GENERALIZED SOLVABILITY OF PSEUDO-PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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The method of a priori inequalities in negative norms is one of the approaches used to investigate various issues in mathematical modeling, including correctness of the initial-boundary value problem formulation, existence of optimal control, convergence of computational methods, and more.

Using this approach, S.I. Lyashko and his colleagues obtained various results regarding optimization problems for different models described by partial differential equations with partial derivatives [1]. Subsequently, it was found that the method of a priori inequalities in negative norms can be effectively applied to linear integro-differential equations with Volterra-type integral terms as well. For instance, in the work [2], the existence of optimal control for a system described by a pseudo-parabolic integro-differential equation was proven.

Let us assume that the evolution of the system be described by the equation $\mathcal{L}u = f$, with a linear integro-differential operator given by

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{n} (a_{ij}(x)\, u_{x_j})_{x_i t} + a(x)u_t - \sum_{i,j=1}^{n} (b_{ij}(x)\, u_{x_j})_{x_i} + b(x)u+$$

$$+\int_0^t \sum_{i=1}^{n} (K_i(x,t,\tau)\, u_{x_i}(x,\tau))_{x_i} d\tau.$$

Here, the unknown function $u(x,t)$ describes the system state in the domain $Q = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded spatial domain with smooth boundary $\partial\Omega$. The function u satisfies homogeneous Dirichlet-type initial and boundary conditions

$$u|_{t=0} = 0, \; u|_{x\in\partial\Omega} = 0. \tag{1}$$

It is assumed that $\{a_{ij}\}_{i,j=1}^n$, $\{b_{ij}\}_{i,j=1}^n \subset C^1(\overline{\Omega})$, $a, b \in C(\overline{\Omega})$, and the integral kernels $K_i(x,t,\tau)$ are continuously differentiable with respect to all variables. Further, for all $x \in \Omega$, we suppose that the relations $a_{ij}(x) = a_{ji}(x)$, $a(x) \geq 0$, hold and that there exists $\alpha > 0$, such that the functions $a_{ij}(x)$ satisfy the inequality $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^n \xi_i^2$ for all $\xi_i \in \mathbb{R}$, $i = \overline{1,n}$.

The domain of definition of the operator $\mathcal{L}$ is considered to be the space C^∞_{BR}, the set of infinitely differentiable functions in the domain $\overline{Q}$, satisfying the conditions (1). Similarly, let $C^{\infty}_{BR^+}$ be the set of the smooth functions in the domain $\overline{Q}$, satisfying the adjoint conditions

$$v|_{t=T} = 0,\ v|_{x\in\partial\Omega} = 0. \tag{2}$$

The domain of definition of the adjoint operator $\mathcal{L}^*$ is the space $C^{\infty}_{BR^+}$, composed of smooth functions in the domain $\overline{Q}$, satisfying the conditions (2).

Let W_{BR}, H_{BR} denote the completions of the space of smooth functions C^{∞}_{BR}, satisfying (1) in the norms

$$\|u\|_{W_{BR}} = \left(\int_Q u_t^2 + \sum_{i=1}^n u_{x_i t}^2 dQ\right)^{\frac{1}{2}},\ \|u\|_{H_{BR}} = \left(\int_Q u^2 + \sum_{i=1}^n u_{x_i}^2 dQ\right)^{\frac{1}{2}}.$$

Similarly, let W_{BR^+}, H_{BR^+} be the completions of $C^{\infty}_{BR^+}$, satisfying (2) in the same norms. Denote the corresponding negative spaces (with respect to $L_2(Q)$) with W^-_{BR}, H^-_{BR}, $W^-_{BR^+}$, $H^-_{BR^+}$.

In the work [2], a priori inequalities in negative norms were proven for the formulated problem. In particular, the following theorem was established:

Theorem 1. *There exists constants $C_1 > 0$, $C_2 > 0$ such that for any function $u \in W_{BR}$, the inequalities $\|\mathcal{L}u\|_{W^-_{BR^+}} \geq C_1 \|u\|_{H_{BR}}$, $C_2 \|u\|_{W_{BR}} \geq \|\mathcal{L}u\|_{W^-_{BR^+}}$ hold, and for any function $v \in W_{BR^+}$, the inequalities $\|\mathcal{L}^*v\|_{W^-_{BR}} \geq C_1 \|v\|_{H_{BR^+}}$, $C_2 \|v\|_{W_{BR^+}} \geq \|\mathcal{L}^*v\|_{W^-_{BR}}$ hold.*

Additionally, in the works [1] and [2], for all $x \in \Omega$, it was assumed that

$$b_{ij}(x) = b_{ji}(x), \tag{3}$$

$$b(x) \geq 0, \tag{4}$$

$$\sum_{i,j=1}^n b_{ij}(x)\xi_i\xi_j \geq 0. \tag{5}$$

In our work, we prove the theorem 1 without requiring the fulfillment of conditions (3)–(5). Thus, it is demonstrated that the theorems regarding the well-possedned of the initial-boundary value problem and the existence of optimal control from [2] remain valid under weaker assumptions.

References

[1] Lyashko S.I. *Generalized optimal control of linear systems with distributed parameters.* — London: Kluwer Academic Publishers, 2002. — 466 p.

[2] Anikushyn A.V., Andaral A.I. *Generalized optimal control of pseudo-parabolic integro-differential systems* // Nonlinear Oscillations. — 2024. — V. 27, No 1. — P. 3–18. (in Ukrainian)

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