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INTERPOLATION PROBLEMS FOR RANDOM FIELDS ON SIERPINSKI'S CARPET

The prediction of stochastic processes and the estimation of random fields of different natures is becoming an increasingly common field of research among scientists of various specialties. However, an analysis of papers across different estimating problems shows that a dynamic approach over an iterative and recursive interpolation of random fields on fractal is still an open area of investigation. There are many papers related to the interpolation problems of stationary sequences, estimation of random fields, even on the perforated planes, but all of this still provides a place for an investigation of a more complicated structure like a fractal, which might be more beneficial in appliances of certain industry fields. For example, there has been a development of mobile phone and WiFi fractal antennas based on a first few iterations of the Sierpinski carpet. In this paper, we introduce an estimation for random fields on the Sierpinski carpet, based on the usage of the known spectral density, and calculation of the spectral characteristic that allows an estimation of the optimal linear functional of the omitted points in the field. We give coverage of an idea of stationary sequence estimating that is necessary to provide a basic understanding of the approach of the interpolation of one or a set of omitted values. After that, the expansion to random fields allows us to deduce a dynamic approach on the iteration steps of the Sierpinski carpet. We describe the numerical results of the initial iteration steps and demonstrate a recurring pattern in both the matrix of Fourier series coefficients of the spectral density and the result of the optimal linear functional estimation. So that it provides a dependency between formulas of the different initial sizes of the field as well as a possible generalizing of the solution for N-steps in the Sierpinski carpet. We expect that further evaluation of the mean squared error of this estimation can be used to identify the possible iteration step when further estimation becomes irrelevant, hence allowing us to reduce the cost of calculations and make the process viable.

Keywords: interpolation, Sierpinski carpet, spectral characteristic, spectral density, random fields.

Introduction

The investigation of Brownian motion, homogeneous random fields on the Sierpinski carpet has attracted significant attention in recent years. Notably, there has been a development of mobile phone and WiFi fractal antennas based on a first few iterations of the Sierpinski carpet. These antennas exhibit remarkable properties such as self-similarity and scale invariance, which enable them to effectively accommodate multiple frequencies. Additionally, their ease of manufacturing and compact size, compared to conventional antennas of similar performance, render them highly suitable for pocket-sized mobile devices.

The primary objective of this study is to illustrate the procedure of estimating unknown values of random fields on the Sierpinski carpet. In other words, we analyze the estimating procedure in dynamic, using two iterations of the Sierpinski carpet as a case study. The overall aim is to identify potential similarities or patterns between iterations and analyze the likelihood of further iterations' emergence. For each iteration, we solve the interpolation problem for linear functionals of random fields on a "perforated" plane. Initially, estimation problems were investigated by A. Kolmogorov [1] and R. Wiener [2] for stationary sequences. Later the problem was extended for linear functionals, which depends on the unknown values of stationary sequences and a random field based on observations in some regions of the plane (see [3], [7]). The problem was considered for different special cases, like with noise or from observations in discrete moments of time or for random fields with unknown spectral densities (see [4], [5], [6], [8]-[10] and [14]). Additionally, the papers of Moklyachuk, Shchestyuk, and Florenko (see [11], [12], [13]) proposed solutions of the interpolation problems using observations in the perforated plane for the sum of uncorrelated random fields.

The paper is organized as follows:

In the second section, we present the theoretical framework for the stationary case. Firstly, the Kolmogorov approach is presented to understand the general idea of the interpolation. The approach is based on the assumption that the spectral density is known and all observed values of the stationary sequence are the elements of the Hilbert space. Then optimal estimate of the missed unknown value of the sequence can be found as its projection on the Hilbert space. Secondly, the following theory is an extension from stationary sequences to homogeneous fields. The given formulas for calculating the root mean square error and the spectral characteristic of the optimal linear estimation of the functionals are derived under spectral certainty, where the spectral density of the random field is precisely known. The third part of the second section is devoted to the dynamic approach. First consider the dynamic procedure of fractal grinding of the estimation area K at each moment of discrete time $t = 1, 2, ...n$. The dynamic procedure of estimation for each time is based on the stationary of interpolation of unknown values of fields. We show how the problem of estimating unknown values in random fields can be solved through observations of perforations in the Sierpinski carpet. Moreover, we explore the dependency of the obtained formulas on the size and number of iterations, which are common concerns in forecasting theory applied in domains such as the oil industry, geodesy, and other related fields.

The third section presents numerical results derived from interpolating the first two iterations of the Sierpinski carpet. We assume that spectral density is a product of two densities of the autoregressions of the first orders. In this context, we compare matrices with Fourier series coefficients and conduct analyses of further iterations along with potential solutions.

In conclusion, we summarise the main aspects of the dynamic approach and have highlighted the main patterns in the matrix that are present in the further iteration steps on the Sierpinski carpet. Also, we have carried out the possible research directions that can be continued from this paper.

Theoretical Framework

Kolmogorov approach to interpolation problem for stationary sequence. One of the first interpolation problems posed and solved was finding the optimal mean square estimate of the unknown (unobserved) value ξ_0 of the stationary sequence $\xi = \{\xi_n\}$, $n = 0, \pm 1, \pm 2, \ldots$ with the spectral density $f(\lambda)$.

Mathematically speaking this problem can be formulated as following one ([1], [2]). Let $L_2(\xi)$ is linear subspace of the Hilbert space $H = L_2(\Omega, F, P)$ generated by random variables $\xi(t)$:

$$\xi(t) = \int_{-\pi}^{\pi} e^{it\lambda} Z d\lambda = \int_{-\pi}^{\pi} \varphi(\lambda) Z d\lambda.$$

$\varphi(\lambda) \in L_2(F)$, where $L_2(F)$ is the space of square-integrable functions on the interval $-\pi < \lambda < \pi$ by measure $F(\Delta) = E|Z(\Delta)|^2$. So we need to find the optimal in the mean square estimate $\tilde{\varphi_0}(\lambda)$, which minimizes the value of the mean square error:

$$\Delta = E|\tilde{\varphi_0}(\lambda) - \varphi_0(\lambda)|^2$$

In other words in the space $L_2(F)$ one need to find the projection of $\varphi_0(\lambda) \in L_2$ onto the subspace $H^0(F)$ generated by the functions $e^{i\lambda k \varphi(\lambda)}$, $k = \pm 1, \pm 2, \ldots$.

$$\xi_n \leftrightarrow e^{i\lambda n}$$

From the properties of "perpendiculars" in the Hilbert space $H^0(F)$ [citing 13] we obtain that the function $\hat{\varphi}_0(\lambda)$ is completely determined by two conditions:

1. $\hat{\varphi}_0(\lambda) \in H^0(F)$
2. $1 - \hat{\varphi}_0(\lambda) \perp H^0(F)$

The formula for estimation of unknown value ξ_0 and the mean square error can be easily obtained from these conditions (see [1]). In the next subsection, we will show how these conditions can be applied to the interpolation problem for linear functional of the random fields.

Interpolation problem for random fields. Before consideration the dynamic procedure of fractal grinding of the estimation area K let's solve problem for the optimal estimating of the linear functional

$$A_K\xi = \sum_{(k,j)\in K} a(k,j)\xi(k,j) \tag{1}$$

of unknown values of the homogeneous field $\xi(k,j)$, $(k,j) \in K$ from observations of the $\xi(k,j)$ where $(k,j) \in \mathbb{Z}^2 \backslash K$ and $K = \{(k,j) : k \in \{0, 1, ...n\},\ j \in \{0, 1, ...m\}\}$. So we need to find the optimal in the mean square estimate $\tilde{A}_K\xi$, which minimizes the value of the mean square error:

$$\Delta = M|A_K\xi - \tilde{A}_K\xi|^2 \tag{2}$$

Linear estimate $\tilde{A}_K\xi$ of the unknown functional $A_K\xi$ from observations of field $\xi(k,j) \in \mathbb{Z}^2 \backslash K$ is of the form:

$$\widehat{A_K\xi} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(\lambda,\mu) dZ(\lambda,\mu), \tag{3}$$

where $h(\lambda,\mu)$ is the spectral characteristic of the estimate $\widehat{A_K\xi}$. We will suppose that the condition of "minimality" holds true

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{1}{f(\lambda,\mu)} d\lambda d\mu < \infty, \tag{4}$$

The spectral characteristic $h(\lambda,\mu)$ is characterized by the two conditions (see [4]-[8]).:

1) $h(\lambda,\mu) \in L_2^{K-}(f(\lambda,\mu))$
2) $\Big(A_k(\lambda,\mu) - h(\lambda,\mu)\Big) \perp L_2^{K-}(f(\lambda,\mu))$
namely

$$\Big(A_K(\lambda,\mu) - h(\lambda,\mu)\Big) \perp e^{i(k\lambda+j\mu)}, \quad \forall (k,j) \in \mathbb{Z}^2 \backslash K$$

where

$$A_K(\lambda,\mu) = \sum_{(k,j)\in K} a(k,j) e^{i(k\lambda+j\mu)}$$

This condition can be rewritten into the following form:

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Big(A_K(\lambda,\mu) - h(\lambda,\mu)\Big) f(\lambda,\mu) \cdot \cdot e^{-i(k\lambda+j\mu)} d\lambda d\mu = 0$$

Then we obtain

$$\big(A_K(\lambda,\mu) - h(\lambda,\mu)\big) f(\lambda,\mu) = C_K(\lambda,\mu)$$

where

$$C_K(\lambda,\mu) = \sum\sum_{(k,j)\in K} c_{k,j} e^{i(k\lambda+j\mu)}$$

$c(k,j)$ is vector of unknown coefficients. And we get the following formula:

$$h(\lambda,\mu) = \frac{A_K f(\lambda,\mu) - C_K(\lambda,\mu)}{f(\lambda,\mu)} = A_K(\lambda,\mu) - \frac{C_K(\lambda,\mu)}{f(\lambda,\mu)} \quad (5)$$

Let us extend the function $\frac{1}{f(\lambda,\mu)}$ in Fourier series:

$$\frac{1}{f(\lambda,\mu)} = \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} b_{kj} e^{i(k\lambda+j\mu)},$$

$$b_{kj} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k\lambda+j\mu)} \frac{1}{f(\lambda,\mu)} d\lambda,$$

we obtain a system of equations for finding a vector of unknown coefficients $c(k,j)$

$$a(\lambda,\mu) = R(\lambda,\mu) c(\lambda,\mu), \quad (6)$$

where

$$c(\lambda,\mu) = R(\lambda,\mu)^{-1} a(\lambda,\mu),$$

where $a(\lambda,\mu)$ is vector of known weighting coefficients. $R(\lambda,\mu)$ is an operator, which is determined by matrices:

$$R(\lambda,\mu)(k,j) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(k\lambda+j\mu)} \frac{1}{f(\lambda,\mu)} d\lambda d\mu,$$

with elements that are the Fourier coefficients of the function.

The mean square error $\Delta(h;f)$ may be calculated by the formula (2) after calculating vector:

$$\Delta(h,f) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |A_K(\lambda,\mu) - h(\lambda,\mu)|^2 f(\lambda,\mu) d\lambda d\mu \quad (7)$$

Thus, in order to find the estimation of the functional (3) we are using the spectral characteristics (5) and we can find the mean square error for the estimation (7).

Dynamic procedure for interpolation problem on Sierpinski's carpet.

There is an iterative approach to building the Sierpinski carpet. First consider the dynamic procedure of fractal grinding of the estimation area K at each moment of discrete time $t = 1, 2, ...n$. The square $N \times N$ is cut into 9 congruent subsquares in a 3-by-3 grid. The central square is removed from the estimated area. The procedure is repeated recursively to the rest of 8 sub-squares that belong to the observation zone, and this process is an example of a finite subdivision rule. This variation of Sierpinski's carpet you can see on Figure 1.

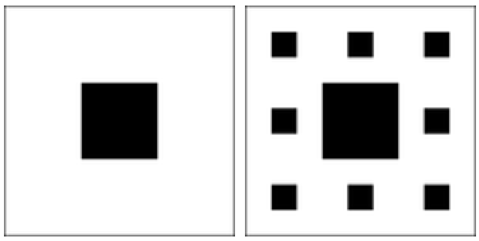

Figure 1. First and second iteration

For the first iteration of the Sierpiski carpet, we find values of the omitted field $N/3 \times N/3$. We observe homogeneous field $\xi(k,j)$, set on the area $K = \{(k,j) : \ k \in \mathbb{Z} \setminus \{0,1,2\}\}$. In order to find unknown values in this area we can compose linear functional when coefficients A are known:

$$A_K \xi = \sum_{(k,j)\in K} a_{(k,j)} \xi(k,j)$$

The value of the functional $\widehat{A_K}$ needs to be estimated.

Spectral characteristic of the estimation H_{NM} can be found with the formula:

$$H_{NM}(e^{i(\lambda k+\mu j)}) = \sum_{(k,j)\in K} h_{(k,j)} e^{i(\lambda k+\mu j)},$$

where K is a square of $3^n \times 3^n$. We use (5) for the spectral characteristics and we can find the mean square error for the estimation (7). For the all next iterations, we need to change area K according to building procedure of the Sierpinski carpet and use again (5) for the spectral characteristics and (5) the mean square error for the estimation. We explore this theory in the next section for the special case, under the assumption that spectral density is a product of two densities of the auto-regressions of the first orders. In this context, we compare matrices with Fourier series coefficients and conduct analyses of further iterations along with potential solutions.

Numerical Results

Let the spectral density $f(\lambda, \mu)$ be of the form :

$$f(\lambda,\mu) = \frac{1}{4\pi^2}\frac{1}{|1-\alpha e^{-i\lambda}|^2}\frac{1}{|1-\beta e^{-i\mu}|^2}$$

The interpolation of the first iteration.

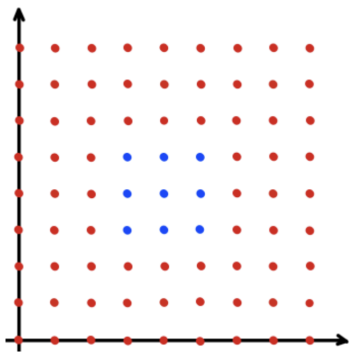

Figure 2. The first iteration of the Sierpinski carpet

We explore the dynamic procedure of fractal grinding of the observation area at each moment of discrete time $t = 1, 2, ...n$.

Firstly, we assume that for $t = 1$ the estimation area is K. In order to form the first iteration we set a central square 3×3 (where the number is a number of points of one side) for estimation on the plane of size 6×6. Following the mentioned above procedure, firstly, we need to find the Fourier series of the spectral density.

$$\frac{1}{f(\lambda,\mu)} = 4\pi^2(|1-\alpha e^{-i\lambda}|^2)(|1-\beta e^{-i\mu}|^2)$$

$$\frac{1}{f(\lambda,\mu)} = 4\pi^2(1+\alpha^2-\alpha(e^{-i\lambda}+e^{i\lambda}))\cdot$$
$$\cdot(1+\beta^2-\beta(e^{-i\mu}+e^{i\mu}))$$

Fourier coefficients can be written in the following form:

$$r_0 = r_{0,0} = 4\pi^2(1+\alpha^2)(1+\beta^2)$$
$$r_1 = r_{0,-1} = r_{0,1} = -4\pi^2\beta(1+\alpha^2)$$
$$r_2 = r_{-1,0} = r_{1,0} = -4\pi^2\alpha(1+\beta^2)$$
$$r_3 = r_{1,1} = r_{1,-1} = r_{-1,1} = r_{-1,-1} = 4\pi^2\alpha\beta$$

Then using an approach proposed in the [11], [12] for perforated planes, the equation (6) can be written in the form:

$$\overrightarrow{a_K} = R \cdot \vec{c},$$

where $\overrightarrow{a_K}$ is the vector determined by given functional $A_K\xi$,

$\vec{c}$ is a vector from unknown coefficients,

$$R = \begin{pmatrix} V & E & 0 \\ E & V & E \\ 0 & E & V \end{pmatrix},$$

where

$$V = \begin{pmatrix} r_0 & r_1 & 0 \\ r_1 & r_0 & r_1 \\ 0 & r_1 & r_0 \end{pmatrix}, \quad E = \begin{pmatrix} r_2 & r_3 & 0 \\ r_3 & r_2 & r_3 \\ 0 & r_3 & r_2 \end{pmatrix}$$

Using the formula (8) for finding linear functional of the problem:

$$A_K(\lambda,\mu) = \sum_{k=3,j=3}^{5,5} a_{(k,j)} e^{i(k\lambda+j\mu)}$$

$$f^{-1}(\lambda,\mu) = \sum_{k=-1,j=-1}^{1,1} r_{(k,j)} e^{i(k\lambda+j\mu)}$$

We can present C_K as sum of our coordinates with coefficients:

$$C_K(\lambda,\mu) = \sum_{k=3,j=3}^{5,5} c_{(k,j)} e^{i(k\lambda+j\mu)}$$

The spectral characteristic coefficient will be found from the following formula:

$$h(\lambda,\mu) = A_K(\lambda,\mu) - f^{-1}(\lambda,\mu) \times C_K(\lambda,\mu)$$

Therefore, the estimation of the linear functional will take a look of:

$$A_K\xi = \sum_{k=2}^{5}\xi_{2,k}\gamma_{2,k} + \sum_{k=3}^{5}\xi_{k,2}\gamma_{k,2} + \sum_{k=2}^{5}\xi_{k,6}\gamma_{k,6} + \sum_{k=2}^{6}\xi_{6,k}\gamma_{6,k}$$

where $\gamma_{2,2} = -r_3c_{3,3}$, $\gamma_{2,3} = -r_2c_{3,3} - r_3c_{3,4}$, $\gamma_{2,4} = -r_3c_{3,3} - r_2c_{3,4} - r_3c_{3,5}$, $\gamma_{2,5} = -r_2c_{3,5} -$

$r_3c_{3,4}$), $\gamma_{2,6} = -r_3c_{3,5}$, $\gamma_{3,6} = -r_1c_{3,5} - r_3c_{4,5}$, $\gamma_{4,6} = -r_3c_{3,5} - r_1c_{4,5} - r_3c_{5,5}$, $\gamma_{5,6} = -r_3c_{4,5} - r_1c_{5,5}$, $\gamma_{6,6} = -r_3c_{5,5}$, $\gamma_{6,5} = -r_3c_{5,4} - r_2c_{5,5}$, $\gamma_{6,4} = -r_3c_{5,3} - r_2c_{5,4} - r_3c_{5,5}$, $\gamma_{6,3} = -r_2c_{5,3} - r_3c_{5,4}$, $\gamma_{6,2} = -r_3c_{5,3}$, $\gamma_{5,2} = -r_3c_{4,3} - r_1c_{5,3}$, $\gamma_{4,2} = -r_3c_{3,3} - r_1c_{4,3} - r_3c_{5,3}$, $\gamma_{3,2} = -r_1c_{3,3} - r_3c_{4,3}$

As expected, we have received values around the defined points using the input spectral density, which is a product of the auto-regression of the first order.

The interpolation of the second iteration.

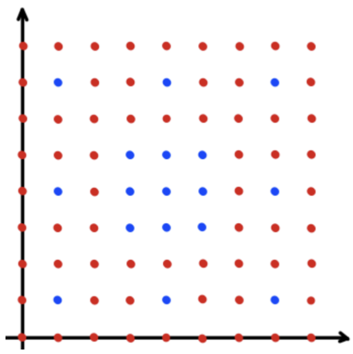

Figure 3. Second iteration of the Sierpinski carpet

Let's consider the situation for $t = 2$, where D is a set of red dots on Figure 3 (blue dots represent omitted values, and red ones contain known coefficients). Then the problem is to find the optimal estimate of the linear functional:

$$A_D(\lambda,\mu) = \sum_{(k,j)\in D} a_{(k,j)} e^{i(k\lambda+j\mu)} = \sum_{k=3,j=3}^{5,5} a_{(k,j)} e^{i(k\lambda+j\mu)} + \sum_{\substack{n=0\\m=0}}^{2,2} \sum_{\substack{k=3n+1\\j=3m+1}} a_{(k,j)} e^{i(k\lambda+j\mu)}$$

The following spectral density will be used:

$$f(\lambda,\mu) = \frac{1}{4\pi^2}\frac{1}{|1-\alpha e^{-i\lambda}|^2}\frac{1}{|1-\beta e^{-i\mu}|^2}$$

Then using an approach proposed in the section 2, the equation (6) can be written in the form:

$$\vec{c} = R^{-1}\cdot\overrightarrow{a_D}$$

The matrix R has a size of 17×17 and can be presented in the form of submatrices:

$$V = \begin{pmatrix} r_0 & r_1 & 0\\ r_1 & r_0 & r_1\\ 0 & r_1 & r_0\end{pmatrix}, \quad E = \begin{pmatrix} r_2 & r_3 & 0\\ r_3 & r_2 & r_3\\ 0 & r_3 & r_2\end{pmatrix}$$

$$D_0 = \begin{pmatrix} r_0 & 0 & 0\\ 0 & r_0 & 0\\ 0 & 0 & r_0\end{pmatrix}, \quad H_0 = \begin{pmatrix} r_0 & 0 & 0\\ 0 & r_0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Then matrix R of coefficients of the Fourier expansion of spectral density can be written in the form:

$$R = \begin{pmatrix} V & E & 0 & 0 & 0 & 0\\ E & V & E & 0 & 0 & 0\\ 0 & E & V & 0 & 0 & 0\\ 0 & 0 & 0 & D_0 & 0 & 0\\ 0 & 0 & 0 & 0 & D_0 & 0\\ 0 & 0 & 0 & 0 & 0 & H_0\end{pmatrix}$$

We can present C_D as a sum of our coordinates with coefficients:

$$C_D(\lambda,\mu) = \sum_{(k,j)\in D} c_{(k,j)} e^{i(k\lambda+j\mu)} = \sum_{k=3,j=3}^{5,5} c_{(k,j)} e^{i(k\lambda+j\mu)} + \sum_{\substack{n=0\\m=0}}^{2,2} \sum_{\substack{k=3n+1\\j=3m+1}} c_{(k,j)} e^{i(k\lambda+j\mu)}$$

Now we can find the spectral characteristic coefficient from the following formula:

$$h(\lambda,\mu) = A_D(\lambda,\mu) - f^{-1}(\lambda,\mu)\times C_D(\lambda,\mu)$$

In the result, the estimation of the linear functional will take look as:

$$A_K\xi = \sum_{k=2}^{5}\xi_{2,k}\gamma_{2,k} + \sum_{k=3}^{5}\xi_{k,2}\gamma_{k,2} + \sum_{k=2}^{5}\xi_{k,6}\gamma_{k,6} + \sum_{k=2}^{6}\xi_{6,k}\gamma_{6,k} + \sum_{m,n\in\{1,4,7\}}\sum\Big(-c_{m,n}\big(r_3 \sum_{\substack{k=\{m-1,n+1\}\\ j=\{m-1,n+1\}}}\sum \xi_{(k,j)} + r_1 \sum_{j=\{m-1,n+1\}} \xi_{m,j} + r_2 \sum_{j=\{m-1,n+1\}} \xi_{j,n}\big)\Big)$$

where all $\gamma_{k,j}$ are taken from the previous iteration step in the section 3.1.

The interpolation of further iterations.

For the general case, we consider the square field of $3^{n+1} \times 3^{n+1}$.

The order of the coordinates of squares is written from bottom to up column by column from the biggest square to the smallest one.

R is square symmetric matrix from Fourier coefficients of spectral density $\frac{1}{f(\lambda,\mu)}$:

$$R = \begin{pmatrix} R_{3^n} & 0 & 0 & \cdots & 0 \\ 0 & R_{3^{n-1}} & 0 & \cdots & 0 \\ 0 & \cdots & \ddots & \cdots & \cdots \\ 0 & ... & 0 & R_{3^1} & 0 \\ 0 & ... & 0 & 0 & R_{3^0} \end{pmatrix},$$

where R_k is s a square symmetric submatrix of size $k^2 \times k^2$ which is repeated 8^k-times representing the amount of subsquares for each iteration of the Sierpinski carpet.

$$R_k = \begin{pmatrix} V_k & E_k & 0 & 0 & \cdots & 0 \\ E_k & V_k & E_k & 0 & \cdots & 0 \\ 0 & E_k & V_k & E_k & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & ... & 0 & E_k & V_k & E_k \\ 0 & ... & 0 & 0 & E_k & V_k \end{pmatrix},$$

V_k is a square symmetric submatrix of k-size with r_0 as the main diagonal element and r_1 as an upper diagonal element.

$$V_k = \begin{pmatrix} r_0 & r_1 & 0 & \cdots & 0 \\ r_1 & r_0 & r_1 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & r_1 & r_0 & r_1 \\ 0 & \cdots & 0 & r_1 & r_0 \end{pmatrix}$$

E_k is a square symmetric submatrix of k-size with r_2 as the main diagonal element and r_3 as an upper diagonal element.

$$E_k = \begin{pmatrix} r_2 & r_3 & 0 & \cdots & 0 \\ r_3 & r_2 & r_3 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & r_3 & r_2 & r_3 \\ 0 & \cdots & 0 & r_3 & r_2 \end{pmatrix}$$

Singular element squares are represented by r_0 in the matrix R_1. After that, we can find the spectral characteristic of the estimation as in previous cases.

Conclusion

The paper explores the possibility of iterative estimating unknown values in random fields, using two iterations of the Sierpinski carpet as a case study. The paper reviews interpolation problems related to the estimation of unknown values in random fields, based on previous research on stationary sequences and functionals in homogeneous random fields.

There was an investigation of the problem of estimation areas for random fields using the Sierpinski carpet at the first and second iterations. We apply the spectral characteristic function and spectral density function to our problem which results in the linear functional of the estimation. The paper presents the estimation formula and outlines a pattern for retrieving solutions for further iterations.

The paper concludes by presenting numerical results derived from interpolating the first two iterations of the Sierpinski carpet and discussing potential solutions for further iterations. Overall, the study aims to identify patterns and similarities between iterations and analyze the likelihood of their emergence. I can be noticed that enlarging the size of the initial central square allows to write down recursively linear functional and the matrix of Fourier coefficients.

In future research, there are prospects to examine the error of the estimation in order to determine when the following iterations are becoming ineffective and precision will no longer be relevant for fields estimation. Besides, extending different variations of the Sierpinski carpet with higher orders of autoregression functions can be explored.

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ПРОБЛЕМИ ІНТЕРПОЛЯЦІЇ ДЛЯ ВИПАДКОВИХ ПОЛІВ НА КИЛИМІ СЕРПІНСЬКОГО

Прогнозування випадкових процесів та оцінка випадкових полів різної природи стає все більш поширеним напрямом досліджень серед науковців різних спеціальностей. Проте аналіз статей щодо різних проблем оцінювання показує, що динамічний підхід до ітераційної та рекурсивної інтерполяції випадкових полів на фракталі все ще є відкритою сферою дослідження. Є багато робіт, пов'язаних з інтерполяцією стаціонарних послідовностей, оцінкою випадкових полів, навіть на перфорованих площинах, але виникають нові виклики для дослідження на більш складній структурі, як фрактал, що може бути більш корисним у застосуваннях в окремих галузях промисловості. Наприклад, було розроблено фрактальну антену мобільного телефону та WiFi на основі перших кількох ітерацій килима Серпінського. У цій статті ми представляємо динамічну процедуру для оцінювання випадкових полів на килимі Серпінського на основі використання відомої спектральної щільності та розрахунку спектральної характеристики, що дозволяє оцінити оптимальний лінійний функціонал від пропущених точок випадкових полів для кожної ітерації. Спочатку ми представляємо підхід Колмогорова для забезпечення базового розуміння ідеї, що використовується в задачах інтерполяції одного чи набору пропущених значень стаціонарних послідовностей. Після цього поширюємо цей підхід на випадкові поля, що дає нам змогу надалі вивести формули для динамічного оцінювання випадкових полів на перших кількох ітераціях килима Серпінського. Ми описуємо чисельні результати початкових кроків ітерації та демонструємо повторювану закономірність як у матриці коефіцієнтів ряду Фур'є спектральної щільності, так і в формулах оптимальної оцінки лінійного функціоналу від випадкових полів. Таким чином, ця закономірність забезпечує залежність між формулами різних початкових розмірів поля, а також можливе узагальнення рішення для N-кроків у килимі Серпінського. Ми очікуємо, що дослідження середньоквадратичної помилки цієї оцінки може бути використано для визначення можливого кроку ітерації, коли подальша оцінка стає нерелевантною, що дозволить нам зменшити вартість обчислень.

Ключові слова: інтерполяція, килим Серпінського, спектральна характеристика, спектральна щільність, випадкові поля.

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