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COHERENCE ISOMORPHISMS FOR TRIANGULATED HOPF CATEGORY SL(2)

We construct examples of coherence isomorphisms for a triangulated Hopf category related to SL(2). It is an equivariant derived category equipped with multiplication and comultiplication functors and structure isomorphisms.

1. Introduction

Crane and Frenkel proposed a notion of a Hopf category [2]. It was motivated by Lusztig's approach to quantum groups — his theory of canonical bases. In particular, Lusztig obtains braided deformations $U_q n_+$ of universal enveloping algebras Un_+ for some nilpotent Lie algebras n_+ together with canonical bases of these braided Hopf algebras [3,4,5]. The elements of the canonical basis are identified with isomorphism classes of simple perverse sheaves — certain objects of equivariant derived categories. They are contained in a semisimple abelian category of semisimple complexes. One of the proposals of Crane and Frenkel is to study this category rather than its Grothendieck ring $U_q n_+$. Conjectural properties of this category were collected into a system of axioms of a Hopf category, equipped with functors of multiplication and comultiplication, isomorphisms of associativity, coassociativity and coherence which satisfy four equations [2].

Crane and Frenkel [2] gave an example of a Hopf category resembling the semisimple category encountered in Lusztig's theory corresponding to one-dimensional Lie algebra n_+ — nilpotent subalgebra of SL(2). We want to discuss an example of a related notion — triangulated Hopf category — the whole equivariant derived category equipped with multiplication and comultiplication functors and structure isomorphisms. In particular, it is a monoidal category. The new feature of coherence is that additive relations of [2] are replaced with distinguished triangles. This new structure does not induce a Hopf category structure of Crane and Frenkel on a subcategory of semi-simple complexes. The missing component is a consistent choice of splitting of splittable triangles. Verification of some of the consistency equations is still an open question.

In the present paper we shall discuss coherence at the category level. If one replaces linear mappings in bialgebra axiom with functors and Σ with $\oplus$ the equation fails: the left and the right hand side functors are, in general, not isomorphic. (Restricted to the constant sheaf they give, however, isomorphic results.) One of the results of the present paper is the following. The value of the left hand side functor on an object X of is a repeated extension of the values on X of summands in the right hand side in the sense of distinguished triangles. Precise analogy is as follows: a sheaf S on a topological space W is an extension of its quotient-sheaf S_F supported on closed subset F by subsheaf S_U supported on its open complement $U = W - F$.

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2. Preliminaries

The definition of equivariant derived categories is given by Bernstein and Lunts [1]. First we explain basic terms. With a topological space X is associated the category $Sh(X)$ of sheaves of topological spaces. Its derived category is denoted $D(X)$. The subcategory consisting of bounded complexes of sheaves is denoted $D^b(X)$. If X is a complex algebraic variety, we call a sheaf *constructible* if it is constructible with respect to some stratification by algebraic submanifolds and stalks are finite-dimensional vector spaces. A complex is cohomologically constructible if its cohomology sheaves are constructible. The subcategory of bounded constructible complexes is denoted $D^{b,c}(X)$.

Assume that a complex linear algebraic group G acts algebraically on a complex algebraic variety X. In this setting Bernstein and Lunts [1] defined bounded constructible equivariant derived category $D^{b,c}_G(X)$. It will be denoted also $D_G(X) = D^{b,c}_G(X)$. Notice that, if X is G-free, then $D_G(X)$ is equivalent to $D^{b,c}(G/X)$. Without freeness assumption the former and the latter categories are not equivalent, in general.

For a G-map $f: X \to Y$ of algebraic varieties Bernstein and Lunts [1] defined also the inverse image functor $f^*: D^{b,c}_G(Y) \to D^{b,c}_G(X)$, the direct image functor $f_*: D^{b,c}_G(X) \to D^{b,c}_G(Y)$ and the direct image functor with proper supports $f_!: D^{b,c}_G(X) \to D^{b,c}_G(Y)$. Their definitions encompassed also f-equivariant maps f, where $f: G \to H$ is a group homomorphism.

3. Hopf category $n_+SL(2)$

3.1. Setup and notations

We partially follow Lusztig [4, 5] in notations. Let V be a vector space and

$$G = G_V = GL(V).$$

Let us make the product of $D_G(pt)$ over varying $\dim V$ into a sort of a graded Hopf category.

Assume we are given a decomposition

$$V: \quad V^1 \oplus V^2 \oplus \ldots \oplus V^k = V$$

into vector subspaces. Associate with it a filtration of V

$0 = V^{(0)} \subset V^{(1)} \subset \ldots \subset V^{(n)} = V,$
$V^{(m)} = V^1 \oplus \ldots \oplus V^m.$

P_V is the corresponding parabolic group

$P_V = \{g \in G_V \mid \forall m\, g(V^{(m)}) \subset V^{(m)}\} =$
$= \{g \in G_V \mid \forall m\, g(V^m) \subset V^{(m)}\}$

and U_V is its unipotent radical. The group

$$L_V = \{gOG_V \mid \forall m\, g(V^m) \subset V^m\} = \prod_{m=1}^{k} GL(V^m)$$

is a Levi subgroup of P_V.

Notice that P_V, U_V need only a filtration to be defined unlike L_V, which requires direct sum decomposition.

3.2. Braiding

The categories $D_{Ga^{\cdot}\ldots}(pt)$ are viewed as a monoidal 2-category, where a_i are some vector spaces. Define a braiding in it via the functor

$c: D_{\prod G_{ai}}\prod G_{bj}(pt) \to^{h^*} D_{\prod G_{ai}}\prod G_{bj}(\mathrm{Hom}(\oplus^k_{i=1} a_i, \oplus^l_{j=1} b_j)) \to$

$\to^{h}_{!} D_{\prod G_{ai}}\prod G_{bj}(pt) \to^{\sigma^*} D_{\prod G_{bj}}\prod G_{ai}(pt), \qquad (1)$

where σ is the permutation isomorphism of groups. The action of $\prod G_{ai}\prod G_{bj}$ in

$$\mathrm{Hom}(\oplus^k_{i=1} a_i, \oplus^l_{j=1} b_j) \text{ is } (g,h)x = hxg^{-1}.$$

Proposition. The R-matrix $R: D_G(pt) \to D_G(pt)$ (the product of the first two functors in (1)) is isomorphic to the shift functor $K \to K[-2d]$.

3.3. Operations

Let two decompositions of V into a direct sum be given:

$V: V^1 \oplus V^2 \oplus \ldots \oplus V^k = V,$
$W: W^1 \oplus W^2 \oplus \ldots \oplus W^l = V.$

Let $O \subset G$ be a left P_W-invariant and right P_V-invariant subset. We associate with it an operation $X^{O;V}_W = \cap^{V_0}_W \Psi^V_W$.

The components of it are defined below.

3.3.1. Multiplication

The multiplication operation is

$\Psi^{O,V}_W = (D_{L_V}(pt) \xrightarrow{\phi^*} D_{P_W L_V}(O/U_V) \to$
$\to^{\pi}_{*} D_{P_W}(O/P_V) \to^{\alpha}_{!} D_{P_W}(pt)).$

Here the scheme of multiplication is similar to that in [5]:

$$pt \leftarrow^{\phi} O/U_V \to^{\pi} O/P_V \to^{\alpha} pt, \qquad (2)$$

where π is the canonical projection. The action of $L_V = P_V/U_V$ in O/U_V is induced from the action $p.o = op^{-1}, p \in P_V$.

In particular, for $l = 1$ we have $P_W = G_V$, $O = G_V$ and the multiplication operation is

$\Psi^V_1 = (D_{L_V}(pt) \to^{\phi^*} D_{G_V L_V}(G_V/U_V) \to$
$\to^{\pi}_{*} D_{G_V}(G_V/P_V) \to^{\alpha}_{!} D_{G_V}(pt)),$

Here the scheme of multiplication is precisely that of Lusztig [5]:

$$pt \leftarrow^{\phi} G_V/U_V \to^{\pi} G_V/P_V \to^{\alpha} pt.$$

The particular case $k = 1$, $O = G_V$ is also important. We have then $L_V = P_V = G_V$, $U_V = 1$, and

$$\Psi^1_W = (D_G(pt) \to^{pr_2^*} D_{P_W G}(G) \to^{d_1}_{*} D_{P_W}(pt)),$$

where $d_1: G \to pt$.

Proposition. The functor Ψ^1_W is isomorphic to the restriction functor $\mathrm{Res}_{P_W,G}: D_G(pt) \to D_{P_W}(pt)$.

3.4. Comultiplication

The comultiplication functor is

$$\cap^V_W = (D_{P_W}(pt) \to^{\mathrm{Res}}{}_{L_W,P_W} D_{L_W}(pt)) = i_{L_W,P_W}^*.$$

Together with multiplication definition of Section 3.3.1 it gives the general operation (in the setup of (3.3))

$X^{O;V}_W = (D_{L_V}(pt) \to^{\phi^*} D_{P_W L_V}(O/U_V) \to$
$\to^{\pi}_{*} D_{P_W}(O/P_V) \to^{\alpha}_{!} D_{P_W}(pt) \to^{\mathrm{Res}}{}_{L_W,P_W} D_{L_W}(pt)) \cong$
$\cong (D_{L_V}(pt) \to^{\phi^*} D_{L_W L_V}(O/U_V) \to^{\pi}_{*} D_{L_W}(O/P_V) \to$
$\to^{\alpha}_{!} D_{L_W}(pt)).$

In the particular case $k = 1$ we have $O = L_V = P_V = G_V$, $U_V = 1$, and the comultiplication operation is isomorphic to

$(D_{G_V}(pt) \to^{\mathrm{Res}} D_{P_W}(pt) \to^{\mathrm{Res}} D_{L_W}(pt)) \cong$
$\cong (D_{G_V}(pt) \to^{\mathrm{Res}} D_{L_W}(pt))$

by Proposition 3.3.1.

3.5. *Coherence isomorphism: examples*

Application of two comultiplications, middle braiding, and two multiplications gives one functor

$$D_{G_aG_bG_cG_d}(pt) \xleftarrow{Res} D_{G_eG_f}(pt)$$

$$(23)\downarrow \quad Res$$

$$D_{G_aG_cG_bG_d}(pt) \xrightarrow{h^*} D_{G_aG_cG_bG_d}(Hom(b,c)) \xrightarrow{h_!} D_{G_aG_cG_bG_d}(pt)$$

$$p^* \quad \downarrow f^* \qquad \downarrow \phi\phi^*$$

$$D_{G_gG_aG_cG_hG_bG_d}(P_W/U(P_W \cap P_V)) \xrightarrow{t_!} D_{G_gG_aG_cG_hG_bG_d}(G_g/U_g\, G_h/U_h)$$

$$\downarrow q_* \qquad \downarrow \pi\pi_*$$

$$D_{G_gG_h}(P_W/(P_W \cap P_V)) \xrightarrow{t_!} D_{G_gG_h}(G_g/P_g\, G_h/P_h)$$

$$\alpha_! \qquad \downarrow \alpha\alpha_!$$

$$D_{G_gG_h}(pt)$$

Here we use the isomorphism of $G_gG_hP_gP_h$-spaces

$P_W/(U_W \cap P_V) \to^{\cong} G_gG_h \times \mathrm{Hom}(b, c)$,

where G_g, G_h act by left multiplication on P_W and G_gG_h; P_g, P_h act by inverse right multiplication on P_W and G_gG_h; P_g acts on Hom(b, c) by left multiplication via $P_g \to G_c$; P_h acts on Hom(b, c) by inverse right multiplication via $P_h \to G_b$. It implies the isomorphism of $G_gG_hG_aG_bG_cG_d$-spaces

$P_W/\ U(P_W \cap P_V) \to^{\cong} G_g/U_gG_h/U_h\mathrm{Hom}(b, c)$,

where $U(P_W \cap P_V)$ is the unipotent radical of $P_W \cap P_V$. It follows that

$G_aG_bG_cG_d/(G_g/\ U_gG_h/U_h\mathrm{Hom}(b, c)) \cong$
$\cong P_W/(P_W \cap P_V) \cong (P_W \cdot P_V)/P_V$

are isomorphic as G_gG_h-spaces.

Another single operation-functor is

$$D_{G_eG_f}(pt) \xrightarrow{\phi^*} D_{P_WG_eG_f}((P_WP_V)/U_V) \xrightarrow{\pi_*} D_{P_W}(P_W/(P_W \cap P_V)) \xrightarrow{\alpha_!} D_{P_W}(pt)$$

$$\phi^* \quad \downarrow Res \qquad \downarrow Res \qquad \downarrow Res$$

$$D_{G_gG_hG_eG_f}((P_WP_V)/U_V) \xrightarrow{\pi_*} D_{G_gG_h}(P_W/(P_W \cap P_V)) \xrightarrow{\alpha_!} D_{G_gG_h}(pt)$$

Now the following diagram gives the coherence isomorphism between the functors presented by above two diagrams:

$$D_{G_gG_hG_eG_f}(P_WP_V/U_V) \xleftarrow{\phi^*} D_{G_eG_f}(pt)$$

$$q^* \qquad \phi^*$$

$$D_{G_gG_hP_V}(P_WP_V)$$

$$\pi_*\downarrow \qquad i^*\downarrow \qquad \downarrow r^*$$

$$D_{G_gG_hP_W \cap P_V}(P_W)$$

$$q_* \qquad q^*$$

$$D_{G_gG_h}(P_W/P_W \cap P_V) \xleftarrow{q_*} D_{G_gG_hG_aG_bG_cG_d}(P_W/U(P_W \cap P_V))$$

$$\alpha_!\downarrow$$

$$D_{G_gG_h}(pt)$$

Here all functors with q are quotient equivalences, and i^* is an induction equivalence obtained from

$i: P_W \to^{id} P_{W\, P_W \cap P_V}P_V \to P_WP_V$.

Commutativity of one of the triangles

$$D_{G_gG_hG_aG_bG_cG_d}(P_W/U(P_W \cap P_V))$$

$$r^* \qquad q^*\downarrow$$

$$D_{G_eG_f}(pt) \xrightarrow{u^*} D_{G_gG_h\, P_W \cap P_V}(P_V)$$

follows from the isomorphism of functors

$D_{P_eP_f}(pt) \to \chi^*{}_{\psi}{}^*D_{P_W \cap P_V}(U_W) = D_{P_W \cap P_V}(P_gP_h\backslash B_+)$,

where

$\psi: P_W \cap P_V \to P_eP_f,\ \ \chi: P_W \cap P_V \to G_aG_bG_cG_d \to P_eP_f$

Indeed,

$\psi^* \cong (D_{P_eP_f}(pt) \to^{r^*} D_{G_aG_bG_cG_d}(pt) \to^{p^*} D_{P_eP_f}(pt) \to$
$\to^{\chi^*} D_{P_W \cap P_V}(U_W))$

and the product of the first two functors is isomorphic to identity.

More general case: one functor is a composition of three triple comultiplications, a braided shuffle and of three triple multiplications:

$$D_{G_jG_kG_l}(pt)$$

$$\downarrow Res$$

$$D_{diag}(pt) \xrightarrow{t^*} D_{diag}(B) \xrightarrow{t_!} D_{diag}(pt)$$

$$\downarrow p_1^* \qquad \downarrow p_1^*$$

$$D_{P_mP_nP_p}(pt) \xrightarrow{t^*} D_{P_mP_nP_p}(U_W/U_W \cap P_V) \xrightarrow{t_!} D_{P_mP_nP_p}(pt)$$

$$r^*\downarrow \quad r^* \qquad \downarrow \phi^* \qquad \downarrow \phi\phi\phi^*$$

$$D_{L_W\, P_W \cap P_V}(P_W) \leftrightarrow^{q^*}{}_{q_*} D_{L_WP_mP_nP_p}(P_W/U_W \cap P_V) \xrightarrow{t_!} D_{L_WP_mP_nP_p}(G_m\, G_n\, G_p)$$

$$q_* \qquad \downarrow \pi_* \qquad \downarrow \pi\pi\pi_*$$

$$D_{L_W}(P_W/P_W \cap P_V) \xrightarrow{t_!} D_{L_W}(G_m/P_m\, G_n/P_n\, G_p/P_p)$$

$$\alpha_! \qquad \downarrow \alpha\alpha\alpha_!$$

$$D_{L_W}(pt)$$

where

$B = \mathrm{Hom}(b, d \oplus g) \oplus \mathrm{Hom}(c, d \oplus e \oplus g \oplus h) \oplus$
$\oplus \mathrm{Hom}(e, g) \oplus \mathrm{Hom}(f, g \oplus h)$

is isomorphic to $U_W/\ U_W \cap P_V$ as a vector space.

Another single operation-functor is

$$D_{G_jG_kG_l}(pt) \xrightarrow{\phi^*} D_{P_W\, P_V}(P_W \cdot P_V) \xrightarrow{\pi_*} D_{P_W}(P_W/P_W \cap P_V) \xrightarrow{\alpha_!} D_{P_W}(pt)$$

$$r^*\phi^* \quad \downarrow Res \qquad \downarrow Res \qquad \downarrow Res$$

$$D_{L_W\, P_V}(P_W \cdot P_V) \xrightarrow{\pi_*} D_{L_W}(P_W/P_W \cap P_V) \xrightarrow{\alpha_!} D_{L_W}(pt)$$

$$i^*\downarrow \uparrow i_* \qquad q_*$$

$$D_{L_W\, P_W \cap P_V}(P_W)$$

with $O = P_WP_V$, $L_W = G_mG_nG_p$, where i: $P_W \to$
$\to P_{WP_W \cap P_V}P_V = P_W \cdot P_V$ is an induction map. The above two functors turn out to be isomorphic. Indeed, the exterior left-lower paths in both diagrams above give isomorphic functors after identification of the triple product of inverse image functors with the single one.

Indeed, the isomorphism in question

$$(D_{GjGkGl}(pt) \to^{\mathrm{Res}} D_{diag}(pt) \to^{p_!{}^*} D_{PmPnPp}(pt) \to \to^{r^*} D_{LWPW\cap PV}(P_W)) \cong r^*$$

follows from the isomorphism

$$\begin{array}{ccc} D_{PjPkPl}(pt) & \to & D_{PW\cap PV}(pt) \\ \uparrow & & \uparrow \\ D_{diag}(pt) & \to & D_{PmPnPp}(pt) \end{array}$$

3.6. Distinguished triangles

Consider left P_W-invariant and right P_V-invariant subsets $F \subset Y \subset G_V$, such that F is closed in Y. Denote $S = Y - F$ and consider inclusions

$$F/U_V \to {}^i{}_U Y/U_V \leftarrow {}^j{}_U S/U_V,$$

$$F/P_V \to {}^i{}_P Y/P_V \leftarrow {}^j{}_P S/P_V.$$

Denote ϕ^X, π^X, α^X the maps (2) for $O = X \in \{Y, F, S\}$. Let $K \in D_{LV}(pt)$, $K_U = \phi^{Y*}K \in D_{PWLV}(Y/U_V)$, $K_P = \pi_*{}^Y K_U \in D_{PW}(Y/U_V)$ (see the middle row in the following diagram).

$$\begin{array}{ccccccc} D_{LV}(pt) & \to^{\phi^{S*}} & D_{PWLV}(S/U_V) & \to^{\pi_*{}^S} & D_{PW}(S/P_V) & \to^{\alpha_!{}^S} & D_{PW}(pt) \\ \Downarrow & & j_U{}^*\uparrow & & j_P{}^*\uparrow\downarrow j_{P!} & & \Downarrow \\ D_{LV}(pt) & \to^{\phi^{Y*}} & D_{PWLV}(Y/U_V) & \to^{\pi_*{}^Y} & D_{PW}(Y/P_V) & \to^{\alpha_!{}^Y} & D_{PW}(pt) \\ \Downarrow & & i_U{}^*\downarrow & & i_P{}^*\downarrow\uparrow i_{P!} & & \Downarrow \\ D_{LV}(pt) & \to^{\phi^{F*}} & D_{PWLV}(F/U_V) & \to^{\pi_*{}^F} & D_{PW}(F/P_V) & \to^{\alpha_!{}^F} & D_{PW}(pt) \end{array}$$

Applying $\alpha_!^Y$ to the standard triangle for K_P we get a distinguished triangle

$$\alpha_!^Y j_{P!} j_P{}^* K_P \to \alpha_!{}^Y K_P \to \alpha_!{}^Y i_{P!} i_P{}^* K_P \to$$

The above diagram shows that it is isomorphic to certain following sequences:

$$\alpha_!{}^S\pi_*{}^S j_U^* K_U \to \alpha_!{}^Y\pi_*{}^Y K_U \to \alpha_!{}^F\pi_*{}^F i_U^* K_U \to$$

$$\alpha_!{}^S\pi_*{}^S\phi^{S*}K \to \alpha_!{}^Y\pi_*{}^Y\phi^{Y*}K \to \alpha_!{}^F p_*{}^F\phi^{F*}K \to$$

This is the triangle

$$\Psi_W{}^{S;\,V}K \to \Psi_W{}^{Y;\,V}K \to \Psi_W{}^{F;\,V}K \to$$

from which the structure triangle

$$X_W^{S;\,V}K \to X_W^{Y;\,V}K \to X_W^{F;\,V}K \to \qquad (3)$$

is obtained.

This triangle replaces non-existing isomorphism $X_W^{Y;\,V}K \cong X_W^{S;\,V}K \oplus X_W^{F;\,V}K$ (this would be too much to ask).

Theorem. ***For any pair of closed embeddings $F \subset Z \subset Y$ set $S = Y - F$, $Q = Z - F$, $R = Y - Z$. Then there is an octahedron made with distinguished triangles (3) for these spaces.***

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ПРИКЛАД ТРИАНГУЛЬОВАНОЇ КАТЕГОРІЇ ХОПФА: SL(2)

Будуються приклади ізоморфізмів когерентності для триангульованої категорії Хопфа, пов'язаної з SL(2). Це еквіваріантна похідна категорія, оснащена функторами множення та комноження і структурними ізоморфізмами.