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ON REPRESENTATIONS WITH NONSINGULARITY CONDITIONS FOR CERTAIN CLASS OF POSETS

We consider such representations of a poset $A \cup B$ restrictions of which on A and B are nonsingular matrices, and show how to get rid of these conditions in one partial case.

In this paper we consider such representations of a partially ordered set (poset) of the form $A \cup B$, restrictions of which on the subsets A and B are (on the matrix language [1]) square non-singular matrices. We show how to get rid of these conditions, when A is a linear ordered set. Our results can be almost word for word extended to linear problems that are given by vector space categories [2] (in particular, to the problems considered in [3—6]).

1. Formulation of the main result. Throughout the paper, k denotes an arbitrary field; all posets are finite and all vector spaces are finite-dimensional. Under consideration linear maps, morphisms, functors and so on we use the right-side notations. Single-element subsets (of various sets) are identified with the elements.

For a poset S and a field k we denote by $\mathrm{mod}_S\, k$ (by analogy with the category of finite-dimensional vector k-spaces mod$k$) the category of S-spaces over k[7], i. e. the category with objects the vector k-spaces $U = \oplus_{x \in S} U_x$ and with morphisms $\delta: U \to U'$ those linear maps $\delta \in \mathrm{Hom}_k(U,U')$ for which $\delta_{xy} = 0$ if $x > y$ or x and y are incomparable (such maps are called S-maps); here δ_{xy} denotes (as usual in analogous situations) the linear map of U_x into U'_y induced by δ. The set of all S-maps of U into U' (U and U' are S-spaces) is denoted by $\mathrm{Hom}_{S,k}(U,U')$. If U is a S-space and $C \subset S, U_C$ denotes the subspace $\oplus_{x \in C} U_x \subset U$; if, moreover, V is a k-space and $\gamma \in \mathrm{Hom}_k(V,U)$, γ_C denotes the map of V into U_C induced by γ; if γ is a map of a S-space U into a S-space U' $\gamma_{C,D}$ denotes the map of U_C into U'_D induced by γ.

Representation of a poset S over k[1] is (in our terms) a triple (V,U,γ) formed by spaces $V \in \mathrm{mod}\, k$, $U \in \mathrm{mod}_S\, k$, and a linear map $\gamma \in \mathrm{Hom}_k(V,U)$. A morphism of representations $(V, U,\gamma) \to (V', U',\gamma')$ is given by a pair (μ,ν) of linear maps $\mu \in \mathrm{Hom}_k(V,V')$ and $\nu \in \mathrm{Hom}_{S,k}(U,U')$ such that $\gamma\nu = \mu\gamma'$. Thus defined category is denoted by $R_k(S)$. For a lower subset $C \subset S$ (i.e. $y \in C$ and $x < y$ imply $x \in C$) we denote by $R_k(S,C)$ the full subcategory of $R_k(S)$ consisting of all objects (V,U,γ) with $\gamma_C: V \to U_C$ being isomorphism (in mod$k$); intersection of the full subcategories $R_k(S,C)$ and $R_k(S,D)$ is denoted by $R_k(S,C,D)$. If C is a poset, it representations $(0,U,0)$ and $(k,U,1)$, where $U_x = k$ and $U_y = 0$ for $y \neq x$ ($1: k \to k$ is the identity map), are denoted by I_x and $I_{x,1}$, respectively; the representation $(k,0,0)$ is denoted by I_0. Put $E(C) = I_0 \cup \{I_x, I_{x,1} | x \in C\}$. Denote by $\langle n \rangle$ the linear ordered set (chain) $\{1 < 2 < \cdots < n\}$, $n \geq 0$. The direct sum of posets C and D is denoted by $C \,\|\, D$ (i. e. $C \,\|\, D = C \cup D$, where $C \cap D = \varnothing$ and the order relation of $C \cup D$ is the smallest order relation containing the order relations of C and D). C^{op} denotes the poset dual to C.

If M is a set of indecomposable (in our cases mutually nonisomorphic) objects of an additive category Φ, we denote by Φ/M the full subcategory of Φ consisting of all objects without direct summands from M (Y is a direct summand of X if X is isomorphic to some $Y \oplus Z$).

In this paper we shall prove the following theorem.

Theorem. ***Let B be a poset and*** $n \geq 2$ ***a natural number. Then the category*** $R_k(\langle n \rangle \,\|\, B, \langle n \rangle, B)$ ***is equivalent to the category*** $R_k(\langle n-2 \rangle \,\|\, B^{\mathrm{op}}) / E(\langle n-2 \rangle)$.

The cases $n = 0$ and $n = 1$ (which are not covered by the Theorem) are trivial: the category $R_k(\langle 0 \rangle \,\|\, B, \langle 0 \rangle, B)$ is null, and indecomposable objects of the category $R_k(\langle 1 \rangle \,\|\, B, \langle 1 \rangle, B)$ are exhausted (up to isomorphism) by the following representations $H(x) = (V,U,\gamma)$, $x \in B$: $U_x = k$, $U_y = 0$, for $y \in B$, $y \neq x$, $V = U_1 = U_B$ and $\gamma_1 = g_B = 1_V$.

2. Proof of the theorem. Recall that a functor $F:\Phi\to\Psi$ is called faithful (respectively, full) if for an arbitrary $X,Y\in\mathrm{Ob}\,\Phi$ the map $F:\mathrm{Hom}_\Phi(X,Y)\to\mathrm{Hom}_\Psi(XF,YF)$ is injective (respectively, surjective); a functor F is called dence if each $Y\in\mathrm{Ob}\,\Psi$ is isomorphic to some XF (a special case of a dence functor is a surjective on objects functor, i. e. such that the map $F:\mathrm{Ob}\,\Phi\to\mathrm{Ob}\,\Psi$ is surjective). According to the well-known theorem a functor F is equivalence of categories iff it is full faithful and dence.

First step: a removal of the invertibility condition for $\gamma_{\langle n\rangle}$. For a poset C put $J(C)=\{I_x|x\in C\}$. B denotes (as earlier) an arbitrary poset; n is a natural number.

Proposition 1. *The category* $R_k(\langle n\rangle\,\underline{||}\,B,\langle n\rangle,B)$ *is equivalent to the category* $R_k(\langle n-1\rangle\,\underline{||}\,B,B)/J(\langle n-1\rangle)$.

Proof. Put $B(m)=\langle m\rangle\,\underline{||}\,B$ and define the functor $F=F_n:R_k(\langle n\rangle\,\underline{||}\,B,\langle n\rangle,B)\to R_k(\langle n-1\rangle\,\underline{||}\,B,B)$ in the following way: $(V,U,\gamma)F=(V,U_{B(n-1)},\gamma_{B(n-1)})$ on objects and $(\mu,\nu)F=(\mu,\nu_{B(n-1),B(n-1)})$ on morphisms.

Let $(\mu,\nu)F=0$. It follows from the equalities $(\mu,\nu_{B(n-1),B(n-1)})=0$, $\gamma\nu=\mu\gamma'$ and the definition of $\langle n\rangle\,\underline{||}\,B$-maps that $\mu=0$, $\nu_{B,B}=0$, $\nu_{B,\langle n\rangle}=0$, $\nu_{\langle n\rangle,B}=0$ and $\gamma_{\langle n\rangle}\nu_{\langle n\rangle,\langle n\rangle}=0$; hence $\mu=0$ and $\nu=0$ (since $\gamma_{\langle n\rangle}$ is an isomorphism in $\mathrm{mod}\,k$). Thus, the functor F is faithful.

Let $(\alpha,\beta):(V,U,\gamma)F\to(V',U',\gamma')F$ is a morphism of $R_k(\langle n-1\rangle\,\underline{||}\,B,B)$. Consider the following morphism (μ,ν) of $R_k(\langle n\rangle\,\underline{||}\,B,\langle n\rangle,B)$: $\mu=\alpha$, $\nu_{\langle n\rangle,B}=0$, $\nu_{B,\langle n\rangle}=0$, $\nu_{n,\langle n-1\rangle}=0$, $\nu_{\langle n-1\rangle,\langle n-1\rangle}=\beta_{\langle n-1\rangle,\langle n-1\rangle}$, $\nu_{B,B}=\beta_{B,B}$, and $\nu_{\langle n-1\rangle,n}$, $\nu_{n,n}$ are uniquely given by the equality $\gamma_{\langle n-1\rangle}\nu_{\langle n-1\rangle,n}+\gamma_n\nu_{n,n}=\alpha\gamma'_n$ (since $\gamma_{\langle n\rangle}$ is an isomorphism in $\mathrm{mod}\,k$). It is easy to see that $(\mu,\nu)F=(\alpha,\beta)$, and hence the functor F is full.

Show, finally, that for each object $(V,U,\gamma)\in R_k(\langle n-1\rangle\,\underline{||}\,B,B)/J(\langle n-1\rangle)$ there exist an object $(\overline{V},\overline{U},\overline{\gamma})\in R_k(\langle n\rangle\,\underline{||}\,B,\langle n\rangle,B)$ such that $(\overline{V},\overline{U},\overline{\gamma})F=(V,U,\gamma)$. Since the map $\gamma_{\langle n-1\rangle}:V\to U_{\langle n-1\rangle}$ is an epimorphism (in $\mathrm{mod}\,k$), it can be "continue" to some iso-morphism $\gamma':V\to U_{\langle n-1\rangle}\oplus U'$ (U' is a k-space). Then in the capicity of an object $(\overline{V},\overline{U},\overline{\gamma})$ can take the following representation: $\overline{V}=V$, $\overline{U}=U\oplus U'$ with $\overline{U}_x=U_x$ for $x\in B(n-1)$, $\overline{U}_n=U'$, and $\overline{\gamma}=(\gamma,\gamma'_n)$, where γ'_n is the map of V into $U'=\overline{U}_n$ induced by γ'.

Thus, the functor F realize equivalence between the categories indicated in our Proposition.

Second step: a removal of the invertibility condition for γ_B. Besides the category $R_k(S)=R_k^-(S)$ of representations of the poset S we shall consider the category $R_k^+(S)$ with objects the triples (V,U,γ), where $V\in\mathrm{mod}\,k$ $U\in\mathrm{mod}_S\,k$ and $\gamma\in\mathrm{Hom}_k(U,V)$, and with morphisms $(V,U,\gamma)\to(V',U',\gamma')$ the pairs (μ,ν) of linear maps $\mu\in\mathrm{Hom}_k(V,V')$, $\nu\in\mathrm{Hom}_{S,k}(U,U')$ such that $\gamma\mu=\nu\gamma'$. Denote by $I_0^\pm$ the object $(k,0,0)$ of $R_k^\pm(S)$ (then $I_0=I_0^-$).

For an arbitrary poset C and $x\in C$ introduce the objects I_x^+ and $I_{x,1}^+$ analogous to the objects $I_x=I_x^-$ and $I_{x,1}=I_{x,1}^-$ of $R_k(C)=R_k^-(C)$: $I_x^+=(0,U,0)$ and $I_{x,1}^+=(k,U,1)$, where $U_x=k$ and $U_y=0$ for $y\neq x$. Put $J^+(C)=\{I_x^+\,|\,x\in C\}$, $E^+(C)=I_0^+\cup\{I_x^+,I_{x,1}^+|x\in C\}$, $J^-(C)=J(C)$, $E^-(C)=E(C)$. If S is a poset and $C\subset S$, the map of U_C into V induced by a map $\gamma\in\mathrm{Hom}_k(U,V)$ (where $U\in\mathrm{mod}_S\,k$ and $V\in\mathrm{mod}\,k$) is denoted by γ_C(as for $\gamma\in\mathrm{Hom}_k(V,U)$).

Proposition 2. *There exist a full and faithful functor* $G_n:R_k^-(\langle n\rangle\,\underline{||}\,B,B)\to R_k^+(\langle n-1\rangle\,\underline{||}\,B)$ *that realize equivalence between the categories* $R_k^-(\langle n\rangle\,\underline{||}\,B,B)/J^-(\langle n\rangle)$ *and* $R_k^+(\langle n-1\rangle\,\underline{||}\,B)/E^+(\langle n-1\rangle)$.

Note that Proposition 2 remove the invertibility conditon for γ_B, but we go out beyond the considered categories (the categories of representations of posets); in connection with this see Third step.

Proof of the Proposition. Define first the functor $G=G_n$ on objects: $(V,U,\gamma)G=(\overline{V},\overline{U},\overline{\gamma})$, where $\overline{V}=U_{\langle n\rangle}$, $\overline{U}_{\langle n-1\rangle}=U_{\langle n\rangle/1}$, $\overline{U}_B=U_B$ with $\overline{U}_i=U_{i+1}$ for $i=1,\dots,n-1$, $\overline{U}_x=U_x$, for any $x\in B$, $\overline{\gamma}_{\langle n-1\rangle}$ is the natural inclussion (of $\overline{U}_{\langle n-1\rangle}$ into $\overline{V}$) and $\overline{\gamma}_B=\gamma_B^{-1}\gamma_{\langle n\rangle}$. Define now the functor G on morphisms: $(\mu,\nu)G=(\overline{\mu},\overline{\nu})$, where $\overline{\mu}=\nu_{\langle n\rangle,\langle n\rangle}$, $\overline{\nu}_{\langle n-1\rangle,\langle n-1\rangle}=\nu_{\langle n\rangle/1,\langle n\rangle/1}$, $\overline{\nu}_{B,B}=\nu_{B,B}$, ($\overline{\nu}_{\langle n-1\rangle,B}=0$, $\overline{\nu}_{B,\langle n-1\rangle}=0$). It is easy to see that $(\overline{\mu},\overline{\nu})$ is a morphism of the category $R_k^+(\langle n-1\rangle\,\underline{||}\,B)$ (when it is considered that the equality $\gamma\nu=\mu\gamma'$ is equivalent to the equalities $\gamma_{\langle n\rangle}\nu_{\langle n\rangle,\langle n\rangle}=\mu\gamma'_{\langle n\rangle,\langle n\rangle}$ and $\gamma_B\nu_{B,B}=\mu\gamma'_B$).

The functor G is faithful: $(\mu,\nu)G=0$ implies $\nu=0$ (since $\nu=\nu_{\langle n\rangle,\langle n\rangle}\oplus\nu_{B,B}$) and $\mu=0$ (since $\mu\gamma'_B=\gamma_B\nu_{B,B}$ and γ'_B is an isomorphism in $\mathrm{mod}\,k$).

The functor G is full: if (α,β) is a morphism from $(V,U,\gamma)G$ to $(V',U',\gamma')G$, then $(\mu,\nu)G=(\alpha,\beta)$ for the following morphism (μ,ν) of $R_k^-(\langle n\rangle\,\underline{||}\,B,B)$: $\mu=\gamma_B\beta_{B,B}(\gamma'_B)^{-1}$, $\nu_{\langle n\rangle,\langle n\rangle}=\alpha$, $\nu_{B,B}=\beta_{B,B}$.

It is obvious that objects XG don't contain of direct summands from $J^+(\langle n-1\rangle)$. Show that each object $Y=(V,U,\gamma)$ from $R_k^+(\langle n-1\rangle\,\underline{||}\,B)/J^+(\langle n-1\rangle)$ is isomorphic to some XG. Since any indecomposuble object of $R_k^+(\langle n-1\rangle)$ is isomorphic to an object from $E^+(\langle n-1\rangle)$ (see [1]), the object $Y=(V,U,\gamma)$ is isomorphic to an object $Y'=(V',U',\gamma')$ such that $V'=W\oplus U'_{\langle n-1\rangle}$ (W is a k-space) and $\gamma'_{\langle n-1\rangle}$ is the natural inclusion (of $U'_{\langle n-1\rangle}$ into V'). Then, as is easily seen, in the capacity of an object $X\in R_k^-(\langle n\rangle \,||\, B, B)$ can take the following object $(\overline{V},\overline{U},\overline{\gamma})$: $\overline{V}=U'_B$, $\overline{U}_{\langle n\rangle}=V'$, $\overline{U}_B=U'_B$, with $\overline{U}_1=W$, $\overline{U}_i=U'_{i-1}$ for $i=2,\dots,n$, $\overline{U}_x=U'_x$, for any $x\in B$, and $\overline{\gamma}_{\langle n\rangle}=\gamma'_B$, $\overline{\gamma}_B=1_{\overline{V}}$.

Thus, the functor G realize equivalence between the categories $R_k^-(\langle n\rangle \,||\, B, B)$ and $R_k^+(\langle n-1\rangle \,||\, \underline{B})/$ $/J^+(\langle n-1\rangle)$. Since $X\overline{G}$ is isomorphic to the object $I_0^+, I_{1,1}^+,\dots,I_{n-1,1}^+$ iff X is (respectively) isomorphic to the object $I_1^-, I_2^-,\dots,I_n^-$, the functor G realize equivalence $R_k^-(\langle n\rangle \,||\, \underline{B}, B)/J^-(\langle n\rangle)$ with $R_k^+(\langle n-1\rangle \,||\, \underline{B})/E^+(\langle n-1\rangle)$. The Proposition is proved.

Third, final, step: use of reflections. The poset dual to a poset S is denoted by S^{op}. We identify S and S^{op} as sets (then $x\le y$ for S^{op} iff $x\ge y$ for S).

Proposition 3. ***The categories*** $R_k^+(S)/I_0^+$ ***and*** $R_k^-(S)/I_0^-$ ***are equivalent.***

Proof. With objects of $R_k^+(S)$ (respectively, $R_k^-(S)$) it is possible to associate (in a natural way) the representations of the quiver S^+ (respectively, S^-) consisting of the points $0, x$, where $x\in S$, and the arrows $(x,0): x\to 0$ (respectively, $(0,x): 0\to x$). Hence we may consider the functors of reflections (at the point 0) $S_0^+: R_k(S^+)\to R_k(S^-)$ and $S_0^-: R_k(S^-)\to R_k(S^+)$, where $R_k(S^-)$ and $R_k(S^+)$ are the category of representation (over k) of the quivers S^- and S^+, respectively (see the definition of reflections in [8]). These functors induce the equivalence between the category $R_k^+(S)/I_0^+$ and $R_k^-(S^{op})/I_0^-$ (see [9]). Now our Theorem follows from Propositions 1—3 if take account that $I_1^{\pm}S_0^{\pm}\cong I_{1,1}^{\mp},\dots,I_{n-2}^{\pm}S_0^{\pm}\cong I_{1,n-2}^{\mp}$, $I_{1,1}^{\pm}S_0^{\pm}\cong I_1^{\mp},\dots,I_{1,n-2}^{\pm}S_0^{\pm}\cong I_{n-2}^{\mp}$ (and $\langle n-2\rangle^{op}\cong\langle n-2\rangle$).

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ПРО ЗОБРАЖЕННЯ З УМОВОЮ НЕВИРОДЖЕНОСТІ ДЛЯ ОДНОГО КЛАСУ ЧАСТКОВО ВПОРЯДКОВАНИХ МНОЖИН

Розглядаються зображення частково впорядкованої множини $A \cup B$, обмеження яких на A і B є невиродженими матрицями; показано, як можна звільнитися від цих умов в одному частковому випадку.