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THE p-GROUPS SATISFYING THE CONDITION: EACH CYCLIC SUBGROUP IS CONTAINED IN THE CENTER OR HAS A TRIVIAL INTERSECTION WITH IT

Y. Berkovich proposed the next problem: «Suppose that p-group G satisfies the following condition: if Z is a cyclic subgroup of G then either $Z \le Z(G)$ *or* $Z \cap Z(G) = \{1\}$. *Classify all such groups». We have proved that abelian p-groups and p-groups with exponent p exhaust all regular p-groups satisfying this condition.*

1. Introduction

Let G be a non-trivial finite p-group, $Z(G)$ – the center of G.

In his book [1] Y. Berkovich proposed the next problem: «Suppose that p-group G satisfies the following condition: if Z is a cyclic subgroup of G then either $Z \le Z(G)$ or $Z \cap Z(G) = \{1\}$. Classify all such groups».

It is easy to see all abelian groups satisfy this condition.

Let G be a nonabelian group. If G is a group of exponent p, $\exp G = p$, then every cyclic subgroup Z of G has the prime order p. So the intersection of every cyclic subgroup Z with other subgroup either is equal Z or is a trivial subgroup. We have all non-abelian groups of exponent p satisfy our condition too.

The aim of this work is to prove each regular p-group satisfying this condition is either the group of exponent p or abelian.

The p-group G is called regular if for each g, $h \in G$ we have

$$g^p h^p = (gh)^p \prod_i s_i^p$$

where s_i is the element from the commutator subgroup of the group $\langle g, h \rangle$ generated by g, h.

To answer the question does the regular p-group G satisfy the condition: «For each cyclic subgroup Z of G holds either $Z \le Z(G)$ or $Z \cap Z(G) = \{1\}$», we will describe all regular p-groups having a cyclic subgroup Z which is not contained in the center and which has a non-trivial intersection with it.

2. Proof

Theorem 1. *Let G be a non-abelian regular p-group and let the center $Z(G)$ of G has an exponent greater then p. Then G has the cyclic subgroup Z which is not contained in $Z(G)$ and has a non-trivial intersection with $Z(G)$.*

Proof. Let G be a regular nonabelian group and let $\exp Z(G) > p$.

1) Suppose that there is an element $g \in G$, $g \in Z(G)$ of exponent p. We may choose the element $z_1 \in Z(G)$ such that $\exp z_1 = p^m > p$. The element $z_1 g$ does not belong to the center and has an exponent which is equal to the exponent of element z_1. The center $Z(G)$ does not contain the cyclic subgroup Z_1 generated by the element $z_1 g$. But $Z_1 = \langle z_1 g \rangle$ has the non-trivial subgroup Z_1^p which is generated by the element $(z_1 g)^p = z_1^p g^p = z_1^p \ne 1$ and is contained in the center of G. So the cyclic subgroup Z_1 is not contained in $Z(G)$ but Z_1 and $Z(G)$ have a non-trivial intersection.

2) Suppose each element g from $G \setminus Z(G)$ has the order greater then p. Regard the subgroup $\Omega(G) = \langle x \mid x^p = 1 \rangle$. The subgroup $\Omega(G)$ is characteristic so it has a non-trivial intersection with the center. The assumption that the center $Z(G)$ does not contain $\Omega(G)$ gives the contradiction with the supposition. Really, if $\Omega(G) \setminus (\Omega(G) \cap Z(G)) \ne \{1\}$ then there is an element g from $\Omega(G) \setminus (\Omega(G) \cap Z(G))$. It does not belong to the center $Z(G)$ and has the order equal p.

Therefore $Z(G)$ contains $\Omega(G)$. For regular p-group G the subgroup $\Omega(G)$ coincides with the set of all elements of order p. So there exists g from $G \setminus Z(G$ such that $g^p \in \Omega(G) \subset Z(G)$. We obtain the cyclic subgroup $Z = \langle g \rangle$ of G which has a non-trivial intersection with center $Z(G)$ but $Z(G)$ does not contain Z.

The Theorem 1 is proved.

Theorem 2. *Let G be a nonabelian regular p-group with exponent greater then p and the center $Z(G)$ of G has an exponent equal p. Then G has the cyclic subgroup Z which is not contained in $Z(G)$ and has a non-trivial intersection with $Z(G)$.*

Proof. Suppose that $\exp G > p$, $\exp Z(G) = p$. Regard the characteristic subgroup $□(G) = \langle x^p \mid x \in G \rangle$. It has a non-trivial intersection with the center $Z(G)$. For each regular p-group G the subgroup $□(G)$ coincides with the set of all elements x^p, $x \in G$. So we may find the element $g \in G$ such that $g^p \neq 1, g^p \in □(G) \cap Z(G)$. It is easy to see $g \notin Z(G)$. Hence, the cyclic subgroup $Z = \langle g \rangle$ of G is not contained in the center $Z(G)$ and has a non-trivial intersection with $Z(G)$.

The Theorem 2 is proved.

As an immediate consequence of these theorems we obtain the following result.

Theorem 3. *Each regular p-group G which satisfies the condition if Z is a cyclic subgroup of G then either* $Z \le Z(G)$ or $Z \cap Z(G) = \{1\}$, *is either the group of exponent p or abelian.*

The case with irregular group G is much more complicated.

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1. *Y. Berkovich* Groups of prime power order. In preparation.
2. *B. Huppert* Endliche Gruppen 1. Springer-Verlag, Berlin-Heidelberg-New York. – 1967. – P. 794.
3. *Холл М.* Теория групп. – М.: ИЛ, 1962.

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p-ГРУПИ, ЩО ЗАДОВОЛЬНЯЮТЬ УМОВІ: КОЖНА ЦИКЛІЧНА ПІДГРУПА АБО МІСТИТЬСЯ У ЦЕНТРІ ГРУПИ, АБО МАЄ З НИМ ТРИВІАЛЬНИЙ ПЕРЕТИН

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