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THE BOUNDARY PROBLEM BY VARIABLE t FOR EQUATION OF FRACTAL DIFFUSION WITH ARGUMENT DEVIATION

For a quasilinear pseudodifferential equation with fractional derivative by time variable t with order $\alpha \in (0,1)$, the second derivative by space variable x and the argument deviation with the help of the step method we prove the solvability of the boundary problem with two unknown functions by variable t.

Keywords: boundary problem, fractional diffusion, argument deviation.

Formulation of the problem

We should determine the solution of the boundary value problem

$$D_t^\alpha u(t,x) = a^2 \frac{\partial^2 u(t,x)}{\partial x^2} - B(t)p(t,x) + f(t,x,u(t-h,x)), \quad t > h, x \in \mathbb{R}, \tag{1}$$

$$u(t,x)|_{0\le t\le h} = u_0(t,x), \quad x \in \mathbb{R}, \tag{2}$$

$$u((k+1)h,x) = \varphi(x), \quad x \in \mathbb{R}, \tag{3}$$

where

$$D_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \times \left[\frac{\partial}{\partial t}\int_h^t \frac{u(\tau,x)d\tau}{(t-\tau)^\alpha} - (t-h)^\alpha u_0(h,x)\right]$$

is a regularized fractional Riemann–Liouville derivative of $\alpha \in (0,1)$ order, $t > h$, $x \in \mathbb{R}$, h is a number, $k \in \mathbb{N}$, f, u_0, φ are known functions, u, p are unknown functions.

The problem (1)–(3) contains fractal integro-differential equations, which are used in physical, mechanical, and other disciplines. We note that the Cauchy problem for an equation with fractional derivatives is sufficiently complete and thoroughly analyzed in the papers of A. N. Kochubey and S. D. Eidelman [1–3] and many of their later papers.

As a solution of the problem (1)–(3) we mean a pair of functions $u(t,x)$, $p(t,x)$ with such properties [4]:
1. $u \in C_x^2(\Pi)$, $\Pi = (0,T] \times \mathbb{R}$, $T = (k+1)h$; $p \in KC(\Pi)$;
2. fractal integral

$$I_t^{1-\alpha} u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(\tau,x)}{(t-\tau)^\alpha} d\tau$$

belongs to class $C_{t,x}^{1,2}(\Pi)$;
3. function $u(t,x)$ satisfies equation (1) and conditions (2) and (3).

Definition and properties of Green function. The formula of convolution

We denote

$$G_i(t,x,\alpha,h) = F_{\sigma\to x}^{-1}[Q_i(t,\sigma,\alpha,h)], \ i = 1,2, \tag{4}$$

where $F_{\sigma\to x}^{-1}$ is an inverse Fourier transform,

$$Q_1(t,\sigma,\alpha,h) = E_{\alpha,1}(-a^2|\sigma|^2(t-h)^\alpha), \quad t > h, \sigma \in \mathbb{R},$$

$$Q_2(t,\sigma,\alpha,h) = D_t^{1-\alpha} E_{\alpha,1}(-a^2|\sigma|^2(t-h)^\alpha), \quad t > h, \sigma \in \mathbb{R},$$

are expressed in the Mittag–Leffler function

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad t > 0, \alpha > 0, \beta > 0$$

[4, p. 25]. For functions G_i, $i = 1,2$, from (4) such estimates are true [2, lemmas 1, 2]:

$$|D_x^m G_1(t,x,\alpha,h)| \le Ct^{-\frac{\alpha(1+m)}{2}} \times \exp\{-c\rho(t,x,h)\}, \quad m \le 3, \tag{5}$$

$$|D_t^\alpha G_1(t,x,\alpha,h)| \le C(t-h)^{-\frac{3}{2}\alpha} \times \exp\{-c\rho(t,x)\}, \tag{6}$$

$$|D_x^m G_2(t,x,\alpha,h)| \le C(t-h)^{-\frac{\alpha(1+m)}{2}-1+\alpha} \times \\ \times \exp\{-c\rho(t,x,h)\}, \quad m \le 3, \tag{7}$$

$$|D_t^\alpha G_2(t,x,\alpha,h)| \le C(t-h)^{-\frac{\alpha}{2}-1} \times \\ \times \exp\{-c\rho(t,x,h)\}, \tag{8}$$

where $\rho(t,x,h) = \big(|x|(t-h)\big)^{\frac{2}{2-\alpha}}$, $t>0$, $x \in \mathbb{R}$.

Vector-function (G_1, G_2) is called Green function for Cauchy problem without the deviation of the argument:

$$D_t^\alpha u(t,x) = a^2 \frac{\partial^2 u(t,x)}{\partial x^2} + \\ + f(t,x,u_0(t-h,x)), \quad h < t \le 2h, x \in \mathbb{R}, \tag{9}$$

$$u(t,x)|_{t=h} = u_0(h,x), x \in \mathbb{R}, \tag{10}$$

which is obtained from the problem (1), (2) when it is solved by the steps method when $B(t) \equiv 0$. Let us prove the convolution formula for Green function's component. The Cauchy problem for homogenious equation (9) with initial condition $u(t,x)|_{t=0} = \psi(x)$, $x \in \mathbb{R}$ have only one solution in continuously and bounded functions [2] which is represented as convolution

$$u(t,x) = \int_{-\infty}^{\infty} G_1(t-0, x-\xi, \alpha, 0)\psi(\xi)d\xi. \tag{11}$$

Let us write initial condition as

$$u(t,x)|_{t=\beta} = G_1(\beta, x, \alpha, h), \quad x \in \mathbb{R}.$$

The Cauchy problem with this condition corresponds to a solution of the form (11)

$$u_1(t,x) = \int_{-\infty}^{\infty} G_1(t-\beta, x-y, \alpha, h) \times \\ \times G_1(\beta, y, \alpha, h)dy. \tag{12}$$

On the other hand, when $t \ge \beta$ the function $u_2(t,x) = G_1(t,x,\alpha,h)$ is also the solution of equation (9) when $f=0$ with initial condition

$$u_2(t,x)|_{t=\beta} = G_1(\beta, x, \alpha, h), \quad x \in \mathbb{R}.$$

From the theorem of uniqueness of Cauchy problem from [2] follows that $u_1(t,x) \equiv u_2(t,x)$, $0 < \beta < t$, $x \in \mathbb{R}$. So the formula (12) takes the view

$$G_1(t,x,\alpha,h) = \int_{-\infty}^{\infty} G_1(t-\beta, x-y, \alpha, h) \times \\ \times G_1(\beta, y, \alpha, h)dy, \tag{13}$$

where $0 < \beta < t$, $x \in \mathbb{R}$.

Since $G_2(t,x,\alpha,h) = D_t^{1-\alpha} G_1(t,x,\alpha,h)$ using operator $D_t^{1-\alpha}$ to (13) we obtain the convolution formula for Green functions

$$G_2(t,x,\alpha,h) = \int_{-\infty}^{\infty} G_2(t-\beta, x-y, \alpha, h) \times \\ \times G_1(\beta, y, \alpha, h)dy, \tag{14}$$

where $0 < \beta < t$, $x \in \mathbb{R}$.

The classical solution of (1)–(3) problem

Let us construct a classical solution of the original problem (1)–(3) when $kh \le t \le (k+1)h$, $x \in \mathbb{R}$ in the band $\Pi = (h,(k+1)h) \times \mathbb{R}$ [4]

$$u(t,x) = \int_{-\infty}^{\infty} G_1(t, x-\xi, \alpha, kh)u_0(kh,\xi)d\xi + \\ + \int_{kh}^{t} d\tau \int_{-\infty}^{\infty} G_2(t-\tau, x-\xi, \alpha, kh) \times \\ \times [f(\tau,\xi,kh) - B(\tau)p(\tau,\xi)]d\xi, \tag{15}$$

where the term $B(t)p(t,x)$ moved to the right-hand side of equation. Let us assume we satisfied the condition (3) from (15), then we obtain the equation

$$\int_{kh}^{(k+1)h} d\tau \int_{-\infty}^{\infty} G_2((k+1)h-\tau, x-\xi, \alpha, kh) \times \\ \times B(\tau)p(\tau,\xi)d\xi = [G_1(t,x,\alpha,kh)u_0(kh,x) + \\ + G_2(t,x,\alpha,kh)f(t,x,kh)]|_{t=(k+1)h} - \varphi(x) \equiv \\ \equiv \Psi(x,\alpha,kh,(k+1)h), \tag{16}$$

which is an integral Fredholm equation of first kind for definition of function $p(t,x)$. The solution of equation (16) we find as

$$p(t,x) = G_1(t,x,\alpha,kh)C, \tag{17}$$

where C is constant.

If we substitute (17) in (16) we obtain linear equation for definition of C, whose solution is

$$C = \Psi(x,\alpha,kh,(k+1)h)\Big\{\int_{kh}^{(k+1)h} B(\tau)d\tau \\ \int_{-\infty}^{\infty} G_2((k+1)h-\tau, x-\xi, \alpha, kh) \times \\ \times G_1(\tau,\xi,\alpha,kh)d\xi\Big\}^{-1}.$$

If we take into account the convolution formula (14), we obtain

$$C = \Psi(x,\alpha,kh,(k+1)h) \times \\ \times \Big\{G_2(t,x,\alpha,h)\int_{kh}^{(k+1)h} B(\tau)d\tau\Big\}^{-1}.$$

So from (17) we obtain

$$p(t,x) = G_1(t,x,\alpha,kh)\Psi(x,\alpha,kh,(k+1)h) \times \\ \times \Big\{G_2(t,x,\alpha,h)\int_{kh}^{(k+1)h} B(\tau)d\tau\Big\}^{-1}. \tag{18}$$

If we substitute (18) in (15) we obtain the formula for $u(t,x)$. So the pair of functions $u(t,x)$ from (15) and $p(t,x)$ from (18) defines the classical solution of the problem (1)–(3). Based on the inequalities (5)–(8) for Green functions, we estimate pair $(u(t,x),p(t,x))$ of searched functions

$$|p(t,x)| \le Ct^{-\frac{\alpha}{2}}|B|(|u_0|_C + |\varphi|_C + |f|_C), \quad (19)$$

$$|u(t,x)| \le C(|u_0|_C + |f|_C + |B|_C + |\varphi|_C), \quad (20)$$

where $|\cdot|_C$ denotes norm in space continuously functions.

Note that function $p(t,x)$ is not uniquely defined, since in (17) we can add a multiplier $m(t)$.

The two-point boundary value problem for the diffusion equation with the operator of fractional differentiation with respect to the t is considered in [5]. So we have the following theorem.

The classical solution of the problem (1)–(3) defines by pair of functions (15), (18) for which the estimates (19), (20) are correct.

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ГРАНИЧНА ПРОБЛЕМА ЗА ЗМІННОЮ t ДЛЯ РІВНЯННЯ ФРАКТАЛЬНОЇ ДИФУЗІЇ З ВІДХИЛЕННЯМ АРГУМЕНТУ

Для а квазілінійного псевдодиференціального рівняння з дробовою похідною за часом t порядку $\alpha \in (0,1)$, другою похідною за просторовою змінною x і з відхиленням аргументу методом кроків ми доводимо розв'язність граничної задачі з двома невідомими функціями за змінною t.

Ключові слова: гранична проблема, фрактальна дифузія, відхилення аргументу.

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