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# A generalized Landau–Lifshitz equation for an isotropic SU(3) magnet

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**Abstract**

In this paper we obtain equations for large-scale fluctuations of a mean field (the field of magnetization and quadrupole moments) in a magnetic system realized by a square (cubic) lattice of atoms with spin $s \geqslant 1$ at each site. We use the generalized Heisenberg Hamiltonian with a biquadratic exchange as a quantum model. A quantum thermodynamical averaging gives classical effective models, which are interpreted as Hamiltonian systems on coadjoint orbits of the Lie group SU(3).

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## 1. Introduction

Being a multiparticle quantum system, a magnet can be considered on different levels of hierarchy: a quantum (microscopic) level and a classical (macroscopic) one. The quantum level is described by means of quantum electrodynamics, or by simpler models like the Hubbard model or the Heisenberg one. The most common model for the classical level is the mean field model. The dynamics of a mean field is described by the equations of the Landau–Lifshitz type.

Each model is suitable for describing certain phenomena. For example, the problems of formation of large-scale structures (domain walls, topological solitons, nonlinear magnetization waves and so on) are naturally investigated from a classical point of view. More tenuous problems, such as renormalization of the order parameter according to a temperature or an effective interaction constant, require a quantum point of view [1].

Here we start from the quantum level described by the Heisenberg model. In addition to the usual Heisenberg bilinear interaction $-J(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_m)$, we consider the biquadratic one $-K(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_m)^2$. With the help of much theoretical and experimental research, it was shown that biquadratic interactions have significant effects on magnetic properties. For example, a new ordered state (a nematic state, with zero magnetization) occurs as a separate phase transition [2]. Note that the biquadratic interaction can be taken into account only if a magnetic system has the spin $s \geqslant 1$.

In this paper, we propose a classical generalization of the isotropic Landau–Lifshitz equation corresponding to the Heisenberg model with biquadratic exchange interaction. A transition from the quantum level to the classical one is performed by the mean field approximation. The classical model can be interpreted as a Hamiltonian system on a coadjoint orbit of the unitary group SU(3). Therefore, we acquire an additional mathematical apparatus, which gives a significant advantage.

The mean field approximation gives a qualitative analysis of ordered states [3, 4], but has no answer about their stability. Moreover, in this approximation the temperature dependences of order parameters considerably differ from the observed dependences. This proves it necessary to take into account the fluctuations of the mean field. The proposed effective classical models describe large-scale (or slow) fluctuations of the mean field. One can reach slow fluctuations by an averaging over high frequencies [1]. However, in the context of theory of magnetism, we choose the models associated with the equations of the Landau–Lifshitz type.

The paper is organized as follows. Section 2 is devoted to the quantum model based on the spin Hamiltonian with biquadratic exchange interactions. We consider the SU(3)-invariant case. In section 3, we construct two effective models that describe large-scale fluctuations of a mean field (the field of magnetization and quadrupole moments). We obtain one of them by an averaging of the quantum Hamiltonian over coherent states. The other effective model is a result of an averaging over mixed states. These classical models appear to be Hamiltonian systems on coadjoint orbits of the group SU(3), which follows from SU(3)-invariance of the original quantum model. Each coadjoint orbit is determined by constraints, which are observed quantities becoming rigid after averaging. In section 4, we summarize results and give some ideas how to extend the proposed scheme to magnetic systems with higher spins.

## 2. Quantum model of the magnetic system

### 2.1. *Description of the model*

The magnetic system in question is realized by a homogeneous lattice of atoms with the spin $s \geqslant 1$ at each site. The lattice can be one, two or three dimensional, and has the distance $l$ between the nearest-neighbor sites. We assign three *spin operators* $\left(\hat{S}_n^1, \hat{S}_n^2, \hat{S}_n^3\right)$ to each site $n$; they obey the standard commutation relations:

$$\left[\hat{S}_n^\alpha, \hat{S}_m^\beta\right] = \mathrm{i}\varepsilon^{\alpha\beta\gamma}\hat{S}_n^\gamma \delta_{nm},$$

where $\alpha, \beta, \gamma$ run over the set $\{1, 2, 3\}$ and $\delta_{nm}$ denotes the Kronecker symbol.

We use the localized spin model for the magnetic system. In many cases this model adequately describes a magnetic system by the Heisenberg Hamiltonian, which includes only the bilinear exchange interaction. Nevertheless, there are many magnets that require taking into account higher powers of the exchange interaction. Our model is applicable to magnets with the spin $s \geqslant 1$.

In the present paper, we consider the Hamiltonian with a biquadratic exchange and call it *bilinear-biquadratic*:

$$\hat{\mathcal{H}} = -\sum_{n,\delta}\{J(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta}) + K(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta})^2\}, \tag{1}$$

where $\hat{\boldsymbol{S}}_n = \left(\hat{S}_n^1, \hat{S}_n^2, \hat{S}_n^3\right)$ is a vector of spin operators at site $n$ and $\delta$ runs over the nearest-neighbor sites. This Hamiltonian was discussed, for example, in [2–5]. The constants $J$ and $K$ serve as exchange integrals. We suppose that $J$ and $K$ are positive. It means that we consider a ferromagnetic interaction in preference.

The operators $\left\{\hat{S}_n^\alpha\right\}$ (here $n$ is fixed) are defined over the $(2s+1)$-dimensional space of irreducible representation of the group SU(2). They generate an associative matrix algebra over this space. The complete matrix algebra can be represented as a direct sum of irreducible sets of tensor operators with respect to the action $\mathrm{ad}_{\hat{S}^\alpha}$. In the case of $s = 1$, we have $\mathrm{Mat}_{3\times 3} \simeq [9] = [1] + [3] + [5]$. Evidently, the operators $\left\{\hat{S}_n^\alpha\right\}$ form a basis in the three-dimensional irreducible set. One can construct a basis in the five-dimensional irreducible set from the tensor operators of weight 2. These are the *quadrupole operators* $\left\{\hat{Q}_n^{12}, \hat{Q}_n^{13}, \hat{Q}_n^{23}, \hat{Q}_n^{[2,2]}, \hat{Q}_n^{[2,0]}\right\}$ defined by the formulae

$$\hat{Q}_n^{\alpha\beta} = \hat{S}_n^\alpha \hat{S}_n^\beta + \hat{S}_n^\beta \hat{S}_n^\alpha, \qquad \alpha \neq \beta,$$
$$\hat{Q}_n^{[2,2]} = \left(\hat{S}_n^1\right)^2 - \left(\hat{S}_n^2\right)^2, \qquad \hat{Q}_n^{[2,0]} = \sqrt{3}\left(\left(\hat{S}_n^3\right)^2 - \tfrac{2}{3}\right).$$

The spin and quadrupole operators are normalized by the following relation:

$$\mathrm{Tr}(\hat{P})^2 = \tfrac{1}{3}s(s+1)(2s+1).$$

As $s = 1$, we have $\mathrm{Tr}(\hat{P})^2 = 2$. The chosen normalization is matched to the relation $\left(\hat{S}_n^1\right)^2 + \left(\hat{S}_n^2\right)^2 + \left(\hat{S}_n^3\right)^2 = s(s+1)$.

Now, fix the canonical basis $\{|+1\rangle, |-1\rangle, |0\rangle\}$ in the space of representation. Then, one obtains the following matrix representation for the spin and quadrupole operators:

$$\hat{S}_n^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \hat{S}_n^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} \\ \mathrm{i} & -\mathrm{i} & 0 \end{pmatrix},$$
$$\hat{S}_n^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{Q}_n^{[2,0]} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$
$$\hat{Q}_n^{12} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{Q}_n^{13} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix},$$
$$\hat{Q}_n^{23} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} \\ \mathrm{i} & \mathrm{i} & 0 \end{pmatrix}, \qquad \hat{Q}_n^{[2,2]} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We denote all spin and quadrupole operators: $\left\{\hat{S}_n^1, \hat{S}_n^2, \hat{S}_n^3, \hat{Q}_n^{12}, \hat{Q}_n^{13}, \hat{Q}_n^{23}, \hat{Q}_n^{[2,2]}, \hat{Q}_n^{[2,0]}\right\}$ by $\left\{\hat{P}_n^a\right\}_{a=1}^8$. The operators $\left\{\hat{P}_n^a\right\}$ obey the following commutation relations:

$$\left[\hat{P}_n^a, \hat{P}_m^b\right] = \mathrm{i}C_{abc}\hat{P}_n^c\delta_{nm},$$

where $C_{abc}$ are structure constants; the nonzero components are

$$C_{123} = C_{145} = C_{167} = C_{264} = C_{257} = C_{356} = 1,$$
$$C_{168} = C_{528} = \sqrt{3}, \qquad C_{437} = 2.$$

The Hamiltonian (1) becomes bilinear in terms of $\left\{\hat{P}_n^a\right\}$:

$$\hat{\mathcal{H}} = -\left(J - \frac{1}{2}K\right)\sum_{n,\delta}\sum_{\alpha}\hat{S}_n^\alpha\hat{S}_{n+\delta}^\alpha - \frac{1}{2}K\sum_{n,\delta}\sum_{a}\hat{Q}_n^a\hat{Q}_{n+\delta}^a - \frac{4}{3}KN, \tag{2}$$

where $N$ denotes the total number of sites. Obviously, the Hamiltonian is SU(2)-invariant, and one can transform the operators $\{\hat{S}_n^\alpha\}$ and $\{\hat{Q}_n^a\}$ by the formulae of adjoint representation:

$$\hat{U}\hat{S}_n^\alpha\hat{U}^{-1} = \sum_\beta \hat{D}^{\alpha\beta}(\hat{U})\hat{S}_n^\beta, \qquad \hat{D}^{\alpha\beta} \in \mathrm{SO}(3),$$
$$\hat{U}\hat{Q}_n^a\hat{U}^{-1} = \sum_b \hat{D}^{ab}(\hat{U})\hat{Q}_n^b, \qquad \hat{D}^{ab} \in \mathrm{SO}(5),$$

where $\hat{D}^{\alpha\beta}(\hat{U})$ and $\hat{D}^{ab}(\hat{U})$ are matrices of the real irreducible three- and five-dimensional representations of the group SU(2), respectively, and $\hat{U} = \exp\left\{\sum_\alpha \varphi_\alpha \hat{S}_n^\alpha\right\}$, where $\{\varphi_\alpha\}$ are group parameters. As $K = J$ the SU(2)-symmetry is extended to the SU(3)-one, and the Hamiltonian (2) gets the form

$$\hat{\mathcal{H}} = -\frac{1}{2}J\sum_{n,\delta}\sum_a \hat{P}_n^a\hat{P}_{n+\delta}^a - \frac{4}{3}JN. \tag{3}$$

## 2.2. *Mean field approach and ordered states*

Instead of interactions between the spin and quadrupole operators $\{\hat{P}_n^a\}$ according to the Hamiltonian (2), we consider effective interactions of the operators $\{\hat{P}_n^a\}$ with a classical mean field. We suppose that the components of the mean field at site $n$ are proportional to averages (quasiaverages) of the quantum operators $\{\hat{P}_n^a\}$.

In the mean field approximation, the Hamiltonian (2) has the form

$$\hat{\mathcal{H}}_{\mathrm{MF}} = -\left(J - \frac{1}{2}K\right)z\sum_n\sum_\alpha \hat{S}_n^\alpha\langle\hat{S}_n^\alpha\rangle - \frac{1}{2}Kz\sum_n\sum_a \hat{Q}_n^a\langle\hat{Q}_n^a\rangle - \frac{4}{3}KNz, \tag{4}$$

where $z$ is a number of the nearest-neighbor sites. We have to give a warning about the averages of $\{\hat{P}_n^\alpha\}$. If one calculates the averages by means of the density matrix $\hat{\rho}(T) = \exp\left\{-\frac{\mathcal{H}}{kT}\right\}$, one obtains zeros. This follows from the SU(2)-symmetry of the Hamiltonian (2). Nonzero values of the averages appear if the symmetry is broken. Symmetry breaking can be stimulated by an external magnetic field that vanishes after specifying an order in the magnetic system. Such averages are called *quasiaverages* [6].

Suppose that the magnetic system in question has nonzero quasiaverages $\{\langle\hat{P}_n^\alpha\rangle\}$. They form a classical 8-component vector field $\{\mu_a(\boldsymbol{x}_n)\}_{a=1}^8$, which we call a *mean field*. Suppose that the mean field is constant over the whole magnetic system. This happens in the case of thermodynamic equilibrium and an infinite lattice. Then under an action of the group SU(2), the Hamiltonian (4) can be reduced to a diagonal form, namely:

$$\begin{aligned}\hat{\mathcal{H}}_{\mathrm{MF}} &= -\left(J - \frac{1}{2}K\right)z\sum_n \hat{S}_n^3\langle\hat{S}_n^3\rangle - \frac{1}{2}Kz\sum_n \hat{Q}_n^{[2,0]}\langle\hat{Q}_n^{[2,0]}\rangle - \frac{4}{3}KNz \\ &= -z\sum_n\left\{\left(J - \frac{1}{2}K\right)\hat{S}_n^3\mu_3 + \frac{1}{2}K\hat{Q}_n^{[2,0]}\mu_8 + \frac{4}{3}K\right\},\end{aligned}$$

where the components $\mu_3 = \langle\hat{S}^3\rangle$ and $\mu_8 = \langle\hat{Q}^{[2,0]}\rangle$ do not depend on the spatial point $\boldsymbol{x}_n$. These components are suitable to be *order parameters*. Evidently, $\mu_3$ describes a normalized magnetization (a ratio of the $z$-projection of magnetic moment to a saturation magnetization) and $\mu_8$ is similarly connected to a quadrupole moment.

Now we briefly show that the proposed quantum model admits ordered states. In the mean field approximation a partition function is calculated by the formula

$$Z(\mu_3, \mu_8, T) = \mathrm{Tr}\, \mathrm{e}^{-\frac{h_{\mathrm{MF}}}{kT}},$$

where $h_{\mathrm{MF}}$ denotes the one-site Hamiltonian:

$$h_{\mathrm{MF}} = -\left(J - \tfrac{1}{2}K\right)\mu_3 \hat{S}^3 - \tfrac{1}{2}K\mu_8 \hat{Q}^{[2,0]} - \tfrac{4}{3}K.$$

The mean field mentioned exists if self-consistent relations are held, in other words, if the system

$$\mu_3 = \langle \hat{S}^3\rangle_{\mathrm{MF}} = \frac{\mathrm{Tr}\ \hat{S}^3\,\mathrm{e}^{-\frac{h_{\mathrm{MF}}}{kT}}}{\mathrm{Tr}\ \mathrm{e}^{-\frac{h_{\mathrm{MF}}}{kT}}}, \qquad \mu_8 = \langle \hat{Q}^{[2,0]}\rangle_{\mathrm{MF}} = \frac{\mathrm{Tr}\ \hat{Q}^{[2,0]}\,\mathrm{e}^{-\frac{h_{\mathrm{MF}}}{kT}}}{\mathrm{Tr}\ \mathrm{e}^{-\frac{h_{\mathrm{MF}}}{kT}}}.$$

has a solution. After calculation of the mean field averages, one obtains the self-consistent relations in the form

$$\mu_3 = \frac{2\sinh\frac{(J-\frac{K}{2})\mu_3}{kT}}{\exp\left\{-\frac{\sqrt{3}K\mu_8}{2kT}\right\} + 2\cosh\frac{(J-\frac{K}{2})\mu_3}{kT}},$$

$$\mu_8 = \frac{2}{\sqrt{3}}\,\frac{\cosh\frac{(J-\frac{K}{2})\mu_3}{kT} - \exp\left\{-\frac{\sqrt{3}K\mu_8}{2kT}\right\}}{\exp\left\{-\frac{\sqrt{3}K\mu_8}{2kT}\right\} + 2\sinh\frac{(J-\frac{K}{2})\mu_3}{kT}}.$$

The solutions of the system correspond to ordered states of the magnetic system in question.

An evident solution is the paramagnetic state ($\mu_3 = 0, \mu_8 = 0$). All other solutions depend on a temperature $T$ and the exchange integrals $J$ and $K$. Note that we consider the ferromagnetic interaction in preference: $J > 0$. Nontrivial solutions appear at temperatures lower than the critical one: $T_{\mathrm{crit}} = \frac{2}{3k}\left(J - \frac{1}{2}K\right)$. As $K<0$ there exists a ferromagnetic state with the values $\left(\mu_3 = 1, \mu_8 = \frac{1}{\sqrt{3}}\right)$ at zero temperature, and a nematic state with the values $\left(\mu_3 = 0, \mu_8 = \frac{1}{\sqrt{3}}\right)$ at zero temperature. As $K > 0$ there exist four nontrivial solutions: two ferromagnetic states with the values $\left(\mu_3 = 1, \mu_8 = \frac{1}{\sqrt{3}}\right)$ and $\left(\mu_3 = \frac{2}{3}, \mu_8 = \frac{-1}{2\sqrt{3}}\right)$ at zero temperature and two nematic states with the values $\left(\mu_3 = 0, \mu_8 = \frac{-2}{\sqrt{3}}\right)$ and $\left(\mu_3 = 0, \mu_8 = \frac{1}{\sqrt{3}}\right)$ at zero temperature. The same states are mentioned in [3, 4]. The states $\left(\mu_3 = 1, \mu_8 = \frac{1}{\sqrt{3}}\right)$ and $\left(\mu_3 = 0, \mu_8 = \frac{-2}{\sqrt{3}}\right)$ are stable. The problem of transient processes in the mean field approach is discussed, for example, in [4]. An analysis of solutions of the self-consistent relations proves that ordered states in the proposed model exist.

In the following, we deal with the case $J = K$, which corresponds to the boundary between the ferromagnetic and the nematic regions (see the phase diagram of the bilinear-biquadratic $s = 1$ model in [5]). In this case, the Hamiltonian (2) and its mean field approximation are SU(3)-invariant. The latter gets the form

$$\hat{\mathcal{H}}_{\mathrm{MF}} = -\frac{1}{2}Jz\sum_n\sum_a \hat{P}_n^a\langle\hat{P}_n^a\rangle - \frac{4}{3}JNz = -\frac{1}{2}Jz\sum_n\sum_a \hat{P}_n^a\mu_a - \frac{4}{3}JNz. \tag{5}$$

### *2.3. Motion equations for large-scale fluctuations of the mean field*

Return to the quantum SU(3)-invariant spin model with the Hamiltonian (3). The Heisenberg equation for an evolution of $\hat{P}_n^a$ has the form

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{P}_n^a}{\mathrm{d}t} = \left[\hat{P}_n^a, \hat{\mathcal{H}}\right]. \tag{6}$$

We suppose that the magnetic system is ordered; then we take an average of equation (6) over the Heisenberg (time-independent) coherent states:

$$|\psi(n)\rangle = \frac{1}{\sqrt{N}}(c_1(n)|1\rangle + c_{-1}(n)|-1\rangle + c_0(n)|0\rangle),$$

$$|c_1|^2 + |c_{-1}|^2 + |c_0|^2 = 1.$$

Alternatively, one can take an average by means of the density matrix. In both cases, we neglect correlations between fluctuations of the quantum fields $\{\hat{P}_n^a\}_{a=1}^8$ at distinct sites, that is

$$\langle \hat{P}_n^a \hat{P}_m^b \rangle \approx \langle \hat{P}_n^a \rangle \langle \hat{P}_m^b \rangle = \mu_a(\boldsymbol{x}_n)\mu_b(\boldsymbol{x}_m). \tag{7}$$

An averaging of equation (6) results in the following equation for $\mu_a(\boldsymbol{x}_n)$:

$$\hbar \frac{\partial \mu_a(\boldsymbol{x}_n)}{\partial t} = 2Jl^2 C_{abc}\mu_b(\boldsymbol{x}_n)(\mu_{c,xx}(\boldsymbol{x}_n) + \mu_{c,yy}(\boldsymbol{x}_n)), \tag{8}$$

which is a Hamiltonian one with respect to the Lie–Poisson bracket.

In order to investigate large-scale fluctuations of the mean field $\{\mu_a(\boldsymbol{x}_n)\}_{a=1}^8$, we consider a continuum space instead of the discrete lattice. This can be achieved by the well-known limiting process. In the case of an SU(2)-magnetic system (only bilinear interactions are taken into account), this limiting process underlies the macroscopic phenomenological theory of magnetism [7]. The limiting process replaces quantum operators by densities of their averages, which serve as dynamical variables. In our case, we deal with the densities $M_a$ of averages of the spin and quadrupole moments:

$$M_a(x) = \sum_n \mu_a(\boldsymbol{x}_n)\delta(\boldsymbol{x}, \boldsymbol{x}_n), \qquad \delta(\boldsymbol{x}, \boldsymbol{x}_n) = \begin{cases} \dfrac{1}{V_0} & \boldsymbol{x}_n \in U(\boldsymbol{x}) \\ 0 & \boldsymbol{x}_n \notin U(\boldsymbol{x}), \end{cases}$$

where $V_0$ denotes a physically infinitesimal region of the lattice and $U(\boldsymbol{x})$ is the infinitesimal neighborhood of $\boldsymbol{x}$. The Lie–Poisson bracket for $\{M_a(\boldsymbol{x})\}$ is defined by

$$\{M_a(\boldsymbol{x}), M_b(\boldsymbol{y})\} = C_{abc}M_c(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{y}),$$

where $\delta(\boldsymbol{x})$ is the Dirac function. Since dimensionless quantities are more suitable, we introduce $\mu_a(\boldsymbol{x}) = V_0 M_a(\boldsymbol{x})$ instead of $M_a(\boldsymbol{x})$. Then, equation (8) gets the form

$$\begin{aligned} &\hbar \frac{\partial \mu_a(\boldsymbol{x})}{\partial t} = \{\mathcal{H}_{\text{eff}}, \mu_a(\boldsymbol{x})\} = V_0 C_{abc}\mu_b(\boldsymbol{x}) \frac{\delta \mathcal{H}_{\text{eff}}}{\delta \mu_c}, \\ &\mathcal{H}_{\text{eff}} = \frac{J}{l^{d-2}} \int \sum_a \left\langle \frac{\partial \mu_a}{\partial \boldsymbol{x}}, \frac{\partial \mu_a}{\partial \boldsymbol{x}} \right\rangle \mathrm{d}^d \boldsymbol{x}, \end{aligned} \tag{9}$$

where $l$ is the lattice distance and $d$ is the lattice dimension. Note that in the two-dimensional case, we obtain a scale-invariant Hamiltonian.

Evidently, (9) is a generalization of the well-known Landau–Lifshitz equation to the case of an 8-component vector field $\{\mu_a\}$. In the same way one can obtain the standard Landau–Lifshitz equation, if a spin system with $s = \frac{1}{2}$ over the two-dimensional space of representation of SU(2) is considered.

We rewrite (9) in the matrix form

$$\hbar \frac{\partial \hat{\mu}}{\partial t} = \frac{2JV_0}{l^{d-2}}[\hat{\mu}, \Delta\hat{\mu}], \qquad \hat{\mu} = \sum_a \mu_a \hat{P}^a. \tag{10}$$

Here $\hat{\mu}$ is a Hermitian $3 \times 3$ matrix, $[\cdot, \cdot]$ denotes the matrix commutator and $\Delta$ is the Laplas operator. Being SU(3)-invariant equation (10) as well as (9) preserves the quantities $h_0 = \frac{1}{2}\,\mathrm{Tr}\,\hat{\mu}^2$ and $f_0 = \frac{1}{2}\,\mathrm{Tr}\,\hat{\mu}^3$, which we call invariants. They serve as constraints for the Hamiltonian system and define the manifold where the vector field $\{\mu_a\}$ lives. At the same time, this manifold is an orbit of the coadjoint representation of the group SU(3).

## 3. Classical Hamiltonian systems on coadjoint orbits of SU(3)

In the one-dimensional case, the Hamiltonian system (9) appears to be integrable, which is shown below by means of the orbital approach.

*3.1. Phase space for an SU(3)-symmetric generalization of the Landau–Lifshitz equation*

In this section, we briefly construct the orbital interpretation of a finite-zone phase space for the SU(3)-symmetric generalization of the Landau–Lifshitz equation.

Consider an algebra of polynomials in $\lambda$ with coefficients from the Lie algebra $\mathfrak{su}(3)$. Denote by $\widetilde{\mathfrak{g}}_+$ the algebra $\mathfrak{su}(3) \otimes \mathcal{P}(\lambda)$, where $\mathcal{P}(\lambda)$ is a ring of polynomials in $\lambda$ with the standard multiplication. Let $A, B \in \widetilde{\mathfrak{g}}_+$ have the form

$$A(\lambda) = \sum_{n=0}^{N+1} \hat{A}^n \lambda^n, \qquad B(\lambda) = \sum_{k=0}^{N+1} \hat{B}^k \lambda^k, \qquad \hat{A}^n, \hat{B}^k \in \mathfrak{su}(3).$$

Then

$$[A, B] = \sum_{n,k} [\hat{A}^n, \hat{B}^k] \lambda^{n+k} \in \widetilde{\mathfrak{g}}_+. \tag{11}$$

The operation (11) turns $\widetilde{\mathfrak{g}}_+$ into a graded Lie algebra.

Let $\hat{P}^{a,n} = \lambda^n \hat{P}^a$, where $a$ runs from 1 to 8. The set $\{\hat{P}^{a,n}\}$ serves as a basis in $\widetilde{\mathfrak{g}}_+$. Recall that $[\hat{P}^a, \hat{P}^b] = \mathrm{i} C_{abc} \hat{P}^c$; the nonzero components $C_{abc}$ have the following values:

$$C_{123} = C_{145} = C_{167} = C_{264} = C_{257} = C_{356} = 1,$$
$$C_{168} = C_{528} = \sqrt{3}, \qquad C_{437} = 2.$$

Introduce a bilinear ad-invariant form on $\widetilde{\mathfrak{g}}_+$ by

$$\langle A, B \rangle = \tfrac{1}{2} \operatorname{res} \lambda^{-N-2} \operatorname{Tr} A(\lambda) B(\lambda). \tag{12}$$

The basis $\{\hat{P}^{a,n}\}$ is orthonormal with respect to the bilinear form. Let $\mathcal{M} = \widetilde{\mathfrak{g}}_+^*$ be a dual space to the algebra $\widetilde{\mathfrak{g}}_+$ with respect to (12). Orthonormality of $\{\hat{P}^{a,n}\}$ implies that $\{\hat{P}^{a,n}\}$ also form a basis in $\mathcal{M}$. Consider the following elements of $\mathcal{M}$:

$$\hat{\mu}(\lambda) = \sum_{n=0}^{N} \sum_{a=1}^{8} \mu_a^n \lambda^n \hat{P}^a + \left(\mu_3^{N+1} \hat{P}^3 + \mu_8^{N+1} \hat{P}^8\right) \lambda^{N+1}.$$

The functions $\hat{\mu}(\lambda)$ form a closed ad-invariant subset of $\mathcal{M}$; we denote it by $\mathcal{M}^{N+1}$. One can compute the coordinate $\mu_a^n$ of $\hat{\mu}(\lambda)$ by the formula

$$\mu_a^n = \langle \hat{\mu}(\lambda), \hat{P}^{a,-n+N+1} \rangle.$$

Define a Lie–Poisson bracket in $\mathcal{C}(\mathcal{M}^{N+1})$ as

$$\{f_1, f_2\} = \sum_{m,n} \sum_{a,b}^{8} W_{ab}^{mn} \frac{\partial f_1}{\partial \mu_a^m} \frac{\partial f_2}{\partial \mu_b^n} \tag{13}$$

with the Poisson tensor field

$$W_{ab}^{mn} = \langle \hat{\mu}(\lambda), [\hat{P}^{a,-m+N+1}, \hat{P}^{b,-n+N+1}] \rangle.$$

Also introduce two ad-invariant functions $I_2(\lambda)$ and $I_3(\lambda)$ by the formulae

$$I_2(\lambda) = \frac{1}{2} \operatorname{Tr} \hat{\mu}^2(\lambda) = \sum_a \mu_a^2(\lambda),$$

$$I_3(\lambda) = \frac{1}{2} \operatorname{Tr} \hat{\mu}^3(\lambda) = \sqrt{\frac{5}{3}} d_{abc} \mu_a(\lambda) \mu_b(\lambda) \mu_c(\lambda),$$

where $d_{abc} = \frac{\sqrt{3}}{4\sqrt{5}} \operatorname{Tr} (\hat{P}^a \hat{P}^b \hat{P}^c + \hat{P}^b \hat{P}^a \hat{P}^c)$, and $\mu_a(\lambda)$ denotes the polynomial

$$\mu_a(\lambda) = \mu_a^0 + \mu_a^1 \lambda + \mu_a^2 \lambda^2 + \cdots + \mu_a^{N+1} \lambda^{N+1}.$$

The invariant functions are also polynomials in $\lambda$:

$$I_2(\lambda) = h_0 + h_1\lambda + \cdots + h_{2N+2}\lambda^{2N+2},$$

$$I_3(\lambda) = f_0 + f_1\lambda + \cdots + f_{3N+3}\lambda^{3N+3}.$$

It is easy to prove that the coefficients $\{h_0, \ldots, h_{N+1}, f_0, \ldots, f_{N+1}\}$ are annihilators with respect to the bracket (13). We fix these coefficients and obtain the system of algebraic equations:

$$h_n = \text{const}, \qquad f_n = \text{const}, \qquad n = 0, \ldots, N+1, \tag{14}$$

which determines an embedding of an orbit $\mathcal{O}^{N+1}$ of dimension $6(N+1)$ into $\mathcal{M}^{N+1}$. The coefficients $\{h_{N+2}, \ldots, h_{2N+2}, f_{N+2}, \ldots, f_{3N+3}\}$ are pairwise commutative integrals of motion. We call them Hamiltonians. In the one-dimensional case, the number of Hamiltonians is sufficient for integrability of the Hamiltonian system on an orbit.

Here we are interested in two Hamiltonians: $h_{N+2}, h_{N+3}$, and the corresponding Hamiltonian flows. The Hamiltonian $h_{N+2}$ gives rise to the stationary flow

$$\frac{\partial \mu_a^n}{\partial x} = \left\{\mu_a^n, h_{N+2}\right\} = 2C_{abc}\mu_b^0\mu_c^{n+1}, \qquad a = 1, \ldots, 8. \tag{15}$$

The Hamiltonian $h_{N+3}$ gives rise to the evolutionary flow

$$\frac{\partial \mu_a^n}{\partial t} = \left\{\mu_a^n, h_{N+3}\right\} = 2C_{abc}\left(\mu_b^0\mu_c^{n+2} + \mu_b^1\mu_c^{n+1}\right), \qquad a = 1, \ldots, 8. \tag{16}$$

Equations (15) and (16) are compatible, for the corresponding Hamiltonians commute: $\{h_{N+2}, h_{N+3}\} = 0$. Thus, (16) describes an evolution on the trajectories of (15), that is, the dynamical variables $\left\{\mu_a^n\right\}$ in (16) depend on $x$. From (15) and (16), we have

$$\frac{\partial \mu_a^0}{\partial t} = 2C_{abc}\mu_b^0\mu_c^2 = \frac{\partial \mu_a^1}{\partial x}. \tag{17}$$

The variables $\left\{\mu_a^1\right\}$ can be expressed in terms of $\left\{\mu_a^0\right\}$ and $\left\{\frac{\partial}{\partial x}\mu_a^0\right\}$; then (17) becomes a closed system of partial equations for $\left\{\mu_a^0\right\}$. In order to compute the variables $\left\{\mu_a^1\right\}$, one has to solve the following degenerate system of equations of the stationary flow:

$$\frac{\partial \mu_a^0}{\partial x} = 2C_{abc}\mu_b^0\mu_c^1, \qquad a = 1, \ldots, 8. \tag{18}$$

It becomes possible if one restricts the system to the orbit $\mathcal{O}^{N+1} \subset \mathcal{M}^{N+1}$.

### *3.2. Classification of orbits of SU(3)*

It is evident that the orbit $\mathcal{O}^{N+1}$ defined by (14) is a vector bundle over a coadjoint orbit of the group SU(3). That is why we need to classify orbits of SU(3).

The group SU(3) is simple [8]; hence, its algebra $\mathfrak{g} \simeq \mathfrak{su}(3)$ coincides with the dual space $\mathfrak{g}^*$. Consequently, the coordinates $\{\mu_a\}$ in $\mathfrak{g}^*$ can be regarded as coordinates in $\mathfrak{su}(3)$ as well as in $\mathfrak{su}^*(3)$. A generic element $\hat{\mu} \in \mathfrak{su}^*(3)$ has the form

$$\hat{\mu} = \begin{pmatrix} \mu_3 + \frac{1}{\sqrt{3}}\mu_8 & \mu_7 - \mathrm{i}\mu_4 & \frac{1}{\sqrt{2}}(\mu_1 - \mathrm{i}\mu_6 + \mu_5 - \mathrm{i}\mu_2) \\ \mu_7 + \mathrm{i}\mu_4 & -\mu_3 + \frac{1}{\sqrt{3}}\mu_8 & \frac{1}{\sqrt{2}}(\mu_1 - \mathrm{i}\mu_6 - \mu_5 + \mathrm{i}\mu_2) \\ \frac{1}{\sqrt{2}}(\mu_1 + \mathrm{i}\mu_6 + \mu_5 + \mathrm{i}\mu_2) & \frac{1}{\sqrt{2}}(\mu_1 + \mathrm{i}\mu_6 - \mu_5 - \mathrm{i}\mu_2) & -\frac{2}{\sqrt{3}}\mu_8 \end{pmatrix}. \tag{19}$$

Let $\mathfrak{h}$ be the maximal commutative subalgebra (also called the Cartan subalgebra) of $\mathfrak{g}$. The dual space $\mathfrak{h}^*$ to the Cartan subalgebra $\mathfrak{h}$ coincides with $\mathfrak{h}$.

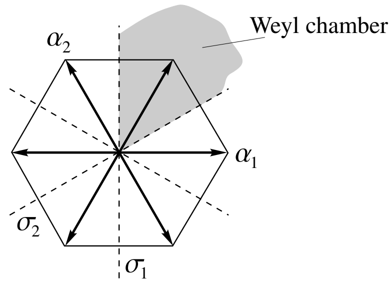


**Figure 1.** Root diagram of SU(3).

By definition, the set $\mathcal{O}_{\hat{\mu}_{\text{in}}} = \{g^{-1}\hat{\mu}_{\text{in}}g, \forall g \in \text{SU}(3)\}$ is the *coadjoint orbit* of SU(3) through an *initial point* $\hat{\mu}_{\text{in}} \in \mathfrak{su}^*(3)$. All elements $g' \in \text{SU}(3)$ such that $g'^{-1}\hat{\mu}_{\text{in}}g' = \hat{\mu}_0$ form the stationary subgroup $\text{S}_{\hat{\mu}_{\text{in}}}$ at $\hat{\mu}_{\text{in}}$. The orbit $\mathcal{O}_{\hat{\mu}_{\text{in}}}$ is a homogeneous space, which is diffeomorphic to the coset space $\text{SU}(3)/\text{S}_{\hat{\mu}_{\text{in}}}$. There exist two types of orbits of SU(3): the *generic* $\mathcal{O}_{\text{gen}} = \frac{\text{SU}(3)}{\text{U}(1)\times\text{U}(1)}$ of dimension 6 and the *degenerate* $\mathcal{O}_{\text{deg}} = \frac{\text{SU}(3)}{\text{SU}(2)\times\text{U}(1)}$ of dimension 4.

It is proven by R Bott that each orbit of the coadjoint action of a semisimple group G intersects $\mathfrak{h}^*$ precisely in an orbit of the Weyl group W(G).

The full Weyl group of SU(3) consists of six elements $\{e, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$, where $\sigma_1$ and $\sigma_2$ are reflections across the hyperplanes orthogonal to the simple roots $\alpha_1$ and $\alpha_2$ respectively (see figure 1). The open domain $C = \{\hat{\mu} \in \mathfrak{h}^*, \langle\hat{\mu}, \alpha\rangle > 0, \forall\alpha \in \Delta^+\}$ is called a positive Weyl chamber. Here, $\Delta^+$ denotes the set of positive roots. We call the set $\Gamma_\alpha = \{\hat{\mu} \in \mathfrak{h}^*, \langle\hat{\mu}, \alpha\rangle = 0\}$ a wall of the Weyl chamber. An orbit of the Weyl group W(G) is obtained by the action of W(G) on a point of $\overline{C}$. Each orbit of the Weyl group W(G) and, consequently, each coadjoint orbit of G intersects the positive Weyl chamber at only one point. That is why we can classify the coadjoint orbits of G by the points of the positive Weyl chamber.

In the case of group SU(3), there exist two types of orbits of the Weyl group. A generic orbit contains six elements and passes through the interior of the positive Weyl chamber. A degenerate orbit contains three elements and passes through a wall of the positive Weyl chamber. According to this, we call an orbit of SU(3) a *generic* one if $\hat{\mu}_{\text{in}}$ lies in the interior of the positive Weyl chamber and a *degenerate* one if $\hat{\mu}_{\text{in}}$ belongs to a wall of the positive Weyl chamber.

In our case, $\hat{\mu}_{\text{in}}$ has the following diagonal form:

$$\hat{\mu}_{\text{in}} = \text{diag}\big(m + \tfrac{1}{\sqrt{3}}q, -m + \tfrac{1}{\sqrt{3}}q, -\tfrac{2}{\sqrt{3}}q\big),$$

where $m$ and $q$ denote initial values of the variables $\mu_3$ and $\mu_8$, respectively, or boundary values (at zero temperature) of the corresponding components of the mean field. As $m > 0, q > 0$, the coadjoint action of SU(3) gives a generic orbit. If $m = 0$ or $m = \sqrt{3}q$, we obtain a degenerate orbit. In the following, we consider degenerate orbits with $m = 0$.

### 3.3. Hamiltonian equations on orbits of SU(3)

Return to the system of equations (18), which is degenerate in $\mathcal{M}^{N+1}$. However, it can be solved if one restricts the system to the orbit $\mathcal{O}^{N+1} \subset \mathcal{M}^{N+1}$. Each orbit is determined by the following equation [9]:

$$\chi_{\min}(\hat{\mu}) = 0, \tag{20}$$

where $\chi_{\min}(\hat{\mu})$ is the minimal characteristic polynomial in $\hat{\mu} \in \mathcal{O}^{N+1}$. Equation (20) serves as a constraint for the system (18), which has the form

$$\frac{\partial \hat{\mu}^0}{\partial x} = \mathrm{Ad}_{\hat{\mu}^0} \hat{\mu}^1. \tag{21}$$

Now we solve (21) on orbits of the group SU(3).

A degenerate orbit is determined by the equation

$$\hat{\mu}^2 + \sqrt{\frac{h_0}{3}}\hat{\mu} - \frac{2h_0}{3} = 0,$$

where $h_0 = q^2 = \text{const}$. Using this constraint, one obtains the following solution of (21): $\mu_a^1 = \frac{1}{6h_0} C_{abc}\mu_b^0\mu_{c,x}^0 + \frac{h_1}{2h_0}\mu_a^0$, where $\frac{h_1}{2h_0}\mu_a^0$ is an element of $\mathrm{Ker}\,\mathrm{Ad}_{\hat{\mu}^0}$. The motion equation (17) on the degenerate orbit has the form

$$\frac{\partial \mu_a}{\partial t} = \frac{8\mathcal{A}}{3h_0} C_{abc}\mu_b\mu_{c,xx} + \frac{8\mathcal{A}h_1}{h_0}\mu_{a,x}, \tag{22}$$

where we write $\mu_a$ instead of $\mu_a^0$ and scale the flow parameter $t$ by $16\mathcal{A}$. The dimensional constant $\mathcal{A}$ provides a correspondence between (22) as $h_1 = 0$ and (10) as $d = 1$. That is, (22) describes large-scale fluctuations of the mean field $\{\mu_a\}$.

A generic orbit is determined by the characteristic equation

$$\hat{\mu}^3 - h_0\hat{\mu} - \tfrac{2}{3} f_0 = 0,$$

where $h_0 = m^2 + q^2$ and $f_0 = \frac{1}{\sqrt{3}}(3m^2 q - q^3)$. On this orbit, we obtain the following solution of (21):

$$\mu_a^1 = \frac{1}{8\left(h_0^3 - 3f_0^2\right)}\left(h_0^2 C_{abc}\mu_b^0\mu_{c,x}^0 - 2\sqrt{3} f_0 C_{abc}\eta_b^0\mu_{c,x}^0 + h_0 C_{abc}\eta_b^0\eta_{c,x}^0\right) \\ + \frac{2f_0 f_1 - 3h_0^2 h_1}{6\left(f_0^2 - h_0^3\right)}\mu_a^0 + \frac{3f_0 h_1 - 2h_0 f_1}{6\sqrt{3}\left(f_0^2 - h_0^3\right)}\eta_a^0,$$

where $\eta_a^0 = \sqrt{5} d_{abc}\mu_b^0\mu_c^0$. The motion equation (17) on the generic orbit has the form

$$\begin{aligned}\frac{\partial \mu_a}{\partial t} = {} & \frac{2\mathcal{A}}{h_0^3 - 3f_0^2}\left(h_0^2 C_{abc}\mu_b\mu_{c,xx} - \sqrt{3} f_0 C_{abc}\mu_b\eta_{c,xx}\right. \\ & \left. - \sqrt{3} f_0 C_{abc}\eta_b\mu_{c,xx} + h_0 C_{abc}\eta_b\eta_{c,xx}\right) \\ & + \frac{8\mathcal{A}}{3}\frac{2f_0 f_1 - 3h_0^2 h_1}{f_0^2 - h_0^3}\mu_{a,x} + \frac{8\mathcal{A}}{3\sqrt{3}}\frac{3f_0 h_1 - 2h_0 f_1}{f_0^2 - h_0^3}\eta_{a,x}.\end{aligned} \tag{23}$$

As $h_1 = 0$, $f_1 = 0$, equation (23) also describes large-scale fluctuations of the mean field. One can obtain (23) from (6) by averaging with a more complicate correlation rule.

Equations (22) and (23) imply the following Hamiltonians respectively:

$$\mathcal{H}_{\text{deg}} = \frac{4\mathcal{A}/\hbar}{3h_0}\int \sum_{a=1}^{8} (\mu_{a,x})^2 \,\mathrm{d}x,$$

$$\mathcal{H}_{\text{gen}} = \frac{\mathcal{A}/\hbar}{h_0^3 - 3f_0^2}\int \sum_{a=1}^{8}\left(h_0^2(\mu_{a,x})^2 + h_0(\eta_{a,x})^2 - 2\sqrt{3} f_0\mu_{a,x}\eta_{a,x}\right)\mathrm{d}x.$$

In addition to the one-dimensional case, one can consider the corresponding two- or three-dimensional Hamiltonian systems with the effective Hamiltonians

$$\mathcal{H}^{\text{eff}} = \mathcal{J} \int H(\boldsymbol{\mu})\, \mathrm{d}^d \boldsymbol{x}, \tag{24}$$

where $\boldsymbol{\mu}$ denotes $\{\mu_a\}_{a=1}^8$. The exchange integral $\mathcal{J} = \mathcal{A}/\hbar$ gives the Hamiltonian the required physical dimension. By $H$, we denote the Hamiltonian density

$$H_{\text{deg}} = \frac{4}{3h_0} \sum_{k=1}^{d} \sum_{a=1}^{8} (\mu_{a,x_k})^2, \qquad \text{or}$$

$$H_{\text{gen}} = \frac{1}{h_0^3 - 3f_0^2} \sum_{k=1}^{d} \sum_{a=1}^{8} \left(h_0^2(\mu_{a,x_k})^2 + h_0(\eta_{a,x_k})^2 - 2\sqrt{3} f_0 \mu_{a,x_k} \eta_{a,x_k}\right).$$

One can use these effective Hamiltonians for describing the magnetic system considered in section 2. Note that $\mathcal{H}_{\text{deg}}$ is the same as the Hamiltonian of (9).

The proposed Hamiltonians describe large-scale (slow) fluctuations of the mean field $\boldsymbol{\mu}$. After averaging over high frequencies, some observed quantities become rigid (or invariant); these quantities are $h_0 = \delta_{ab}\mu_a\mu_b$ and $f_0 = \sqrt{5/3} d_{abc}\mu_a\mu_b\mu_c$. They serve as constraints for the Hamiltonian systems and are equivalent to (20). The constraints determine the orbit where the system has to be considered.

In the case of an SU(3)-invariant model, we deal with the magnet whose ferromagnetic and nematic states are equiprobable. A generic orbit corresponds to a state with the ferromagnetic order at zero temperature because of nonzero magnetization ($m \neq 0$). A degenerate orbit ($m = 0$) corresponds to a state with the nematic order at zero temperature. So equations (22) and (23) describe fluctuations of the mean field $\boldsymbol{\mu}$ near nematic and ferromagnetic ordered states respectively.

### *3.4. SU(3)-invariance of effective Hamiltonians*

As mentioned in section 2, the quantum Hamiltonian (2) and the mean field Hamiltonian (4) are SU(3)-invariant as $K = J$. Here we show that the proposed classical effective Hamiltonians (24) are also SU(3)-invariant.

Recall that the mean field $\{\mu_a\}$ belongs to the real eight-dimensional space of the coadjoint representation of SU(3). Hence, an action of SU(3) transforms $\{\mu_a\}$ by the formula

$$\mu_a = \hat{D}_{ab}\mu_b, \qquad \hat{D}_{ab} \in \mathrm{SO}(8),$$

where $\hat{D}_{ab}$ is a matrix of the real irreducible eight-dimensional representation of the group SU(3).

Note that the tensor $d_{abc}$ satisfies the relation $d_{abc}d_{qbc} = \delta_{aq}$. The components $\{d_{abc}\}$ serve as Clebsch–Gordon coefficients for a decomposition of the tensor square of the coadjoint representation into irreducible components. In this connection, we have the following relation, well known in theory of representations, $\hat{D}_{bb'}\hat{D}_{cc'} = d_{qbc}d_{q'b'c'}\hat{D}_{qq'}$. Then as a result of the action of SU(3) on $\{\eta_a\}$, we get

$$\eta_a = \sqrt{5} d_{abc}\hat{D}_{bb'}\mu_{b'}\hat{D}_{cc'}\mu_{c'} = \hat{D}_{qq'}\eta_{q'}.$$

The action of SU(3) on the vector fields $\{\mu_{a,x}\}$ and $\{\eta_{a,x}\}$ is the same. Therefore, the densities $H_{\text{gen}}$ and $H_{\text{deg}}$ are SU(3)-invariant.

The densities of the effective Hamiltonians can be expressed as

$$H = \sum_{jk} \sum_{ab} g_{ab}(\boldsymbol{\mu}) \frac{\partial \mu_a}{\partial x_j} \frac{\partial \mu_b}{\partial x_k} G_{jk}(\boldsymbol{x}), \tag{25}$$

where $g_{ab}(\boldsymbol{\mu})$ serves as metrics invariant under an action of the group that transforms $\boldsymbol{\mu}$ and $G_{jk}(\boldsymbol{x})$ is metrics in the $\boldsymbol{x}$-space. For the proposed effective Hamiltonians, the $\boldsymbol{x}$-space is Euclidean: $G_{jk}(\boldsymbol{x}) = \delta_{jk}$. The metrics in the $\boldsymbol{\mu}$-space is trivial: $g_{ab}(\boldsymbol{\mu}) = \frac{4}{3h_0}\delta_{ab}$ in the case of a degenerate orbit, and has a more complicate form:

$$g_{ab}(\boldsymbol{\mu}) = \frac{1}{h_0^3 - 3f_0^2}\left(h_0^2\delta_{ab} + 20h_0 d_{cpa}d_{cqb}\mu_p\mu_q - 4\sqrt{15}f_0 d_{abc}\mu_c\right)$$

in the case of a generic orbit.

The density (25) can be interpreted as a Lagrangian density of a relativistic $\sigma$-model; in this case, $G_{jk}$ is the metrics of the Minkowski space. After quantization, one obtains a Hamiltonian system that describes slow fluctuations. Quick fluctuations can be taken into account by means of a renormalization group [1]. It makes the coefficients $\frac{1}{h_0^3-3f_0^2}$ and $\frac{4}{3h_0}$ dependent on the parameters of the renormalization group, for example on a temperature.

### *3.5. Parametrization of orbits*

Remarkably, the effective models are entirely defined by geometry of orbits. We will prove this statement by performing a parametrization of orbits and expressing the effective Hamiltonians in terms of these parameters.

A generalized stereographic projection gives a suitable way of parametrization for coadjoint orbits of a semisimple Lie group [10]. In the case of group SU(3), we have

$$\mu_a = -\frac{m - \sqrt{3}q}{2}\zeta_a + m\xi_a, \qquad \eta_a = \frac{\sqrt{3}(m^2 - q^2) - 2mq}{2}\zeta_a + 2mq\xi_a,$$

where

$$\begin{aligned}
\zeta_1 &= -\frac{1}{\sqrt{2}}\frac{w_2 + w_3 + \bar{w}_2 + \bar{w}_3}{1 + |w_2|^2 + |w_3|^2} & \xi_1 &= -\frac{1}{\sqrt{2}}\frac{(1-w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2) + (1-\bar{w}_1)(w_3 - w_1w_2)}{1 + |w_1|^2 + |w_3 - w_1w_2|^2},\\
\zeta_2 &= \frac{-\mathrm{i}}{\sqrt{2}}\frac{w_2 - w_3 - \bar{w}_2 + \bar{w}_3}{1 + |w_2|^2 + |w_3|^2} & \xi_2 &= \frac{-\mathrm{i}}{\sqrt{2}}\frac{(1+w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2) - (1+\bar{w}_1)(w_3 - w_1w_2)}{1 + |w_1|^2 + |w_3 - w_1w_2|^2},\\
\zeta_3 &= \frac{|w_2|^2 - |w_3|^2}{1 + |w_2|^2 + |w_3|^2} & \xi_3 &= \frac{1 - |w_1|^2}{1 + |w_1|^2 + |w_3 - w_1w_2|^2},\\
\zeta_4 &= \mathrm{i}\frac{\bar{w}_2w_3 - w_2\bar{w}_3}{1 + |w_2|^2 + |w_3|^2} & \xi_4 &= \mathrm{i}\frac{w_1 - \bar{w}_1}{1 + |w_1|^2 + |w_3 - w_1w_2|^2},\\
\zeta_5 &= \frac{1}{\sqrt{2}}\frac{w_2 - w_3 + \bar{w}_2 - \bar{w}_3}{1 + |w_2|^2 + |w_3|^2} & \xi_5 &= -\frac{1}{\sqrt{2}}\frac{(1+w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2) + (1+\bar{w}_1)(w_3 - w_1w_2)}{1 + |w_1|^2 + |w_3 - w_1w_2|^2},\\
\zeta_6 &= \frac{\mathrm{i}}{\sqrt{2}}\frac{w_2 + w_3 - \bar{w}_2 - \bar{w}_3}{1 + |w_2|^2 + |w_3|^2} & \xi_6 &= \frac{\mathrm{i}}{\sqrt{2}}\frac{(1-\bar{w}_1)(w_3 - w_1w_2) - (1-w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2)}{1 + |w_1|^2 + |w_3 - w_1w_2|^2},\\
\zeta_7 &= -\frac{\bar{w}_2w_3 + w_2\bar{w}_3}{1 + |w_2|^2 + |w_3|^2} & \xi_7 &= -\frac{w_1 + \bar{w}_1}{1 + |w_1|^2 + |w_3 - w_1w_2|^2},\\
\zeta_8 &= \frac{1}{\sqrt{3}}\frac{2 - |w_2|^2 - |w_3|^2}{1 + |w_2|^2 + |w_3|^2} & \xi_8 &= \frac{1}{\sqrt{3}}\frac{1 + |w_1|^2 - 2|w_3 - w_1w_2|^2}{1 + |w_1|^2 + |w_3 - w_1w_2|^2}.
\end{aligned}$$

Here $w_1, w_2, w_3$ are complex parameters on a generic orbit, and $m$ and $q$ are initial values of $\mu_3$ and $\mu_8$ respectively. The initial values fix an orbit. For a degenerate orbit, one has to assign $m = 0$ and $w_1 = 0$.

After this parameterization, the effective Hamiltonians get the form

$$\mathcal{H}^{\text{eff}} = \int \sum_{k=1}^{d}\sum_{\alpha,\beta} g_{\alpha\beta}(\boldsymbol{w})\frac{\partial w_\alpha}{\partial x_k}\frac{\partial w_\beta}{\partial x_k}\,\mathrm{d}^d\boldsymbol{x},$$

$$g^{\mathrm{deg}}_{\alpha\beta} = \sum_a \frac{\partial \zeta_a}{\partial w_\alpha} \frac{\partial \zeta_a}{\partial w_\beta},$$

$$g^{\mathrm{gen}}_{\alpha\beta} = \sum_a \left( \frac{\partial \zeta_a}{\partial w_\alpha} \frac{\partial \zeta_a}{\partial w_\beta} - \frac{\partial \zeta_a}{\partial w_\alpha} \frac{\partial \xi_a}{\partial w_\beta} + \frac{\partial \xi_a}{\partial w_\alpha} \frac{\partial \xi_a}{\partial w_\beta} \right).$$

The tensors $g^{\mathrm{gen}}$ and $g^{\mathrm{deg}}$ serve as metrics on orbits in terms of the complex parameters $\boldsymbol{w} = \{w_1, \bar{w}_1, w_2, \bar{w}_2, w_3, \bar{w}_3\}$ for a generic orbit and $\boldsymbol{w} = \{w_2, \bar{w}_2, w_3, \bar{w}_3\}$ for a degenerate orbit. Note that the metrics do not depend on the initial values $m$ and $q$, fixing an orbit. All generic orbits have the same metrics, as well as degenerate orbits.

## 4. Results and discussion

Our main result is the following. For a magnetic system with the spin $s \geqslant 1$, we propose two effective classical models that describe fluctuations of the mean field by the Landau–Lifshitz-like equations. We consider the 8-component mean field $\boldsymbol{\mu} = \{\mu_a\}_{a=1}^{8}$, taking into account not only magnetization but also quadrupole moments.

The effective models deal with large-scale (slow) fluctuations of the mean field. Small-scale (quick) fluctuations are cut off by quasiaveraging. In this process, some observed quantities become rigid and serve as constraints determining the manifold where the mean field lives. This manifold appears to be a coadjoint orbit of the group SU(3).

Also, we propose a complex parametrization for the manifold and reduce the mean field and the Hamiltonian density to complex parameters. Remarkably, in terms of the complex parameters the density becomes independent of the boundary values of $\boldsymbol{\mu}$. Moreover, the Hamiltonian density serves as Riemannian metrics on the manifold.

In the case of an SU(3)-invariant model, we deal with the magnet whose ferromagnetic and nematic states at zero temperature are equiprobable. That is why we propose two effective Hamiltonians: $\mathcal{H}_{\mathrm{gen}}$ for states with the ferromagnetic order at zero temperature and $\mathcal{H}_{\mathrm{deg}}$ for states with the nematic order (when magnetization is zero) at zero temperature. Also, we produce equations (22) and (23) describing large-scale fluctuations of the mean field $\boldsymbol{\mu}$ near nematic and ferromagnetic ordered states respectively.

The proposed classical models can be used to construct topological excitations [11], which are stationary solutions of the Landau–Lifshitz-like equations. These excitations realize the destruction of a long-range order in two-dimensional spin systems at nonzero temperatures, according to the Mermin–Wagner theorem.

The considered scheme is easily extended to the case with higher powers of exchange interaction. For an arbitrary spin $s$, the spin operators $\{\hat{S}_n^\alpha\}$ are defined over the $(2s + 1)$-dimensional space of representation of the group SU(2). The complete matrix algebra generated by the spin operators is $\mathrm{Mat}_{(2s+1)\times(2s+1)}$. Then one can consider a spin Hamiltonian with powers of exchange interaction up to $2s$. Such a Hamiltonian admits a bilinear form if one takes into account multipole moments. In the mean field approximation, this quantum model corresponds to a Hamiltonian system on a coadjoint orbit of the group $\mathrm{SU}(2s + 1)$. Each orbit has a Hamiltonian system, which serves as an effective classical model.

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