

Representations of solutions of Lamé system with real coefficients via monogenic functions in the biharmonic algebra

Serhii Gryshchuk

Dedicated to the memory of Professor Oleksandr Bakhtin

Abstract. New representations of solutions of Lamé system with real coefficients via monogenic functions in the biharmonic algebra are found.

Анотація. В роботі знайдено нові зображення розв’язків системи Ламе з дійсними коефіцієнтами через “аналітичні” функції зі значеннями у комутативній асоціативній двовимірній алгебрі над полем комплексних чисел.

1. INTRODUCTION

Partial solutions of Lamé system of equilibrium equations for displacements of isotropic plane theory of elasticity by use of “analytic” functions taking values in commutative and associative over the complex field algebra are found in the paper [15]. In the present paper, the solutions are considered in domains possessing some convexity properties. In [7], these results are generalized to solutions in arbitrary simply connected domains.

2010 Mathematics Subject Classification: 30G35, 35E20, 74B05

Keywords: Lamé-type system of partial differential equations with real coefficients, Lamé system of equilibrium equations for displacements, biharmonic equation, biharmonic function, commutative and associative algebras, biharmonic algebra, monogenic functions

Ключові слова: система Ламе диференціальних рівнянь із частинними похідними з дійсними коефіцієнтами, система Ламе рівняння рівноваги, бігармонійне рівняння, бігармонійна функція, комутативна та асоціативна алгебра, бігармонійна алгебра, моногенні функції

DOI: <http://dx.doi.org/10.15673/tmgc.v16i1.2400>

Certain solutions of Lamé system of equilibrium equations for displacements of orthotropic plane theory of elasticity via “analytic” functions taking values in commutative and associative over the complex field semisimple algebra are found in [8, 9, 16].

Expressions of Lamé system of equilibrium equations for displacements of isotropic space theory of elasticity by use of quaternion-valued “analytic” functions are found, for example, in the papers [3, 6, 12, 25].

Matrix functions being analytic in the sense of Douglis are used for descriptions of solutions of Lamé system of equilibrium equations for displacements of isotropic plane theory of elasticity in [22], for solutions of Lamé system of equilibrium equations for displacements of orthotropic plane theory of elasticity in [1], and for general anisotropic media in the paper [23].

Despite numerous publications, most of the cited works are devoted to the investigation of solutions of three-dimensional Lamé system of equilibrium equations for displacements of isotropic theory of elasticity and their representations in terms of quaternion-valued functions. It is worth noting that these approaches are based on generalizations of the Kolosov-Muskhelishvili formulas (cf., e.g., [19] or [20]) to the case of quaternion-valued “analytic” functions (cf., e.g., [6]). At the same time, a question of representations of solutions of a system of equilibrium equations for displacements of the plane isotropic theory via “analytic” functions, taking values in the arbitrary commutative and associative over the complex field n -dimensional algebra, $n \geq 2$, has not been opened yet.

Our aim is to generalize Lamé system of equilibrium equations for displacements of isotropic plane theory of elasticity by replacing its coefficients (depending on Lamé constants) to any non-zero real numbers. We call any system of this type as *Lamé-type system with real coefficients* (*Lamé-type system* or *Lamé system* briefly). Such systems are considered by several authors in the papers [5, 13]. More precisely, in the paper [13] polynomial solutions of a general Lamé-type system of any dimension are found, a plane generalization of the plane Lamé-type system with complex coefficients and homogeneous Dirichlet boundary value problem for it were considered in the paper [5]. Expressions of regular solutions of arbitrary elliptic type systems of the second order and any dimension by means of analytic functions of complex variable are obtained in [2].

An answer to the question of building solutions of the plane Lamé-type system by use of hypercomplex analytic function by another researchers is not known to the author.

In the present work, we will find new expressions of general solutions of the plane Lamé system (as well as for isotropic media and for a case of the general Lamé-type system with real coefficients) in the form of the

first real component (under some basis of the algebra) of some “analytic” (monogenic) functions taking values in certain commutative and associative two-dimensional algebra over the complex field (so-called biharmonic algebra).

2. MONOGENIC FUNCTION IN THE BIHARMONIC ALGEBRA

We say that an associative commutative two-dimensional algebra \mathbb{B} with unit e over \mathbb{C} is *biharmonic* (this notion is proposed in [14]) if in \mathbb{B} there exists a *biharmonic* basis, i.e., a basis $\{e_1, e_2\}$ satisfying the conditions

$$(e_1^2 + e_2^2)^2 = 0, \quad e_1^2 + e_2^2 \neq 0. \quad (2.1)$$

We restrict our attention onto the biharmonic basis $\{e_1, e_2\}$ having the following multiplication table

$$e_1 = e, \quad e_2^2 = e_1 + 2ie_2, \quad (2.2)$$

where i is the imaginary complex unit. The biharmonic basis (2.2) is found by V. F. Kovalev and I. P. Mel’nichenko in the paper [14].

E. Study [24] proved that there exist only two types (up to isomorphism) of two-dimensional algebra with e over the field \mathbb{C} .

In the paper [18] I. P. Mel’nichenko proved that there exists a unique biharmonic algebra \mathbb{B} with a non-biharmonic basis $\{e, \rho\}$ for which

$$\rho = 2e_1 + 2ie_2 \quad (2.3)$$

and $\rho^2 = 0$.

Note that the algebra \mathbb{B} is isomorphic to four-dimensional algebras over \mathbb{R} considered in [4, 21].

We use the Euclidean norm $\|a\| := \sqrt{|z_1|^2 + |z_2|^2}$ in the algebra \mathbb{B} , where $a = z_1e_1 + z_2e_2$ and $z_1, z_2 \in \mathbb{C}$.

Similarly to the paper [14], consider a *biharmonic plane*

$$\mu_{e_1, e_2} := \{\zeta = x e_1 + y e_2 : x, y \in \mathbb{R}\},$$

which is a linear span over \mathbb{R} of the elements of biharmonic basis $\{e_1, e_2\}$.

To a domain D of the Cartesian plane xOy we associate the congruent domain $D_\zeta := \{\zeta = xe_1 + ye_2 : (x, y) \in D\}$ in the biharmonic plane μ_{e_1, e_2} .

In what follows, $\zeta = xe_1 + ye_2$ and $z = x + iy$, where $(x, y) \in D$.

Note that every function $\Phi: D_\zeta \rightarrow \mathbb{B}$ has an expansion

$$\Phi(\zeta) = U_1(x, y) e_1 + U_2(x, y) ie_1 + U_3(x, y) e_2 + U_4(x, y) ie_2, \quad (2.4)$$

where $U_l: D \rightarrow \mathbb{R}$, $l = \overline{1, 4}$ are real-valued component-functions. We use the following notation:

$$U_l[\Phi] := U_l.$$

Taking into account that every non-zero point ζ in the biharmonic plane μ_{e_1, e_2} has the inverse element $\zeta^{-1} \in \mathbb{B}$, we define a notion of a derivative for functions $\Phi: D_\zeta \rightarrow \mathbb{B}$ similarly to the complex derivative for complex-valued holomorphic functions.

We say that a function $\Phi: D_\zeta \rightarrow \mathbb{B}$ is *monogenic* in a domain D_ζ if it has the classical derivative

$$\Phi'(\zeta) := \lim_{\substack{h \rightarrow 0 \\ h \in \mu_{e_1, e_2}}} (\Phi(\zeta + h) - \Phi(\zeta)) h^{-1}$$

at every point $\zeta \in D_\zeta$.

It is proved in [14] that a function $\Phi: D_\zeta \rightarrow \mathbb{B}$ is monogenic in D_ζ if and only if each of its real-valued component-function in (2.4) is real differentiable in D and the following analogue of the Cauchy-Riemann conditions is satisfied:

$$\frac{\partial \Phi(\zeta)}{\partial y} e_1 = \frac{\partial \Phi(\zeta)}{\partial x} e_2. \quad (2.5)$$

It is proved in [10, 11] that every monogenic function $\Phi: D_\zeta \rightarrow \mathbb{B}$ has derivatives $\Phi^{(n)}(\zeta)$ of any natural order in the domain D_ζ and satisfies the following *biharmonic equation*:

$$\Delta^2 u(x, y) \equiv \left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) u(x, y) = 0,$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in the domain D due to the first relation in (2.1) and the equality

$$\Delta^2 \Phi(\zeta) = \Phi^{(4)}(\zeta) (e_1^2 + e_2^2)^2.$$

Therefore, all the components $U_j: D \rightarrow \mathbb{R}$, $j = \overline{1, 4}$, in the expression (2.4) are biharmonic functions in the domain D .

At the same time, every biharmonic function $U(x, y)$ in D is the first component $U_1 \equiv U$ in the expression (2.4) of a certain monogenic function $\Phi: D_\zeta \rightarrow \mathbb{B}$:

$$U(x, y) = U_1 [\Phi(\zeta)], \quad (\zeta \in D_\zeta). \quad (2.6)$$

Moreover, all such functions Φ are described in [10, 11] for a bounded and simply connected domain D . Using the uniqueness theorem (see [11]) this result is generalized to unbounded and simply connected domains as well. By the proof of this fact we use a description of monogenic functions $P := \Phi$ with $U_1 [P] \equiv 0$ (see [10, Lemma 3]), that can be rewritten with the help of the equalities (2.2), (2.3) and their corollaries:

$$\zeta^2 = (x^2 + y^2) e_1 + 2xy e_2 + 2iy^2 e_2, \quad ye_1 + xe_2 = e_2 \zeta - 2ie_2 y,$$

in the form of a polynomial in the indeterminate ζ having at most second order:

$$P(\zeta) = -ia\zeta^2 + (k - be_2)i\zeta + nie_1 + ce_2 + mie_2, \quad (2.7)$$

where a, b, c, k, m, n are arbitrary real numbers.

3. MAIN RESULTS

Consider a Lamé system

$$\begin{cases} \Delta u(x, y) + p \frac{\partial \theta(x, y)}{\partial x} = 0, \\ \Delta v(x, y) + p \frac{\partial \theta(x, y)}{\partial y} = 0, \end{cases} \quad (3.1)$$

where $(x, y) \in D$, $p \in \mathbb{R} \setminus \{0\}$, and $\theta := \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$.

By a solution of the Lamé system (3.1) we mean its *regular* solution [2], i.e., a pair of twice continuously differentiable functions (u, v) satisfying the equalities (3.1).

Since the characteristic polynomial of the system (3.1) is

$$\chi(s) = (1 + p)(1 + s^2)^4,$$

we simply conclude that the system (3.1) is of elliptic type for $p \neq -1$, is degenerate for $p = -1$. Moreover, for $p > -1$ a Lamé system (3.1) is *uniformly elliptic* ([2]) and *strongly elliptic* by the notion of the paper [26].

For some positive values of p the system (3.1) is a Lamé system of equilibrium equations of isotropic plane theory of elasticity with respect to the vector of displacements $(u(x, y), v(x, y))$ ([19, 20]).

At the same time, the components of the vector of displacements u and v are biharmonic functions in the domain D , and $p = 1 + \lambda/\mu$, where Lamé constants λ and μ depend on elastic properties of an isotropic solid and satisfy inequalities $\lambda > 0$, $\lambda + (2/3)\mu > 0$ ([17, p. 22-23]). In particular, we see that $p > 1/3$.

Isotropic solids are known currently in practice only with $p > 1$ corresponding to the case $\lambda > 0$ and $\mu > 0$ simultaneously ([17, p. 26]). Hence, we assume that $p > 1$ for Lamé system (3.1) of equilibrium equations of isotropic plane theory of elasticity with respect to the vector of displacements.

Let $p \neq -1$ ($p \neq 0$). Using ellipticity of the system (3.1) we differentiate the first equation of the system with respect to the variable x , and the second with respect to the variable y . Then, adding obtained equations, we the following equality

$$(1 + p) \Delta \theta = 0.$$

Hence the function θ is harmonic in the domain D , and components u, v of any solution (u, v) of the system (3.1) satisfy the following equalities

$$\Delta^2 u = \Delta^2 v \equiv 0, \quad (p \neq -1). \tag{3.2}$$

Therefore, similarly to the case of Lamé system of equilibrium equations of isotropic plane theory of elasticity, u and v are biharmonic in D functions.

If $p = -1$, then we say that (u, v) is a *biharmonic solution* of Lamé system (3.1) if it is a solution of (3.1) such that its components u and v are biharmonic functions in D .

In what follows, by the domain D of Cartesian plane xOy we mean such domain that its congruent domain $\{z = x + iy \in \mathbb{C} : (x, y) \in D\}$ of the complex plane \mathbb{C} (or of the extended complex plane) is simply connected (though not necessarily bounded).

Theorem 3.1. *A general solution of Lamé system (3.1) with $p \neq -1$ is expressed by the formula*

$$u(x, y) = U_1 [\Phi(\zeta)], \quad v(x, y) = U_1 [A\Phi(\zeta) + Q(\zeta)], \tag{3.3}$$

where $\Phi: D_\zeta \rightarrow \mathbb{B}$ is an arbitrary monogenic function,

$$\begin{aligned} A &:= i \left(e + \left(\frac{1}{2} + \frac{1}{p} \right) \rho \right), \\ B &:= \left(-\frac{c}{2} + \frac{m}{p} i \right) e - \left(\frac{2+p}{4p} \right) c \rho, \\ Q(\zeta) &:= \frac{B}{2} \zeta^2 + b_1 \zeta + b_0, \quad (\zeta \in D_\zeta), \end{aligned} \tag{3.4}$$

c, m are arbitrary real numbers, b_1 and b_0 are arbitrary elements of the algebra \mathbb{B} .

A general biharmonic solution of Lamé system (3.1) with $p = -1$ is expressed in the form

$$u(x, y) = U_1 [\Phi(\zeta)], \quad v(x, y) = U_1 \left[A\Phi(\zeta) + \frac{ib}{6} \zeta^3 + Q(\zeta) \right], \tag{3.5}$$

where $\Phi: D_\zeta \rightarrow \mathbb{B}$ is any monogenic function, A and Q are defined by (3.4) with $p = -1$, and b is an arbitrary real number.

Proof. First we prove that a general solution of Lamé system (3.1) with $p \neq -1$ is represented in the form (3.3). Due to (3.2) and (2.6), we see that any solution (u, v) of Lamé system (3.1) can be represented by the following formulas:

$$u(x, y) = U_1 [\Phi(\zeta)], \quad v(x, y) = U_1 [\Psi(\zeta)] \tag{3.6}$$

where $\zeta \in D_\zeta$, and Φ and Ψ are monogenic functions in D_ζ . Our aim is to find all these pairs Φ and Ψ . Then we can write a general solution of the Lamé system by the formula (3.6).

Applying (2.5) to (3.6) we get the following equalities:

$$\begin{aligned}\Delta u &= U_1[(e + e_2^2) \Phi''], & \Delta v &= U_1[(e + e_2^2) \Psi''], \\ \theta &= U_1[\Phi' + e_2 \Psi'], \\ \frac{\partial \theta}{\partial x} &= U_1[\Phi'' + e_2 \Psi''], & \frac{\partial \theta}{\partial y} &= U_1[e_2 \Phi'' + e_2^2 \Psi''],\end{aligned}$$

after that the system (3.1) turns into

$$\begin{cases} U_1[(e + e_2^2 + p) \Phi''(\zeta) + p e_2 \Psi''(\zeta)] = 0, \\ U_1[(e + e_2^2 + p e_2^2) \Psi''(\zeta) + p e_2 \Phi''(\zeta)] = 0, \end{cases} \quad (\zeta \in D_\zeta). \quad (3.7)$$

By the help of (2.7), the system (3.7) is equivalent to the following one:

$$\begin{cases} (e + e_2^2 + p) \Phi''(\zeta) + p e_2 \Psi''(\zeta) = P(\zeta), \\ (e + e_2^2 + p e_2^2) \Psi''(\zeta) + p e_2 \Phi''(\zeta) = \tilde{P}(\zeta), \end{cases} \quad (\zeta \in D_\zeta), \quad (3.8)$$

where P and \tilde{P} are polynomials of the form (2.7). Coefficients of the polynomial \tilde{P} in (3.8) are obtained from the formula (2.7) by replacing coefficients with appropriate coefficients with upper tilde. For example, the coefficient near ζ^2 in \tilde{P} is $(-i)\tilde{a}$, $\tilde{a} \in \mathbb{R}$.

As a consequence of (2.3) we get the following equalities:

$$e_2 = i \left(e - \frac{\rho}{2} \right), \quad e_2^{-1} = -i \left(e + \frac{\rho}{2} \right). \quad (3.9)$$

Substituting them into the first equation of the system (3.8), we get that

$$\Psi''(\zeta) = A \Phi''(\zeta) - \frac{i}{p} \left(e + \frac{\rho}{2} \right) P(\zeta), \quad (\zeta \in D_\zeta), \quad (3.10)$$

where the constant $A \in \mathbb{B}$ is the same as in (3.3).

Substituting (3.10) into the second equation of the system (3.8) and applying simple transformations, we get the following equality

$$i \left(e - \left(\frac{1}{2} + \frac{1}{p} \right) \rho \right) P(\zeta) = \tilde{P}(\zeta), \quad (\zeta \in D_\zeta). \quad (3.11)$$

Taking into account expressions of P and \tilde{P} and differentiating twice the equality (3.11) with respect to variable ζ , we get the relation between coefficients of polynomials P and \tilde{P} near ζ^2 :

$$a \left(e - \left(\frac{1}{2} + \frac{1}{p} \right) \rho \right) = -i\tilde{a}. \quad (3.12)$$

Now from the uniqueness of expansion by the elements of the basis $\{e, \rho\}$, we get from (3.12) that

$$a = \tilde{a} = 0. \quad (3.13)$$

Substituting now (3.13) and (3.9) to (3.11), differentiating then the obtained equality by the variable ζ , we get relations between coefficients of polynomials P and \tilde{P} near ζ :

$$(-k + ib)e + \left(\left(\frac{1}{2} + \frac{1}{p}\right)k - \left(1 + \frac{1}{p}\right)bi\right)\rho = (\tilde{b} + i\tilde{k})e - \frac{\tilde{b}}{2}\rho. \quad (3.14)$$

Using the uniqueness of expansion by the elements of the basis $\{e, \rho\}$, we obtain from (3.14) the following system of equations

$$\begin{cases} -k + ib = \tilde{b} + \tilde{k}i, \\ \left(\frac{1}{2} + \frac{1}{p}\right)k - \left(1 + \frac{1}{p}\right)bi = -\frac{\tilde{b}}{2}. \end{cases} \quad (3.15)$$

Since $p \neq -1$, it follows from (3.15) that

$$b = k = \tilde{b} = \tilde{k} = 0. \quad (3.16)$$

Using (3.13), (3.16) and (3.9), the equality (3.11) after elementary transformations reduces to the following one:

$$\begin{aligned} -(c + n + mi)e + \left(\left(1 + \frac{1}{p}\right)c + \left(\frac{1}{2} + \frac{1}{p}\right)n + \left(1 + \frac{1}{p}\right)mi\right)\rho = \\ = (-\tilde{m} + (\tilde{c} + \tilde{n})i)e + \frac{\tilde{m} - \tilde{c}i}{2}\rho. \end{aligned}$$

Now, taking into account the uniqueness of expansion by the elements of the basis $\{e, \rho\}$, we get the following system of equations:

$$\begin{cases} c + n + mi = \tilde{m} - (\tilde{c} + \tilde{n})i, \\ \left(1 + \frac{1}{p}\right)c + \left(\frac{1}{2} + \frac{1}{p}\right)n + \left(1 + \frac{1}{p}\right)mi = \frac{\tilde{m}}{2} - \frac{\tilde{c}}{2}i. \end{cases} \quad (3.17)$$

A condition of solvability of the system (3.17) is

$$n = -\left(1 + \frac{p}{2}\right)c, \quad (3.18)$$

where c is any real number. Under that condition (3.18) the solutions $(\tilde{c}, \tilde{n}, \tilde{m}, c, m, n)$ of the system (3.17) are given by the following formulas:

$$\tilde{c} = -\left(2 + \frac{2}{p}\right)m, \quad \tilde{n} = \left(1 + \frac{2}{p}\right)m, \quad \tilde{m} = -\frac{p}{2}c,$$

where c and m are any real numbers, and n is obtained from c by the formula (3.18).

Then (2.7) turns into

$$P(\zeta) = -\left(m + \frac{p}{2}c\right)e + \frac{m - ic}{2}\rho, \quad (3.19)$$

where c and m are any real numbers.

Substituting (3.19) into (3.10) and doing elementary transformations we get the equality

$$\Psi''(\zeta) = A\Phi''(\zeta) + B, \quad (\zeta \in D_\zeta). \quad (3.20)$$

The equality (3.20) simply yields the equality

$$\Psi(\zeta) = A\Phi(\zeta) + \frac{B}{2}\zeta^2 + b_1\zeta + b_0 \equiv A\Phi(\zeta) + Q(\zeta), \quad (\zeta \in D_\zeta)$$

where b_1 and b_0 are any elements in \mathbb{B} . It also implies the equality (3.3), which can now be represented as follows:

$$u(x, y) = u_1(x, y) + u_0(x, y), \quad v(x, y) = v_1(x, y) + v_0(x, y), \quad (3.21)$$

where

$$\begin{aligned} u_1(x, y) &:= U_1[\Phi(\zeta)], & v_1(x, y) &:= U_1[A\Phi(\zeta)], \\ u_0(x, y) &:= U_1[P(\zeta)] \equiv 0, & v_0(x, y) &:= U_1[Q(\zeta)], \end{aligned}$$

and P is defined by the formula (2.7) with arbitrary corresponding coefficients.

Natural substitution $u = u_1$, $v = v_1$ in (3.1) yields the equivalent system (3.7) whose left-hand sides equals zero: this can be seen by elementary transformations. Thus, (u_1, v_1) is a solution of Lamé system (3.1).

Similarly, we conclude that the pair of functions $u = u_0 \equiv U_1[P(\zeta)]$, $v = v_0$, satisfies Lamé system (3.1). Summarizing the obtained results, we conclude that a general solution of Lamé system (3.1) with $p \neq -1$ can be expressed by the formula (3.3).

Consider the case $p = -1$. Then for any biharmonic equation of Lamé system (3.1) equalities (3.6)-(3.15) are fulfilled with $p = -1$.

A general solution of the system (3.15) with $p = -1$ is a set of ordered real quadruples $(b, k, \tilde{b}, \tilde{k})$, where

$$k = \tilde{b} = 0, \quad \tilde{k} = b, \quad (3.22)$$

and b is any real number. Therefore the equation (2.7) turns into

$$P(\zeta) = b \left(e - \frac{\rho}{2} \right) \zeta + nie + (-m + ci) \left(e - \frac{\rho}{2} \right), \quad (\zeta \in D_\zeta), \quad (3.23)$$

while the equality (3.11) reduces to the following one:

$$bi\zeta - (c + n + mi)e - \frac{n}{2}\rho = bi\zeta + (-\tilde{m} + (\tilde{c} + \tilde{n})i)e + \frac{\tilde{m} - \tilde{c}i}{2}\rho.$$

Thus, we obtain the following system of equations:

$$\begin{cases} c + n + mi = \tilde{m} - (\tilde{c} + \tilde{n})i, \\ -\frac{n}{2} = \frac{\tilde{m}}{2} - \frac{\tilde{c}}{2}i. \end{cases} \quad (3.24)$$

A general solution of the system (3.24) is a set of ordered 6-tuples $(\tilde{m}, \tilde{c}, \tilde{n}, n, c, m)$, where

$$\tilde{m} = \frac{c}{2}, \quad \tilde{c} = 0, \quad \tilde{n} = -m, \quad n = -\frac{c}{2},$$

here c and m are any real numbers.

Substituting a general solution of the system (3.24) into (3.23), we will see that $P(\zeta)$ turns into the form

$$P(\zeta) = b \left(e - \frac{\rho}{2} \right) \zeta + \left(-m + \frac{c}{2} i \right) e + \frac{m-ic}{2} \rho.$$

Thus, (3.10) will be written as follows:

$$\Psi''(\zeta) = A\Phi''(\zeta) + ib\zeta + B, \quad (\zeta \in D_\zeta), \quad (3.25)$$

where A and B are given by the formula (3.4) with $p = -1$.

Now, we get from (3.25) that

$$\Psi(\zeta) = A\Phi(\zeta) + \frac{ib}{6}\zeta^3 + Q(\zeta), \quad (\zeta \in D_\zeta), \quad (3.26)$$

where b is any real number, and Q is defined by (3.4) with $p = -1$.

Substituting (3.26) into (3.6), we get

$$\begin{aligned} u(x, y) &= u_2(x, y) + u_1(x, y) + u_0(x, y), \\ v(x, y) &= v_2(x, y) + v_1(x, y) + v_0(x, y), \end{aligned} \quad (3.27)$$

where u_k and v_k , $k = 0, 1$, are defined in the same way as in the formula (3.21) (but with $p = -1$), and

$$u_2(x, y) := u_0(x, y), \quad v_2(x, y) := U_1 [ib\zeta^3/6].$$

Analogously to the case $p \neq -1$, we obtain that a pair of functions (u_k, v_k) , $k = 0, 1$, satisfy the system (3.1) with $p = -1$.

It easy to show that the pair of functions $u = u_2$, $v = v_2$ satisfies the system (3.1) as well.

Summarizing obtained results, we conclude that a general biharmonic solution of of Lamé system (3.1) with $p = -1$ is expressed in the form of (3.5). Theorem 3.1 is proved. \square

Using (2.3) and (3.9) it is easy to establish following consequences of Theorem 3.1.

Corollary 3.2. *Componentwise the equality (3.3) has a form*

$$\begin{aligned} u(x, y) &= U_1 [\Phi(\zeta)], \\ v(x, y) &= -2 \left(1 + \frac{1}{p} \right) U_2 [\Phi(\zeta)] - \\ &\quad - \left(1 + \frac{2}{p} \right) U_3 [\Phi(\zeta)] + Q_1(x, y), \quad (\zeta \in D_\zeta), \end{aligned} \quad (3.28)$$

where

$$Q_1(x, y) := \frac{1}{2^p} \operatorname{Re} \left((mi - (1 + p)c)z^2 + (2m + cpi)yz \right) + U_1[b_1]x + U_3[b_1]y + U_1[b_0], \quad (3.29)$$

and the componentwise form of (3.5) is obtained from (3.28) by replacing $Q_1(x, y)$ in (3.29) with the sum $(b/6) \operatorname{Re}(iz^3 + 3yz^2) + Q_1(x, y)$.

Corollary 3.3. *Theorem 3.1 still holds if in the equalities (3.3) and (3.5) the symbol Φ is replaced with the product $C\Phi$, where C is any non-zero element in \mathbb{B} , while an element A is replaced by the product CA . In particular, if $C = e_2$, then (3.28) can be written as follows:*

$$\begin{aligned} u(x, y) &= U_3[\Phi(\zeta)], \\ v(x, y) &= -\left(1 + \frac{2}{p}\right) U_1[\Phi(\zeta)] + \frac{2}{p} U_4[\Phi(\zeta)] + Q_1(x, y), \quad (\zeta \in D_\zeta), \end{aligned} \quad (3.30)$$

and a componentwise analogue of (3.5) is obtained from (3.30) by replacing Q_1 in (3.29) with the sum $(b/6) \operatorname{Re}(iz^3 + 3yz^2) + Q_1(x, y)$.

Remark 3.4. The case $p = -1$ in Theorem 3.1 has a great importance. Because when we pass through the value $p = -1$ to another value $p \neq -1$, we see that appropriate expressions of solutions (possessing a property that components of solutions (u, v) are biharmonic functions) of the system (3.1) change their form (compare (3.3) and (3.5)). A similar situation occurs in the paper [13] during the process of building polynomial solutions of the Lamé system (3.1).

REFERENCES

- [1] E. A. Abapolova and A. P. Soldatov. Lamé system of elasticity theory in a plane orthotropic medium. *J. Math. Sci. (N.Y.)*, 157(3):387–394, 2009. doi:10.1007/s10958-009-9319-0.
- [2] A. V. Bitsadze. *Boundary value problems for second order elliptic equations*. North-Holland Publishing Company, 1968.
- [3] S. Bock, K. Gürlebeck, D. Legatiuk, and H. M. Nguyen. ψ -hyperholomorphic functions and a Kolosov–Muskhelishvili formula. *Mathematical Methods in the Applied Sciences*, 38(18):5114–5123, 2015. doi:10.1002/mma.3431.
- [4] A. Douglis. A function-theoretic approach to elliptic systems of equations in two variables. *Comm. Pure Appl. Math.*, 6:259–289, 1953. doi:10.1002/cpa.3160060205.
- [5] G. V. Goncharova. Certain properties of the solutions of the second fundamental problem of elasticity theory. *Izv. Akad. Nauk Az. SSR, Ser. Fiz.-Tekh. Mat. Nauk*, 1972(3):13–19, 1972. in Russian.
- [6] Y. Grigor’ev. Three-dimensional quaternionic analogue of the Kolosov–Muskhelishvili formulae. In *Hypercomplex analysis: new perspectives and applications*, pages 145–166. Birkhäuser, 2014.

- [7] S. V. Gryshchuk. Hypercomplex monogenic functions of biharmonic variable in some problems of plane elasticity theory. *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki*, (6):7–12, 2015. in Ukrainian. doi:10.15407/dopovidi2015.06.007.
- [8] S. V. Gryshchuk. Commutative complex algebras of the second rank with unity and some cases of plane orthotropy. II. *Ukr. Math. J.*, 70(10):1594–1603, 2019. doi:10.1007/s11253-019-01592-0.
- [9] S. V. Gryshchuk. Monogenic functions in commutative complex algebras of the second rank and the Lamé equilibrium system for some class of plane orthotropy. *J. Math. Sci. (N.Y.)*, 246(1):30–38, 2020. Translation of *Ukr. Mat. Visn.* 16 (2019), no. 3, 345–356. doi:10.1007/s10958-020-04720-5.
- [10] S. V. Gryshchuk and S. A. Plaksa. Monogenic functions in a biharmonic algebra. *Ukr. Math. J.*, 61(12):1865–1876, 2009. doi:10.1007/s11253-010-0319-5.
- [11] S. V. Gryshchuk and S. A. Plaksa. Basic properties of monogenic functions in a biharmonic plane. In Mark L. Agranovsky, Matania Ben-Artzi, Greg Galloway, Lavi Karp, Vladimir Maz'ya, Simeon Reich, David Shoikhet, Gilbert Weinstein, and Lawrence Zalemán, editors, *Proceedings of the 5th International Conference (CA&DS V) held in Akko, May 22–27, 2011*, volume 591 of *Contemporary Mathematics*, pages 127–134. American Mathematical Society, Providence, RI; Bar-Ilan University, Ramat Gan, 2013. doi:10.1090/conm/591/11831.
- [12] K. Gürlebeck, K. Habetha, and W. Spröbig. *Boundary value problems for second-order partial differential equations*. Springer Basel, Basel, 2016.
- [13] V. V. Karachik. Polynomial solutions to the Lamé equations [translation of *Mat. Tr.* 5 (2002), no. 2, 155–169; mr1944069]. *Siberian Adv. Math.*, 13(2):66–79, 2003.
- [14] V. F. Kovalev and I. P. Mel'nichenko. Biharmonic functions on the biharmonic plane. *Dokl. Akad. Nauk Ukrain. SSR Ser. A*, (8):25–27, 1981.
- [15] V. F. Kovalev and I. P. Mel'nichenko. Biharmonic potentials and plane isotropic displacement fields. *Ukr. Math. J.*, 40(2):197–199, 1988. doi:10.1007/BF01056476.
- [16] V. F. Kovalev and I. P. Mel'nichenko. Algebras of functional-invariant solutions of the p-biharmonic equation, 1991. in Russian.
- [17] L. D. Landau and E. M. Lifshits. *Theory of elasticity: Course of Theoretical Physics, volume 7*. Moscow: FIZMATLIT, 5 edition, 2003. in Russian.
- [18] I. P. Mel'nichenko. Biharmonic bases in algebras of the second rank. *Ukr. Math. J.*, 38(2):224–226, 1986. doi:10.1007/BF01058492.
- [19] S. G. Mikhlin. The plane problem of the theory of elasticity. In *Trans. Inst. of seismology, Acad. Sci. USSR*, number 65. Acad. Sci. USSR Publ. House: Moscow–Leningrad, 1935.
- [20] N. I. Muskhelishvili. Some basic problems of the mathematical theory of elasticity, 1975. Fundamental equations, plane theory of elasticity, torsion and bending, Translated from the fourth corrected and augmented Russian edition by J. R. M. Radok, Reprint of the second English edition of 1963.
- [21] L. Sodbero. Nuovo metodo per lo studio dei problemi di elasticità, con applicazione al problema della piastra forata. *Ric. Ing.*, 13(2):255–264, 1934. in Italian.
- [22] A. P. Soldatov. Hyperanalytic functions and their applications. *J. Math. Sci. (N.Y.)*, 132(6):827–881, 2006. doi:10.1007/s10958-006-0023-z.
- [23] A. P. Soldatov. On the theory of anisotropic flat elasticity. *J. Math. Sci. (N.Y.)*, 235(4):484–535, 2018. URL: dSPACE.BSU.EDU.RU/handle/123456789/31540, doi:10.1007/s10958-018-4083-7.

- [24] E. Study. On systems of complex numbers and their use in the theory of transformation groups. *Monatsh. f. Mathematik und Physik*, 1:283–354, 1890. in German. doi:10.1007/BF01692479.
- [25] A. Tsalik. Quaternionic representation of the 3d elastic and thermoelastic boundary problems. *Math. Methods Appl. Sci.*, 18(9):697–708, 1995. doi:10.1002/mma.1670180904.
- [26] M. I. Vishik. On strongly elliptic systems of differential equations. *Mat. Sb., Nov. Ser.*, 29:615–676, 1951. in Russian.

Received: December 13, 2022, accepted: May 13, 2023.

Serhii Gryshchuk

INSTITUTE OF MATHEMATICS OF THE NATIONAL ACADEMY OF SCIENCE OF UKRAINE,
TERESCHENKIVS'KA STR. 3, KYIV, UKRAINE, 01024;

NATIONAL UNIVERSITY OF KYIV-MOHYLA ACADEMY, FACULTY OF NATURAL SCIENCES,
SKOVORODY STR. 2, KYIV, UKRAINE, 04070

Email: serhii.gryshchuk@gmail.com

ORCID: 0000-0001-8683-445X