Diffusion model in Image Transforms Inversion tasks

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Solving modern machine learning tasks requires development of new methods of solving corresponding inverse problems. Majority of real-world inverse problems are ill-posed and therefore require regularization. For some digital signal processing tasks, such as image de-noising, image restoration, super-resolution, image improvement, the choice of regularization technique is non-trivial, whereas significantly influences the corresponding solution.

In our work we study diffusion model for inversion of image transforms. For inverse problem

$$Ax = y \tag{1}$$

we consider Bayesian approach, or maximum a posteriori probability (MAP) estimate, which finds such an x, that maximises the conditional probability $p(x|y)$. According to Bayes rule

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{p(y|x)p(x)}{\int p(y|x)p(x)dx} \propto p(y|x)p(x),$$

therefore maximisation of $p(x|y)$ corresponds to the following problem:

$$\arg\min_x(-\log p(y|x) - \log p(x)).$$

Obviously, real probability distribution functions are unknown. Therefore instead of it we solve the following heuristics

$$\hat{x} = \arg\min_x\{l(x,y) + \alpha\rho(x)\}, \tag{2}$$

where $l(x,y)$ is a loss function and $\rho(x)$ is a regularization term.

Let's slightly modify (2):

$$\hat{x} = \arg\min_{x,v}\{l(x,y) + \alpha\rho(v)\}, x = v.$$

It allows us to apply Alternating Direction Method of Multipliers (ADMM) from the paper [2], using Lagrangian:

$$L_\lambda(x,v,u) = l(x,y) + \alpha\rho(v) + \frac{\lambda}{2}\|x - v + u\|^2 - \frac{\lambda}{2}\|u\|^2.$$

It leads to iterative solving following minimization tasks till convergence:

$$\begin{aligned}\hat{x} &\longleftarrow \arg\min_x L(x, \hat{v}, u)\\ \hat{v} &\longleftarrow \arg\min_x L(\hat{x}, v, u)\\ u &\longleftarrow u + (\hat{x} - \hat{v})\end{aligned}$$

or in terms of (2)

$$\begin{aligned}\hat{x} &= \min_x l(x, y) + \beta\|x - v\|^2,\\ \hat{v} &= \min_v \alpha\rho(v) + \beta\|x - v\|^2.\end{aligned}$$

In such a way, instead of one inverse problem with regularization scheme we've got two interconnected minimization problems, iterative solving of which allows us to find solution for the initial problem. Having some initial x_0 and v_0 we iterate

$$\begin{aligned}x_{i+1} &= \min_x l(x, y) + \beta\|x - v_i\|^2,\\ v_{i+1} &= \min_v \alpha\rho(v) + \beta\|x_i - v\|^2.\end{aligned}$$

Let's consider some operator $D : X \mapsto X$, that preserves x as a solution, i.e.

$$AD(x, \sigma) = y,$$

for example, for super-resolution task instead of D a denoising operator may be used.

For diffusion model regularization term $\rho(x)$ is $\alpha\rho(x) = \alpha x^T[x - D(x, \sigma)]$. Under mild conditions (differentiability, local homogeneity, and symmetric Jacobian for D) we may apply gradient descent:

$$x_{k+1} = x_k - \mu[A^T(Ax_k - y) - \alpha[x_k - D(x_k, \sigma)]].$$

In our work we study convergence rates of the proposed diffusion model and approximation error, illustrating it with numerical experiments.

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