(G) - d_G(t,x) \le 2\,\mathrm{rad}(G) - \mathrm{rad}(G) = \mathrm{rad}(G)$. Hence, $d_G(z_0,y) = d_G(z_0,x) = \mathrm{rad}(G)$. However, G has exactly two central vertices, u and v. Since $z_0 \ne u, v$, we obtain a contradiction. This means that G is a uep-graph. $\square$

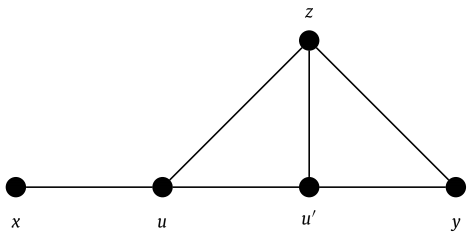

Fig. 5. Non-complete 2-connected subgraph $G[\{u, u', y, z\}]$.

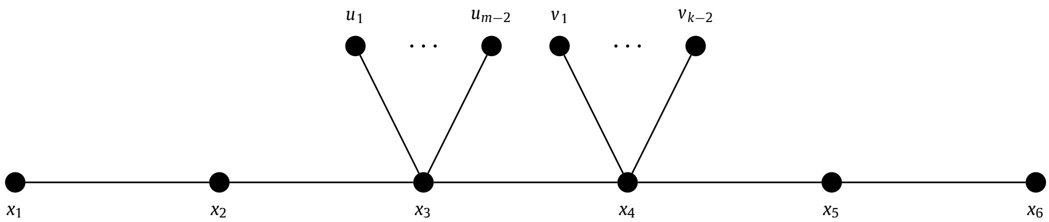

Fig. 6. The tree T with $\mathrm{ED}(T) \simeq D_{m,k}$ for $m, k \geq 2$.

Using Theorem 3.8, we can completely describe eccentric digraphs of uep block graphs.

Proposition 3.9. *Let G be a uep block graph with $n \geq 2$ vertices. Then $\mathrm{ED}(G) \simeq D_{m,k}$ for $m = k = 0$ or $m = k = 1$ or $m, k \geq 2$. Conversely, for every such a pair of integers m, k there exists a uep block graph (even a tree) with $\mathrm{ED}(G) \simeq D_{m,k}$.*

Proof. Combining Lemma 3.7 and Theorem 3.8, we can conclude that G has exactly two peripheral vertices which are the only eccentric vertices in G. Hence, $\mathrm{ED}(G) \simeq D_{m,k}$ for some $m, k \geq 0$. If G has two vertices, then $G \simeq K_2$ and $\mathrm{ED}(G) \simeq D_{0,0}$. Now let $n \geq 3$. Then $m + k \neq 0$ and G is not complete, implying $\mathrm{diam}(G) \geq 2$. Let $x, y \in V(G)$ be a diametral pair in G. Fix two vertices $u \in [x, y]_G \cap N_G(x)$ and $v \in [x, y]_G \cap N_G(y)$. It is clear that $\mathrm{ecc}_G(u) = \mathrm{ecc}_G(v) = \mathrm{diam}(G) - 1$ and $u \to y$, $v \to x$ in $\mathrm{ED}(G)$. This means that $m, k \geq 1$.

Finally, assume that $d^-_{\mathrm{ED}(G)}(y) = 2$ (similarly, we can consider the case $d^-_{\mathrm{ED}(G)}(x) = 2$). Again, fix a vertex $u \in [x, y]_G \cap N_G(x)$ and a vertex $u' \in [x, y]_G$ with $d_G(x, u') = 2$. If $u' = y$, then $\mathrm{diam}(G) = 2$ and $\mathrm{ecc}_G(u) = 1 = d_G(u, x) = d_G(u, y)$, which is a contradiction. Thus, $u' \neq y$. Since $x \to y$, $u \to y$ in $\mathrm{ED}(G)$ and $d^-_{\mathrm{ED}(G)}(y) = 2$, it holds $u' \to x$ (because we already proved that x, y are the only eccentric vertices in G). Hence, $d_G(u', y) < \mathrm{ecc}_G(u') = d_G(u', x) = 2$ which asserts $d_G(u', y) = 1$. Therefore, $\mathrm{diam}(G) = d_G(x, y) = 3$. If $n = 4$, then $G \simeq P_4$ and $\mathrm{ED}(G) \simeq D_{1,1}$. Otherwise, there is $z \in V(G) \backslash \{x, u, u', y\}$. Using the connectedness of G, we can assume that $N_G(z) \cap \{x, u, u', y\} \neq \emptyset$. It is clear that $\mathrm{ecc}_G(z) = 2$. Without loss of generality, assume that x is the eccentric vertex for z in G. Then $N_G(z) = V(G) \backslash \{x\}$. But in this case the vertices u, u', y, z induce a non-complete 2-connected subgraph in G (see Fig. 5). The obtained contradiction shows that $m = k = 0$ or $m = k = 1$ or $m, k \geq 2$.

Now let the pair of non-negative integers m, k satisfy one of the three conditions above. If $m = k = 0$, then clearly $\mathrm{ED}(P_2) \simeq D_{0,0}$. Similarly, for $m = k = 1$ we have $\mathrm{ED}(P_4) \simeq D_{1,1}$. Finally, let $m, k \geq 2$. Put

$$V(T) = \{x_1, \ldots, x_6\} \cup \{u_1, \ldots, u_{m-2}\} \cup \{v_1, \ldots, v_{k-2}\},$$
$$E(T) = \{x_i x_{i+1} : 1 \leq i \leq 5\} \cup \{u_j x_3 : 1 \leq j \leq m-2\} \cup \{v_l x_4 : 1 \leq l \leq k-2\}.$$

It is easy to see that T is a tree with $\mathrm{diam}(T) = 5$ having a unique diametral pair x_1, x_6 (see Fig. 6). Further, $\mathrm{ecc}_T(x_3) = \mathrm{ecc}_T(x_4) = 3$ and $\mathrm{ecc}_T(w) = 4$ for all $w \in V(T) \backslash \{x_1, x_3, x_4, x_6\}$. Moreover, x is the eccentric vertex for vertices x_4, x_5, x_6 and $v_1, \ldots, v_{k-2}$. Similarly, y is the eccentric vertex for vertices x_1, x_2, x_3 and $u_1, \ldots, u_{m-2}$. Hence, $\mathrm{ED}(T) \simeq D_{m,k}$. □

3.3. *Uep-graphs with diameter three*

Recall that uep-graphs G having $\mathrm{diam}(G) = 2$ are exactly the complete graphs minus a perfect matching (see Theorem 2.2). It turns out that non-self-centered uep-graphs with diameter three also admit nice characterization.

Theorem 3.10. *Let G be a connected graph, which is not self-centered. Then G is a uep-graph with $\mathrm{diam}(G) = 3$ if and only if its complement $\overline{G}$ is a bi-star.*

Proof. Necessity. Let x, y be a diametral pair in G. It is clear that $N_G[x] \cap N_G[y] = \emptyset$. Since G is not self-centered, there exists a vertex $u \in V(G)$ with $\mathrm{ecc}_G(u) = 2$. If $x, y \in N_G(u)$, then $d_G(x, y) \leq 2$, which is a contradiction. Without loss of generality, assume $uy \notin E(G)$. Then $d_G(u, y) \geq 2 = \mathrm{ecc}_G(u)$ implying $d_G(u, y) = 2$. Thus, y is the eccentric vertex for u in G. This means that $N_G(u) = V(G)\backslash\{y\}$.

Now fix a vertex $v \in N_G(u) \cap N_G(y)$. Since $N_G(u) = V(G)\backslash\{y\}$, it holds $\mathrm{ecc}_G(v) \leq 2$. However, $d_G(v, x) = 2$, which means that $\mathrm{ecc}_G(v) = 2$ and hence x is the eccentric vertex for v in G. Therefore, $N_G(v) = V(G)\backslash\{x\}$. Further, since for any $z \in V(G)\backslash\{x, u, v, y\}$ we have $uz, vz \in E(G)$, then $\mathrm{ecc}_G(z) \leq 2$. However, $\mathrm{diam}(G) = 3$ again implies that $\mathrm{ecc}_G(z) = 2$ for all $z \in V(G)\backslash\{x, u, v, y\}$. If $z \notin N_G(x) \cup N_G(y)$, then $d_G(z, x) = d_G(z, y) = 2 = \mathrm{ecc}_G(z)$ contradicting the fact that G is a uep-graph. Hence, the set $\{x, y\}$ is dominating in G.

Finally, for all $z \in V(G)\backslash\{x, y\}$ we have $z \in N_G(x) \triangle N_G(y)$ (as $z \in N_G(x) \cap N_G(y)$ would imply $d_G(x, y) \leq 2$). Hence, the eccentric vertex for any such z is either x or y. This asserts that $V(G)\backslash\{x, y\} \subset N_G(z)$. In other words, the subgraph $G - \{x, y\}$ is complete in G. Therefore, the complement $\overline{G}$ is a bi-star having two central vertices x, y, the set of leaf vertices $N_G(x) \cup N_G(y)$, and $N_{\overline{G}}(x) = N_G[y]$, $N_{\overline{G}}(y) = N_G[x]$.

Sufficiency. Let $\overline{G}$ be a bi-star with two central vertices x and y. Then $N_G[x] \cap N_G[y] = \emptyset$, $\{x, y\}$ is a dominating set in G, and the subgraph $G - \{x, y\}$ is complete in G.

Since $N_G[x] \cap N_G[y] = \emptyset$, it holds $d_G(x, y) \geq 3$. However, $G - \{x, y\}$ is a complete subgraph in G implying that $d_G(x, y) = 3$. Further, fix two vertices $u \in N_G(x)$ and $v \in N_G(y)$. Clearly, $u, v \in V(G)\backslash\{x, y\}$. For any $z \in V(G)\backslash\{x, y\}$ it holds $d_G(x, z) \leq d_G(x, u) + d_G(u, z) \leq 2$. Similarly, $d_G(y, z) \leq 2$ as well. Therefore, $\mathrm{diam}(G) = 3$ and x, y is the unique diametral pair in G. Obviously, $\mathrm{ecc}_G(z) = 2$ for all $z \in V(G)\backslash\{x, y\}$. Hence, for all such z either x or y is the eccentric vertex for z in G. Finally, x, y cannot be eccentric vertices for some z simultaneously as $\{x, y\}$ is a dominating set in G. Thus, G is a uep-graph. □

Corollary 3.11. [10] *A uep-graph of diameter three is either self-centered or upper-diameter critical.*

Proof. If G is a non-self-centered uep-graph with $\mathrm{diam}(G) = 3$, then Theorem 3.10 asserts that $\overline{G}$ is a bi-star. Hence, any new edge $e \notin E(G)$ is incident to at least one of the peripheral vertices of G. Clearly, for any $e \notin E(G)$ the graph $G + e$ has diameter of two. □

Corollary 3.12. *The number of non-isomorphic non-self-centered n-vertex uep-graphs G with* $\mathrm{diam}(G) = 3$ *equals* $\lfloor \frac{n}{2} \rfloor - 1$.

Proof. We can calculate the number of non-isomorphic complements of such graphs G instead. By Theorem 3.10, the complement $\overline{G}$ is a bi-star. And the number of non-isomorphic n-vertex bi-stars equals $\lfloor \frac{n}{2} \rfloor - 1$ (as any such bi-star is given by a non-trivial partition of $(n-2)$-element set into two parts). □

Using Corollary 3.2 and Theorem 3.10, we can completely characterize eccentric digraphs of uep-graphs with diameter three. To do so, we define an auxiliary unary graph operation named *eccentric cloning*. Let H be a connected graph. Take an isomorphic copy H' of H with $V(H') = \{u' : u \in V(H)\}$ and $E(H') = \{u'v' : uv \in E(H)\}$. Now consider the graph G which is obtained from the union $H \cup H'$ by adding new edges of the form uv' provided there is an arc between u and v in $\mathrm{ED}(H)$. The graph G is called *eccentric clone* of H. For example, the eccentric clone of K_n is isomorphic to $K_{2n} - nK_2$ (the $2n$-vertex complete graph minus a perfect matching) and the eccentric clone of C_4 is isomorphic to the 3-cube Q_3.

Proposition 3.13. *For a digraph D there exists a uep-graph G with* $\mathrm{diam}(G) = 3$ *having* $\mathrm{ED}(G) \simeq D$ *if and only if D consists of $l \geq 3$ bald components, or $D \simeq D_{m,k}$ for $m, k \geq 1$.*

Proof. Necessity. If G is self-centered, then Corollary 3.2 asserts that each weak component in $\mathrm{ED}(G)$ is bald. Also, it is clear that $\mathrm{ED}(G)$ has $\frac{|V(G)|}{2} \geq \mathrm{diam}(G) = 3$ weak components. Now let G be non-self-centered. From the proof of Theorem 3.10 it follows that G has a unique diametral pair of vertices x, y, which are the only eccentric vertices in G. Hence, $\mathrm{ED}(G) \simeq D_{m,k}$ for $m, k \geq 0$ and $m + k > 0$. Combining this fact with Proposition 3.3 (as the condition $\mathrm{diam}(G) = 3$ implies $|V(G)| \geq 3$), we conclude that $m, k \geq 1$.

Sufficiency. At first, assume that D consists of $l \geq 3$ bald components. If $l = 3$, then $\mathrm{ED}(C_6) \simeq D$. Further, suppose $l \geq 4$. Consider the graph $H \simeq K_{2,l-2}$ having $V(H) = \{x, y\} \cup \{a_i : 1 \leq i \leq l-2\}$ and $E(H) = \{xa_i, ya_i : 1 \leq i \leq l-2\}$. Let G be the eccentric clone of H (see Fig. 7 for the eccentric clone of $K_{2,3}$). It is easy to observe that $\mathrm{ecc}_G(x) = d_G(x, x') = 3$. Indeed, $d_G(x, a_i) = d_G(x, y') = 1$, $d_G(x, y) = 2$, $d_G(x, a_i') = 2$ (as there are paths $x - y' - a_i'$ and $a_i \notin N_G(x)$), $1 \leq i \leq l-2$, and $d_G(x, x') = 3$ (as there is a path $x - y' - a_1' - x'$ between x and x' in G; and $N_G(x) \cap N_G(x') = (\{a_i : 1 \leq i \leq l-2\} \cup \{y'\}) \cap (\{a_i' : 1 \leq i \leq l-2\} \cup \{y\}) = \emptyset$). Similarly, $\mathrm{ecc}_G(x') = \mathrm{ecc}_G(y) = \mathrm{ecc}_G(y') = 3$. Further, for any $1 \leq i \leq l-2$ we have $\mathrm{ecc}_G(a_i) = d_G(a_i, a_i') = 3$ also. Indeed, $d_G(a_i, x) = d_G(a_i, y) = d_G(a_i, a_j') = 1$ for all $1 \leq j \leq l-2$, $j \neq i$. Furthermore, $d_G(a_i, a_j) = d_G(a_i, x') = d_G(a_i, y') = 2$ for all $1 \leq j \leq l-2$, $j \neq i$. Finally, $d_G(a_i, a_i') = 3$. The same arguments show that $\mathrm{ecc}_G(a_i') = 3$ for all $1 \leq i \leq l-2$. Hence, G is self-centered. Moreover, G is a uep-graph as $\mathrm{ED}(G)$ is the union of the following 2-cycles: $x \leftrightarrow x'$, $y \leftrightarrow y'$, and $a_i \leftrightarrow a_i'$ for $1 \leq i \leq l-2$. Clearly, $\mathrm{ED}(G) \simeq D$ as it has exactly l bald weak components.

Now let $D \simeq D_{m,k}$ for $m, k \geq 1$. Consider the complete graph $H \simeq K_{m+k}$. Fix a partition of $V(H) = A \sqcup B$ with $|A| = m$, $|B| = k$. Now add to H two new vertices x, y with the new edges xa for all $a \in A$ and yb for all $b \in B$ (see Fig. 8). Denote the obtained graph by G. It is clear that $\overline{G}$ is a bi-star. By Theorem 3.10, G is a uep-graph with $\mathrm{diam}(G) = 3$. From the construction of G it follows that $\mathrm{ED}(G) \simeq D_{m,k}$. □

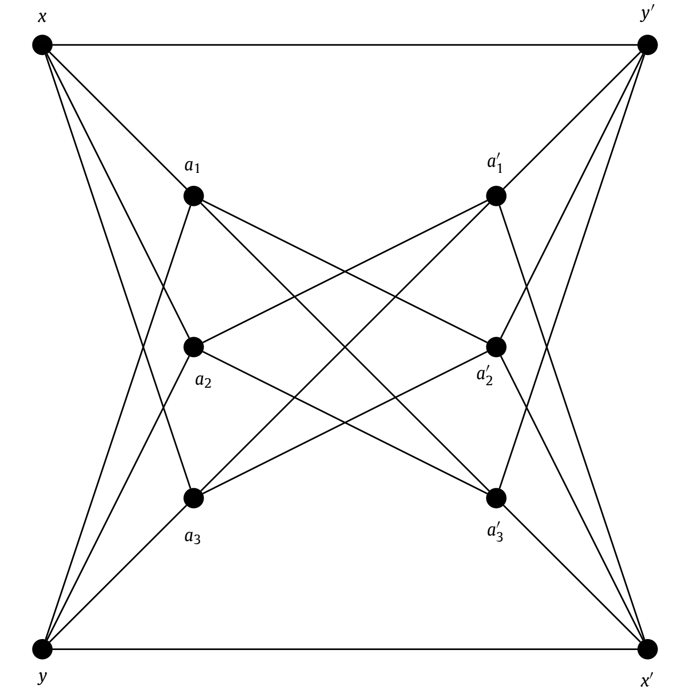

Fig. 7. The eccentric clone of $K_{2,3}$.

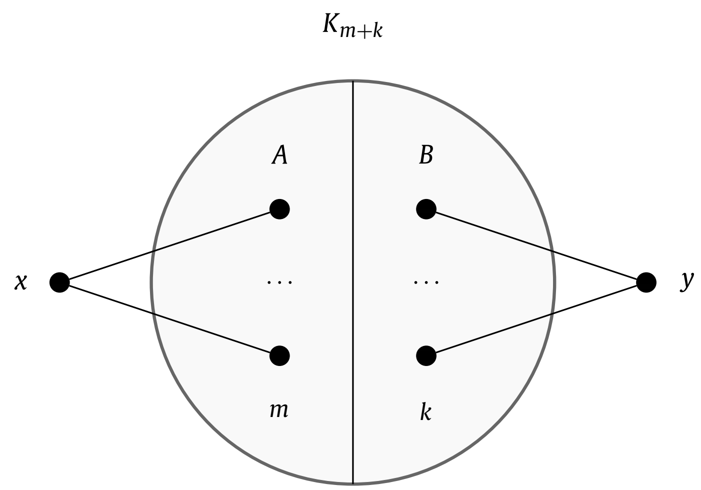

Fig. 8. Non-self-centered uep graph G with $\operatorname{diam}(G) = 3$.

3.4. Uep-graphs with diameter four

The problems of characterizing non-self-centered uep-graphs G having $\operatorname{diam}(G) = 4$ or even obtaining criteria for their eccentric digraphs are considerably harder. The following theorem contains several important results in the direction of tackling the second problem.

Theorem 3.14. *Let G be a non-self-centered uep-graph with* $\operatorname{diam}(G) = 4$. *Then the next statements hold:*

(i) each eccentric vertex in G lies on a cycle in $\operatorname{ED}(G)$*;*
(ii) if $\operatorname{ED}(G)$ *contains a 2-cycle $x \leftrightarrow y$, then x, y induce a bald weak component in* $\operatorname{ED}(G)$ *if and only if $d_G(x, y) = 3$;*
(iii) $\operatorname{ED}(G)$ *does not have half-bald weak components.*

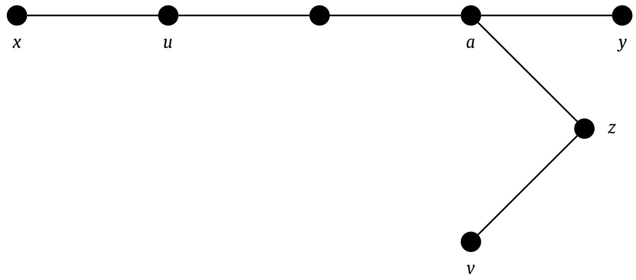

Fig. 9. Two shortest paths between x, y and a, v in G from **Case 1**.

Proof. (i) The first statement directly follows from Proposition 3.4. Indeed, $\operatorname{diam}(G) = 4$ implies that $\mathrm{ED}(G)$ does not have induced paths of length of at least two. Hence, each eccentric vertex in G lies on a cycle in $\mathrm{ED}(G)$.

Before proving the second and the third statements, we note that $\operatorname{ecc}_G(u) \in \{3, 4\}$ for all vertices $u \in V(G)$. Indeed, Proposition 2.6 implies $\operatorname{rad}(G) \geq \frac{\operatorname{diam}(G)+1}{2} = \frac{4+1}{2} = \frac{5}{2}$.

Now we are ready to prove the second statement.

(ii) **Necessity.** Assume that x, y induce a bald weak component in $\mathrm{ED}(G)$. To the contrary, suppose x, y is a diametral pair in G. Since G is not self-centered, there exists a vertex $u \in V(G)$ with $\operatorname{ecc}_G(u) = 3$. Let a be the eccentric vertex for u in G. Clearly, $a \neq x, y$. We have $d_G(u, x) \leq 2$ and $d_G(u, y) \leq 2$. However, if $\min\{d_G(u, x), d_G(u, y)\} = 1$, then using the triangle inequality, we can deduce $d_G(x, y) \leq 3$. The obtained contradiction shows that $d_G(u, x) = d_G(u, y) = 2$. Now fix a vertex $z \in N_G(x) \cap N_G(u)$. If $\operatorname{ecc}_G(z) = 3$, then using Lemma 2.4 for the edge zx, we can conclude that y is the eccentric vertex for z in G. However, this contradicts the fact that x, y induce a bald weak component in $\mathrm{ED}(G)$. Hence, $\operatorname{ecc}_G(z) = 4$. Similarly, we use Lemma 2.4 for the edge zu to conclude that z, a is a diametral pair in G. Further, we fix a vertex $t \in N_G(y) \cap N_G(u)$. Clearly, $z \neq t$. We apply the same argument for t. In other words, if $\operatorname{ecc}_G(t) = 3$, then x is the eccentric vertex for t in G, which is a contradiction. And if $\operatorname{ecc}_G(t) = 4$, then a, t is a diametral pair in G. This contradicts the fact that G is a uep-graph (as the vertex a has two eccentric vertices z and t).

Sufficiency. The fact that a 2-cycle $x \leftrightarrow y$ with $d_G(x, y) = 3$ induces a weak bald component in $\mathrm{ED}(G)$ directly follows from the observation that in such a uep-graph G for all vertices $u \in V(G)$ we have $\operatorname{ecc}_G(u) \in \{3, 4\}$. Thus, x can be an eccentric vertex only for y and vice versa.

Now we prove the third (and the hardest) statement.

(iii) Aiming for a contradiction, assume that there exists a half-bald weak component in $\mathrm{ED}(G)$. Let $x \leftrightarrow y$ be the 2-cycle in it. From the second statement it follows that x, y is a diametral pair in G. Without loss of generality, assume that $d^-_{\mathrm{ED}(G)}(y) = 1$. Hence, $d^-_{\mathrm{ED}(G)}(x) \geq 2$. Fix a vertex $a \in N^-_{\mathrm{ED}(G)}(x) \backslash \{y\}$. It is clear that $\operatorname{ecc}_G(a) = 3$. Furthermore, we can assume that $a \in [x, y]_G$. Indeed, if $ay \notin E(G)$, then $d_G(a, y) = 2$. Consider any vertex $a' \in N_G(a) \cap N_G(y)$. If $\operatorname{ecc}_G(a') = 4$, then using Lemma 2.4 for the edge aa', we obtain that x is the eccentric vertex for a' and y in G, which is a contradiction. Hence, $\operatorname{ecc}_G(a') = 3$ and, applying Lemma 2.4 for the edge ay, we again obtain that x is the eccentric vertex for a' in G. In the latter case, $a' \in [x, y]_G$. Therefore, let $a \in [x, y]_G$ and fix a vertex $u \in [x, y]_G \cap N_G(x)$ with $d_G(u, a) = 2$. It is clear that $\operatorname{ecc}_G(u) = 4$. Let v be the eccentric vertex for u in G. Thus, $u \leftrightarrow v$ is another 2-cycle in $\mathrm{ED}(G)$. We have $d_G(a, v) = 2$ (indeed, the existence of an edge $av \in E(G)$ would imply $d_G(u, v) \leq d_G(u, a) + d_G(a, v) = 3$). Fix a vertex $z \in N_G(a) \cap N_G(v)$. Further, we split the proof into two cases.

Case 1: $z \neq y$ (see Fig. 9).

If $\operatorname{ecc}_G(z) = 4$, then applying Lemma 2.4 for the edge az, we obtain that $z \leftrightarrow x$ in $\mathrm{ED}(G)$, which is a contradiction. Thus, in this case, $\operatorname{ecc}_G(z) = 3$. Again, applying Lemma 2.4 for the edge zv, we obtain $z \to u$ in $\mathrm{ED}(G)$. Hence, $d_G(z, x) = 2$ (if $zx \in E(G)$, then $x - z - a - y$ is the path of length 3 between x and y in G). Further, fix a vertex $w \in N_G(z) \cap N_G(x)$. If $w = v$, then we obtain a shorter path $u - x - w = v$ of length 2 between u and v in G. Hence, $w \neq v$ (see Fig. 10). If $\operatorname{ecc}_G(w) = 3$, then applying Lemma 2.4 for the edge xw, we obtain $w \to y$ in $\mathrm{ED}(G)$. This contradiction asserts that $\operatorname{ecc}_G(w) = 4$. But in this case, apply Lemma 2.4 for the edge wz in order to ensure the existence of the 2-cycle $u \leftrightarrow w$ in $\mathrm{ED}(G)$. Since $w \neq v$, this is a contradiction again.

Case 2: $z = y$.

We have $d_G(x, v) \leq 3$. If $d_G(x, v) \leq 2$, then $d_G(u, v) \leq d_G(u, x) + d_G(x, v) \leq 3$. This means that $d_G(x, v) = 3$. Fix a shortest path $x - p - q - v$ between x and v in G. Since $N^-_{\mathrm{ED}(G)}(y) = \{x\}$, we can conclude that $\operatorname{ecc}_G(p) = 4$. Similarly, using Lemma 2.4 for the edges pq and qv, we obtain that $\operatorname{ecc}_G(q) = 4$. Let $q \leftrightarrow q'$ in $\mathrm{ED}(G)$.

Further, since $\operatorname{ecc}_G(a) = 3$ and $a \to x$ in $\mathrm{ED}(G)$, it holds $d_G(a, q) \leq 2$. If $aq \in E(G)$, then by Lemma 2.4, there is an arc $q \to x$ in $\mathrm{ED}(G)$, which is clearly not the case as $d_G(x, q) = 2$. Therefore, $d_G(a, q) = 2$. As usual, fix a vertex $t \in N_G(a) \cap N_G(q)$. Clearly, $t \neq y$ (otherwise, we obtain a contradiction: $d_G(x, y) \leq d_G(x, q) + d_G(q, y) = 3$), see Fig. 11.

Again, since $\operatorname{ecc}_G(a) = 3$, the equality $\operatorname{ecc}_G(t) = 4$ would lead us to the following contradiction: there would exist the 2-cycle $t \leftrightarrow x$ in $\mathrm{ED}(G)$ with $t \neq y$. Hence, $\operatorname{ecc}_G(t) = 3$. Therefore, $t \to q'$ in $\mathrm{ED}(G)$. Moreover, $d_G(t, x) \leq 2$. If $tx \in E(G)$, then $d_G(x, y) \leq d_G(x, t) + d_G(t, y) = 3$, which is a standard contradiction. Thus, $d_G(t, x) = 2$ and we can fix a vertex $w \in N_G(x) \cap N_G(t)$ (see Fig. 12).

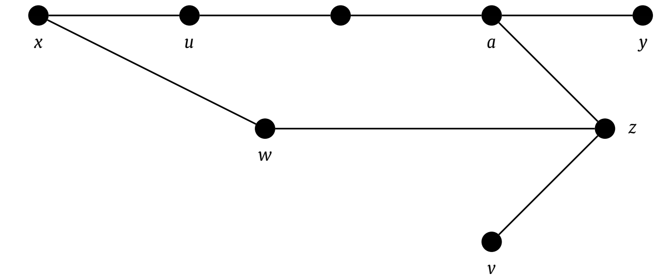

Fig. 10. The existence of a vertex $w \in N_G(z) \cap N_G(x)$ from **Case 1**.

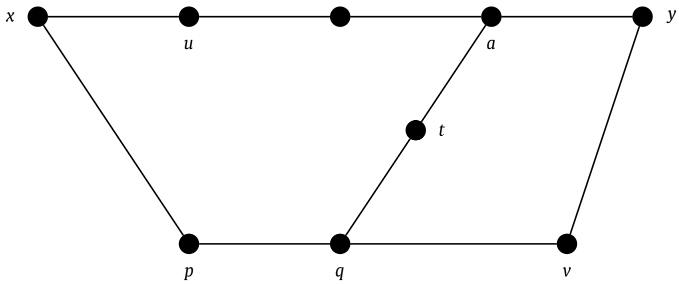

Fig. 11. The existence of a vertex $t \in N_G(a) \cap N_G(q)$ from **Case 2**.

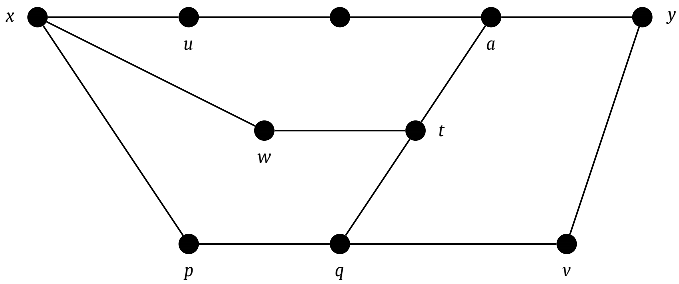

Fig. 12. The existence of a vertex $w \in N_G(x) \cap N_G(t)$ from **Case 2**.

Again, since $N^-_{\mathrm{ED}(G)}(y) = \{x\}$, it holds $\mathrm{ecc}_G(w) = 4$. Therefore, using Lemma 2.4 for the edge wt, we obtain that $w \leftrightarrow q'$ in $\mathrm{ED}(G)$. However, $w \neq q$ as $wx \in E(G)$ and $qx \notin E(G)$. This final contradiction proves the theorem. $\square$

From Theorem 3.14 we can conclude the following facts about the structure of $\mathrm{ED}(G)$ for uep-graphs G with $\mathrm{diam}(G) = 4$:

1. each its weak component is isomorphic to $D_{m,k}$ for $m = k = 0$ or $m, k \geq 1$;
2. the distance in G between two vertices in its bald weak component equals three;
3. the distance in G between two vertices in its full weak component equals four.

Example 3.15. Consider the graph G shown in Fig. 13. One can check that G is a uep-graph with $\mathrm{diam}(G) = 4$. Fig. 14 depicts its eccentric digraph $\mathrm{ED}(G)$, which has two weak full components and one bald weak component.

A direct computer search showed that there are no n-vertex uep-graphs with diameter four for $n \leq 7$. For $n = 8$ there are 4 such graphs, for $n = 9$ the number of these graphs is 16, for $n = 10$ we have 261 such graphs. Finally, for $n = 11$ we have found exactly 4829 of these graphs.

We conclude this section by presenting a uep-graph with diameter five whose eccentric digraph has a half-bald weak component (see Fig. 15 and Fig. 16).

4. Open questions

In this section we present several open questions about the structure of eccentric digraphs for uep-graphs based on the obtained results.

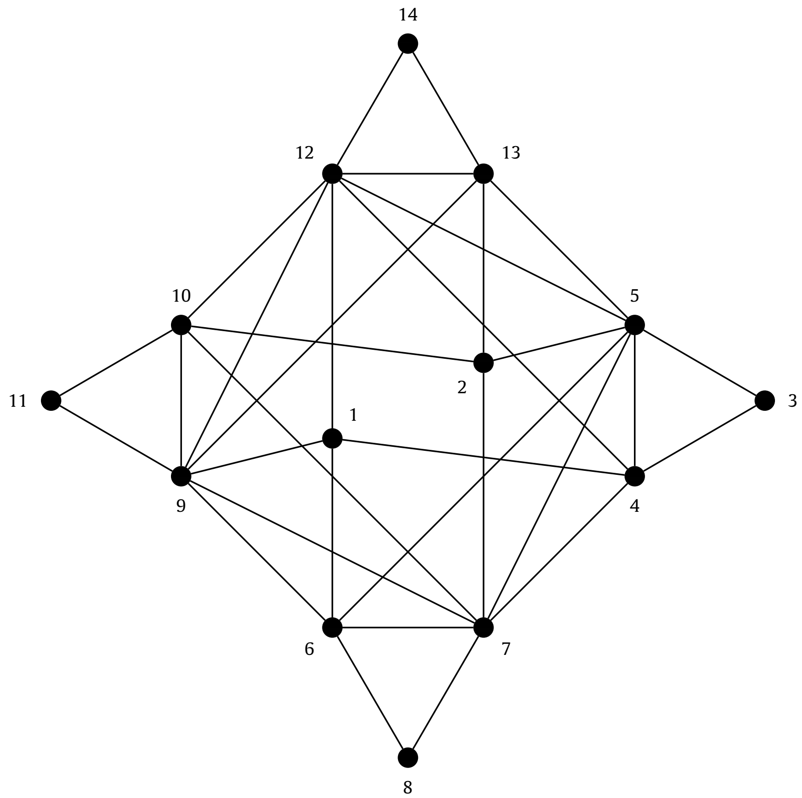

Fig. 13. The uep-graph G with $\operatorname{diam}(G) = 4$ from Example 3.15.

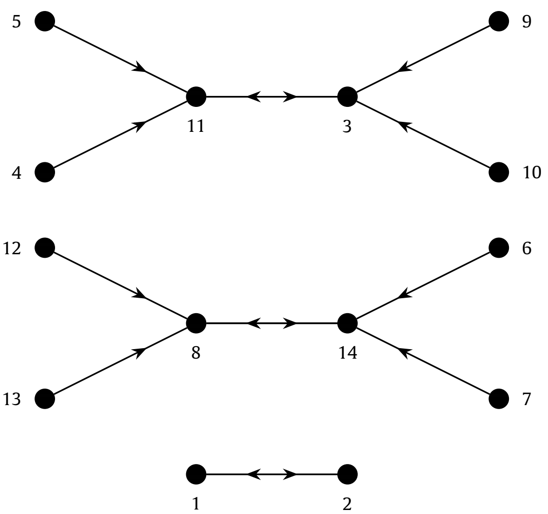

Fig. 14. The eccentric digraph $\operatorname{ED}(G)$ for the graph G from Fig. 13.

Question 1. Can we characterize weakly connected eccentric digraphs of uep-graphs?

This question is not trivial since such eccentric digraphs can contain induced paths of length of at least two (see Fig. 3). Also, note that by Proposition 3.3, any such eccentric digraph is a bald or a full weak component in itself.

The next series of questions concerns the existence and the structure of half-bald weak components.

Question 2. Could there be an induced path of length of at least two in a half-bald weak component in $\operatorname{ED}(G)$ for a uep-graph G?

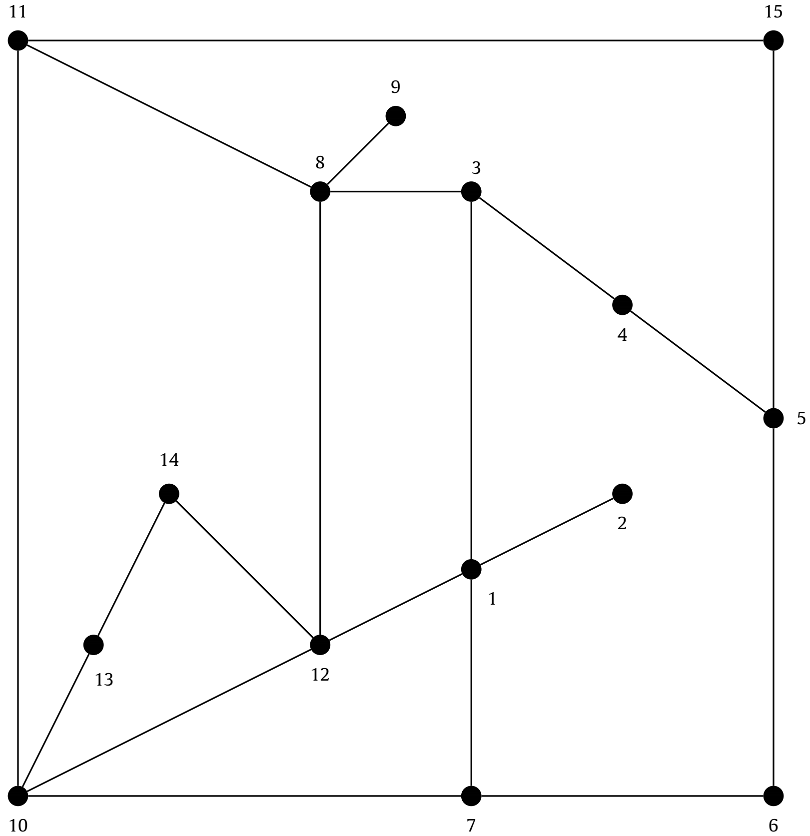

Fig. 15. Uep-graph G with $\mathrm{diam}(G) = 5$ whose eccentric digraph $\mathrm{ED}(G)$ has a half-bald weak component.

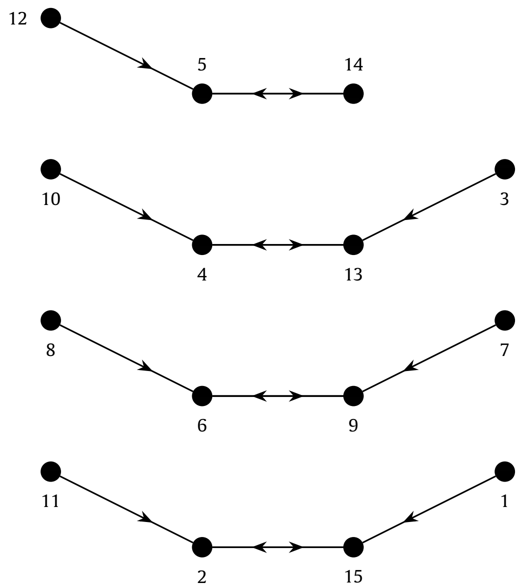

Fig. 16. The eccentric digraph $\mathrm{ED}(G)$ of the uep-graph G from Fig. 15.

Question 3. Does there exist a uep-graph G such that each weak component in $\mathrm{ED}(G)$ is half-bald?

Question 4. Does there exist a non-self-centered uep-graph G without full weak components in $\mathrm{ED}(G)$?

It is clear that if such a graph G exists, then $\mathrm{ED}(G)$ necessarily contains a half-bald weak component (and hence, has another weak component by Proposition 3.3).

Question 5. Can a bald and a half-bald weak components coexist in $\mathrm{ED}(G)$ for a uep-graph G?

Theorem 3.14 implies that if such a graph G exists, then $\mathrm{diam}(G) \geq 5$. Also, note that negative answers to Questions 3, 5 imply negative answer to Question 4.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors would like to express their gratitude to the Ukrainian Armed Forces for keeping Leliukhivka, Mena, and Kyiv safe, which gave us an opportunity to work on this paper. The fourth author thanks the World Central Kitchen organization for providing free meals for refugees, and the PTAK Humanitarian Aid Centre in Warsaw for hospitality. Finally, the authors would also like to acknowledge the anonymous referees for their valuable feedback on the earlier version of the paper, which greatly contributed to the improvement of its quality.

References

[1] J. Boland, F. Buckley, M. Miller, Eccentric digraphs, Discrete Math. 286 (2004) 25–29.
[2] F. Buckley, The eccentric digraph of a graph, Congr. Numer. 149 (2001) 65–76.
[3] J. Gimberta, N. López, M. Miller, J. Ryan, Characterization of eccentric digraphs, Discrete Math. 306 (2006) 210–219.
[4] F. Göbel, V.L. Veldman, Even graphs, J. Graph Theory 10 (1986) 225–239.
[5] A. Hak, V. Haponenko, S. Kozerenko, Eccentric digraphs of unique point eccentric graphs, in: Xth All-Ukrainian Conference of Young Scientists in Physics and Mathematics, April 16-17, Kyiv, Ukraine, 2021, pp. 84–85.
[6] F. Harary, R.Z. Norman, The dissimilarity characteristic of Husimi trees, Ann. Math. 58 (1953) 134–141.
[7] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass, 1969, p. 274.
[8] E. Howorka, On metric properties of certain clique graphs, J. Comb. Theory, Ser. B 27 (1979) 67–74.
[9] H.M. Mulder, n-cubes and median graphs, J. Graph Theory 4 (1980) 107–110.
[10] K.R. Parthasarathy, R. Nandakumar, Unique eccentric point graphs, Discrete Math. 46 (1) (1983) 69–74.
[11] S. Sriram, D. Ranganayakulu, I. Venkat, K.G. Subramanian, Eccentric digraphs of trees, Asia Pac. J. Math. 1 (2) (2014) 141–149.