Received: 28 February 2024 | Revised: 6 May 2024 | Accepted: 29 June 2024

DOI: 10.1111/sapm.12737

ORIGINAL ARTICLE

STUDIES IN APPLIED MATHEMATICS WILEY

Nonlinear systems of PDEs admitting infinite-dimensional Lie algebras and their connection with Ricci flows

Roman Cherniha[1,2] | **John R. King**[1]

[1]School of Mathematical Sciences, University of Nottingham, Nottingham, UK

[2]Institute of Mathematics, NAS of Ukraine, Kyiv, Ukraine

Correspondence
Roman Cherniha, School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK.
Email: roman.cherniha1@nottingham.ac.uk and r.m.cherniha@gmail.com

Funding information
Leverhulme Trust; British Academy

Abstract

A wide class of two-component evolution systems is constructed admitting an infinite-dimensional Lie algebra. Some examples of such systems that are relevant to reaction–diffusion systems with cross-diffusion are highlighted. It is shown that a nonlinear evolution system related to the Ricci flow on warped product manifold, which has been extensively studied by several authors, follows from the above-mentioned class as a very particular case. The Lie symmetry properties of this system and its natural generalization are identified and a wide range of exact solutions is constructed using the Lie symmetry obtained. Moreover, a special case is identified when the system in question is reducible to the fast diffusion equation in one space dimension. Finally, another class of two-component evolution systems with an infinite-dimensional Lie symmetry that possess essentially different structures is presented.

KEYWORDS
exact solution, Lie symmetry, nonlinear PDE, Ricci flow

1 INTRODUCTION

Many nonlinear partial differential equations (PDEs) and systems of PDEs possess Lie symmetries that form a Lie algebra of a finite dimensionality. This is in contrast to linear PDEs: these always admit an infinite-dimensional Lie algebra, reflecting the well-known principle of linear superposition of solutions. However, there are examples of nonlinear PDEs (including some arising in real-world applications) that are invariant with respect to infinite-dimensional Lie algebras. For example, the nonlinear diffusion equation with the fast diffusion

$$u_t = \left(u^{-1}u_x\right)_x + \left(u^{-1}u_y\right)_y \tag{1}$$

admits the Lie symmetry[1]

$$X_\infty = A(x,y)\partial_x + B(x,y)\partial_y - 2A_x u\partial_u,$$

where $A(x,y)$ and $B(x,y)$ are arbitrary functions satisfying the Cauchy–Riemann system $A_x = B_y, A_y = -B_x$. This result can be extended also to systems of reaction–diffusion equations with fast diffusion.[2] Notably, the above result was recently rediscovered by some authors in the context of two-dimensional Ricci flow without citation of Refs. 1 and 2.

Very recently,[3] it was shown that some nonlinear systems of PDEs related to the Ricci flows[4,5] on warped manifolds are invariant with respect to infinite-dimensional Lie algebras, reflecting the geometrical content of such formulations (we note that (1) describes Ricci flow on a two-dimensional manifold[6]). In cases in which symmetry allows the flow to be written as a system of (1+1)-dimensional PDEs in x and t, the corresponding Lie algebra is generated by the Lie symmetry

$$X_\infty = \xi(x)\partial_x + \xi_x(x)I, \tag{2}$$

where $\xi(x)$ is an arbitrary differentiable function and I is a linear operator with respect to unknown functions related to the manifold metrics—see Ref. 3. A natural question arises: what kind of PDEs admit infinite-dimensional Lie algebras generated by operators of the form (2)?

We point out that the best known linear mathematical models arising in physics (the heat and wave equation, the Laplace and Schrödinger equations, etc.) admit nontrivial Lie algebras of invariance, which often reflect the fundamental laws of physics. Constructing nonlinear generalizations preserving these symmetries of the classical equations is one of the most important problems in Lie symmetry analysis. Many such problems were solved in 1980s and 1990s by the prominent Ukrainian researcher Fushchych (Fushchich) and his collaborators (see the monograph,[7] the papers cited therein and the more recent studies[8–10]); see also Refs. 11, 12. Typically, a basic Lie algebra is finite-dimensional (Galilei algebra, Poincare algebra, and Euclid algebra), hence the nonlinear PDE classes to be constructed consist of equations possessing finite-dimensional algebras of invariance. However, the above operator X_∞ generates an infinite-dimensional Lie algebra. To the best of our knowledge, there are not many papers devoted to solving such problems with respect to infinite-dimensional Lie algebras, though some examples can be found in the above cited monograph; see also Refs. 13, 14.

The paper is organized as follows. In Section 2, a wide class of two-component evolution systems is constructed admitting the infinite-dimensional Lie algebra generated by operators of the form (2). Some examples of such systems are highlighted. In Section 3, we show how a known nonlinear evolution system[3,15,16] related to the Ricci flow on warped product manifolds with a given metric follows from the above-mentioned class of systems as a very particular case. The Lie symmetry properties of this system and its natural generalization are identified. In Section 4, a wide range of exact solutions are constructed using the Lie symmetry obtained. Moreover, a special case is identified in which the system in question is reducible to the fast diffusion equation (1) in one space dimension. Finally, we discuss the results obtained and present some conclusions in the last section. In particular, we present another class of two-component evolution systems with an infinite-dimensional Lie symmetry that possess essentially different structures. In our approach, starting from the Lie symmetry of geometric PDEs, we identify wider classes of PDEs that possess exceptional symmetry properties but may lack geometrical interpretation. In this, we are motivated in part by (1), which was known to be exceptional within the class of power-law diffusion equations before its Ricci flow content was elucidated; our analysis proceeds in the opposite direction.

2 | NONLINEAR EVOLUTION SYSTEMS ADMITTING INFINITE-DIMENSIONAL LIE ALGEBRAS

Here are interested in the two-component evolution systems admitting the Lie symmetry (2) with $I = u\partial_u + v\partial_v$, that is,

$$X_\infty = \xi(x)\partial_x + \xi_x(x)(u\partial_u + v\partial_v). \tag{3}$$

The general class of such systems involving the first- and/or second-order PDEs reads as

$$\begin{aligned} u_t &= F(u, v, u_x, v_x, u_{xx}, v_{xx}),\\ v_t &= G(u, v, u_x, v_x, u_{xx}, v_{xx}), \end{aligned} \tag{4}$$

where F and G are arbitrary differentiable functions of their arguments.

Theorem 1. *An evolution system from the class (4) admits the Lie algebra generated by the Lie symmetry (3) if and only if this system possesses the form*

$$\begin{aligned} u_t &= uf(I_0, I_1, I_2),\\ v_t &= vg(I_0, I_1, I_2), \end{aligned} \tag{5}$$

where f and g are arbitrary differentiable functions of their arguments, while $I_0 = \frac{u}{v}$, $I_1 = v(\frac{u}{v})_x$, and $I_2 = u(v(\frac{u}{v})_x)_x$ are absolute differential invariants of order zero, one, and two, respectively.

Proof. According to the classical Lie theory, a system from the class of evolution PDEs (4) admits the Lie symmetry (3) if the following invariance criteria are satisfied (see, e.g., Refs. 17–20)

$$\begin{aligned} \underset{2}{X}\,(u_t - F(u, v, u_x, v_x, u_{xx}, v_{xx}))\Big|_{\mathcal{M}} &= 0,\\ \underset{2}{X}\,(v_t - G(u, v, u_x, v_x, u_{xx}, v_{xx}))\Big|_{\mathcal{M}} &= 0, \end{aligned} \tag{6}$$

where the manifold

$$\mathcal{M} = \{u_t - F(u, v, u_x, v_x, u_{xx}, v_{xx}) = 0,\ v_t - G(u, v, u_x, v_x, u_{xx}, v_{xx}) = 0\}$$

is considered in the prolonged space of independent variables

$$t,\ x,\ u,\ v,\ u_t,\ v_t,\ u_x,\ v_x,\ u_{tx},\ v_{xt},\ u_{tt},\ v_{tt},\ u_{xx},\ v_{xx}.$$

The second prolongation of the operator X reads as

$$\begin{aligned} \underset{2}{X} = X + \rho^1_t \frac{\partial}{\partial u_t} + \rho^2_t \frac{\partial}{\partial v_t} + \rho^1_x \frac{\partial}{\partial u_x} + \rho^2_x \frac{\partial}{\partial v_x} + \\ \sigma^1_{tx} \frac{\partial}{\partial u_{tx}} + \sigma^2_{tx} \frac{\partial}{\partial v_{tx}} + \sigma^1_{tt} \frac{\partial}{\partial u_{tt}} + \sigma^2_{tt} \frac{\partial}{\partial v_{tt}} + \sigma^1_{xx} \frac{\partial}{\partial u_{xx}} + \sigma^2_{xx} \frac{\partial}{\partial v_{xx}}. \end{aligned} \tag{7}$$

Here, the coefficients ρ^k and σ^k with the relevant indices are expressed by the well-known formulas via the coefficients of X and their derivatives. As a result, one obtains

$$\begin{aligned} &\rho^1_t = \xi_x(x)u_t,\ \rho^2_t = \xi_x(x)v_t,\ \rho^1_x = \xi_{xx}(x)u,\ \rho^2_x = \xi_{xx}(x)v, \\ &\sigma^1_{xx} = \xi_{xxx}(x)u + \xi_{xx}(x)u_x - \xi_x(x)u_{xx},\ \sigma^2_{xx} = \xi_{xxx}(x)v + \xi_{xx}(x)v_x - \xi_x(x)v_{xx}, \end{aligned} \tag{8}$$

while all other sigmas with indices vanish.

Thus, inserting the explicit expressions for ρ^k and σ^k into (7), applying the second prolongation of X to the invariance criteria (8), and making straightforward calculations, one arrives at the PDE system

$$\begin{aligned} \xi_x F &= \xi_x u F_u + \xi_x v F_v + \xi_{xx} u F_{u_x} + \xi_{xx} v F_{v_x} \\ &\quad + (\xi_{xxx} u + \xi_{xx} u_x - \xi_x u_{xx}) F_{u_{xx}} + (\xi_{xxx} v + \xi_{xx} v_x - \xi_x v_{xx}) F_{v_{xx}}, \\ \xi_x G &= \xi_x u G_u + \xi_x v G_v + \xi_{xx} u G_{u_x} + \xi_{xx} v G_{v_x} \\ &\quad + (\xi_{xxx} u + \xi_{xx} u_x - \xi_x u_{xx}) G_{u_{xx}} + (\xi_{xxx} v + \xi_{xx} v_x - \xi_x v_{xx}) G_{v_{xx}} \end{aligned} \tag{9}$$

that can be rewritten as

$$\begin{aligned} &uF_u + vF_v - u_{xx}F_{u_{xx}} - v_{xx}F_{v_{xx}} - F \\ &\quad + \frac{\xi_{xx}}{\xi_x}\left(uF_{u_x} + vF_{v_x} + u_x F_{u_{xx}} + v_x F_{v_{xx}}\right) + \frac{\xi_{xxx}}{\xi_x}\left(uF_{u_{xx}} + vF_{v_{xx}}\right) = 0, \\ &uG_u + vG_v - u_{xx}G_{u_{xx}} - v_{xx}G_{v_{xx}} - G \\ &\quad + \frac{\xi_{xx}}{\xi_x}\left(uG_{u_x} + vG_{v_x} + u_x G_{u_{xx}} + v_x G_{v_{xx}}\right) + \frac{\xi_{xxx}}{\xi_x}\left(uG_{u_{xx}} + vG_{v_{xx}}\right) = 0 \end{aligned} \tag{10}$$

(here it is assumed $\xi_x \neq 0$, otherwise (3) reduces to the trivial Lie symmetry $X = \partial_x$). Because $\xi(x)$ is an arbitrary given smooth function and for which the above equations should be satisfied, each of the Equations (10) can be split into three equations. Thus, one obtains the overdetermined linear system of first-order PDEs

$$\begin{aligned} &uF_u + vF_v - u_{xx}F_{u_{xx}} - v_{xx}F_{v_{xx}} = F, \\ &uF_{u_x} + vF_{v_x} + u_xF_{u_{xx}} + v_xF_{v_{xx}} = 0, \\ &uF_{u_{xx}} + vF_{v_{xx}} = 0 \end{aligned} \tag{11}$$

for the function $F(u, v, u_x, v_x, u_{xx}, v_{xx})$. Obviously, the corresponding system for $G(u, v, u_x, v_x, u_{xx}, v_{xx})$ has the same structure.

The last step of Proof consists in the construction of a general solution of the linear system (11), which can be implemented by means of the method of characteristics for solving linear first-order PDEs (see, e.g., the classical book[21]). As a result, one obtains

$$F = uf(I_0, I_1, I_2), \quad G = vg(I_0, I_1, I_2), \tag{12}$$

where the absolute differential invariants I_0, I_1, and I_2 are defined above. □

Remark 1. Because absolute differential invariants are defined only up to functional dependence, each of the above invariants can be presented in equivalent forms. For example, the first-order invariant I_1 can be rewritten as $u(\frac{u}{v})_x$, $u(\frac{v}{u})_x$, or $v(\frac{v}{u})_x$.

Remark 2. One may consider the variable t as an absolute differential invariant as well because the operator X_∞ does not involve the time variable.

Many real-world processes are described by the second-order PDEs or systems of PDEs that are linear with respect to the second-order derivatives, that is, there are no terms like $(u_{xx})^2$, $\exp v_{xx}$, $\log u_{xx}$, and so on. Such systems can be identified from (5) in the form

$$\begin{aligned} u_t &= u^2\Phi_2(I_0)\left(v\left(\frac{u}{v}\right)_x\right)_x + u\Phi_1(I_0, I_1), \\ v_t &= v^2\Psi_2(I_0)\left(v\left(\frac{u}{v}\right)_x\right)_x + v\Psi_1(I_0, I_1). \end{aligned} \tag{13}$$

Hereafter Φ_i and Ψ_i are arbitrary smooth functions of the relevant arguments. The most common examples of this type are reaction–diffusion-convection (RDC) equations and their generalizations. The relevant class of RDC systems possessing the Lie symmetry (3) thus reads as

$$\begin{aligned} u_t &= u^2\Phi_2\left(\frac{u}{v}\right)\left(v\left(\frac{u}{v}\right)_x\right)_x + u^2\Phi_1\left(\frac{u}{v}\right)\left(\frac{u}{v}\right)_x + u\Phi_0\left(\frac{u}{v}\right), \\ v_t &= v^2\Psi_2\left(\frac{u}{v}\right)\left(v\left(\frac{u}{v}\right)_x\right)_x + v^2\Psi_1\left(\frac{u}{v}\right)\left(\frac{u}{v}\right)_x + v\Psi_0\left(\frac{u}{v}\right). \end{aligned} \tag{14}$$

Remark 3. Because (3) does not involve the variable t, similar systems with the same right-hand sides can immediately be written down for wave and telegraph-type equations that admit the Lie symmetry (3).

All the systems derived above have a symmetric structure with respect to u and v. However, one may simplify these systems by introducing the new variables

$$w = \frac{u}{v}, \ v = v. \tag{15}$$

Simple calculations lead to the new form of the operator X_∞:

$$X^*_\infty = \xi(x)\partial_x + \xi_x(x)v\partial_v. \tag{16}$$

For (16), the classes of nonlinear systems (13) and (14) reduce to the forms

$$\begin{aligned} (vw)_t &= v^2\Phi^*_2(w)(vw_x)_x + v\Phi^*_1(w, vw_x), \\ v_t &= v^2\Psi_2(w)(vw_x)_x + v\Psi_1(w, vw_x) \end{aligned} \tag{17}$$

and

$$\begin{aligned} (vw)_t &= v^2\Phi^*_2(w)(vw_x)_x + v^2\Phi^*_1(w)w_x + v\Phi^*_0(w), \\ v_t &= v^2\Psi_2(w)(vw_x)_x + v^2\Psi_1(w)w_x + v\Psi_0(w), \end{aligned} \tag{18}$$

respectively. Here, $\Phi^*_2 = w^2\Phi_2$, $\Phi^*_1 = w^2\Phi_1$, and $\Phi^*_0 = w\Phi_0$ are still arbitrary functions.

It should be noted that all PDE systems belonging to the above subclasses are nonlinear. The simplest system of the form (18) involving second-order PDEs reads

$$\begin{aligned} (vw)_t &= v^2(vw_x)_x, \\ v_t &= v^2(vw_x)_x. \end{aligned} \tag{19}$$

This is a nonlinear diffusion type system with cross-diffusion in the second equation. Interestingly, system (19) possesses a highly nontrivial symmetry.

Theorem 2. *The maximal algebra of invariance of system (19) is spanned by the Lie symmetries*

$$X^*_\infty, \ P_t = \partial_t, \ D = 2t\partial_t + x\partial_x, \ Q = v\partial_v + 2(1-w)\partial_w. \tag{20}$$

Note that the Lie bracket $[D, X^*_\infty] = (x\xi_x - \xi)\partial_x + (x\xi_x - \xi)_x v\partial_v$ is an operator of the same form as in (16).

3 | SYSTEMS RELATED TO THE RICCI FLOW FOR SPECIFIC METRICS

Recently, Ricci flow on warped product manifolds with a given metric has been investigated by the Lie symmetry method in terms of a complicated nonlinear system of PDEs.[3] In the simplest case of reduction to a single space variable, the system can be written in the form (see (46) in

Ref. 3)

$$
\begin{aligned}
(vw)_t &= (1-m)v^2(vw_x)_x + \frac{v}{w}\big((m-1)v^2w_x^2 - \mu\big),\\
v_t &= -m\frac{v^2}{w}(vw_x)_x,
\end{aligned}
\tag{21}
$$

where $m = 1, 2, 3 \ldots$ and μ is a real parameter that we allow to be a function of time. Note that the nonlinear system (21) can also be derived from the earlier studies.[15,16]

If one considers the subclass of systems (17) then it can be noted that the nonlinear system (21) is obtained provided

$$
\Phi_1^* = \frac{1}{w}\big((m-1)(vw_x)^2 - \mu\big),\ \Phi_2^* = 1-m,\ \Psi_1 = 0,\ \Psi_2 = -\frac{m}{w}.
$$

Let us rewrite system (21) in a simpler form, while allowing μ to be a function of t:

$$
\begin{aligned}
w_t &= v(vw_x)_x + \frac{1}{w}\big((m-1)v^2w_x^2 - \mu(t)\big),\\
v_t &= -m\frac{v^2}{w}(vw_x)_x.
\end{aligned}
\tag{22}
$$

Theorem 3. *Depending on the function $\mu(t)$, the nonlinear system (22) admits the maximal Lie algebras of invariance generated by the Lie symmetries listed below.*

1. *If $\mu(t)$ is an arbitrary given function, then*

$$X_\infty^* = \xi(x)\partial_x + \xi_x(x)v\partial_v;$$

2. *If $\mu(t) = 0$, then*

$$X_\infty^*,\quad P_t = \partial_t,\quad Q = w\partial_w,\quad D_1 = 2t\partial_t + w\partial_w - v\partial_v;$$

3. *If $\mu(t) = \mu_0 \neq 0$ is an arbitrary constant, then*

$$X_\infty^*,\quad P_t = \partial_t,\quad D_1 = 2t\partial_t + w\partial_w - v\partial_v;$$

4. *If $\mu(t) = \mu_0(t+t_0)^\alpha$, $\mu_0\alpha \neq 0$, then*

$$X_\infty^*,\quad D_2 = 2t\partial_t + (1+\alpha)w\partial_w - v\partial_v;$$

5. *If $\mu(t) = \mu_0 e^{bt}$, $\mu_0 b \neq 0$, then*

$$X_\infty^*,\quad D_3 = 2\partial_t + bw\partial_w.$$

Proof. Because the nonlinear system (22) contains an arbitrary smooth function $\mu(t)$ as a coefficient, the Lie symmetry classification problem needs to be solved, that is, all possible forms of $\mu(t)$ should be identified that lead to Lie symmetries that cannot be produced by the Lie symmetry (16). Typically, such problems are highly nontrivial (see Chap. 2 in Ref. 20 and references therein) and there are special algorithms for their solution. In the case of system (22), the relevant algorithm can be essentially simplified because the arbitrary function μ depends only on time (no dependence on v and/or w). So, the corresponding routine leading to the four different cases with new Lie symmetries listed above is omitted. □

It should be highlighted that the arbitrary parameters μ_0 and b arising in Cases 3–5 can be reduced to ± 1 by appropriate equivalence transformations, while the parameter t_0 is reducible to zero. For example, the nonlinear system (22) with $\mu_0 e^{bt},\ b > 0$ reduces to the form

$$
\begin{aligned}
w_t &= v(vw_x)_x + \frac{1}{w}\big((m-1)v^2 w_x^2 - e^t\big),\\
v_t &= -m\frac{v^2}{w}(vw_x)_x
\end{aligned}
$$

by the equivalence transformation

$$
t \to bt, \quad v \to \frac{v}{\sqrt{b}}, \quad w \to \frac{w}{\mu_0}.
$$

Remark 4. The Lie symmetries listed in Case 3 were derived in Ref. 3; the other four cases were not considered therein.

4 | REDUCTIONS AND EXACT SOLUTIONS OF THE NONLINEAR SYSTEM (22)

Here, we examine system (22) with $\mu(t) = \mu_0$, that is,

$$
\begin{aligned}
w_t &= v(vw_x)_x + \frac{1}{w}\big((m-1)v^2 w_x^2 - \mu_0\big),\\
v_t &= -m\frac{v^2}{w}(vw_x)_x
\end{aligned}
\tag{23}
$$

with the aim of constructing its reductions to systems of ordinary differential equations (ODEs) and, where possible, to find exact solutions. Note that we assume in what follows that $wv \neq 0$ because these functions determine the metric over time

$$
g = \frac{1}{v^2(t,x)} dx \otimes dx + w^2(t,x) g_{can},
$$

where g_{can} is related to the Einstein manifold (F^m, g_{can})—see Ref. 3.

First of all, we point out that system (23) admits the Lie symmetry X^*_∞ involving an arbitrary differentiable function $\xi(x)$. This allows the construction of a nontrivial formula for the generation of further solutions from any particular solution. Let us assume that $(w_0(t,x), v_0(t,x))$ is an exact solution of (23). Applying to the above solution, the transformations generated by X^*_∞, one derives the following new solution:

$$w(t,x) = w_0\big(t, F^{-1}(\epsilon + F(x))\big), \quad v(t,x) = v_0\big(t, F^{-1}(\epsilon + F(x))\big)\exp\int \frac{F''(x^*)}{\left(F'(x^*)\right)^2}d\epsilon, \tag{24}$$

where $x^* = F^{-1}(\epsilon + F(x))$, F^{-1} is the inverse function to F and $F(x) = \int \frac{dx}{\xi(x)}$.

Formula (24) allows us to construct highly nontrivial solutions from a given solution, because the function F is arbitrary. For example, taking the obvious solution

$$w_0(t,x) = \mathrm{e}^{ax}, \quad v_0(t,x) = \mathrm{e}^{-ax}, \quad a^2 = \frac{\mu_0}{m-1}, m \neq 0$$

of the nonlinear system (23), one can construct the new solution

$$w_0(t,x) = \exp\left(a(\varepsilon + x^n)^{1/n}\right), \quad v_0(t,x) = \left(1 + \frac{\epsilon}{x^n}\right)^{1-\frac{1}{n}} \exp\left(-a(\varepsilon + x^n)^{1/n}\right) \tag{25}$$

using the function $F(x) = x^n,\ n \in \mathbb{R} \setminus \{0\}$.

Notably, the transformations generated by the Lie symmetries P_t and D_1 (see Theorem 3) can also be used for similar purposes but the resulting solutions retain practically the same structure

$$w(t,x) = \epsilon^{-1} w_0(\epsilon^2(t+t_0), x), \quad v(t,x) = \epsilon v_0(\epsilon^2(t+t_0), x), \tag{26}$$

where $\epsilon \neq 0$ and t_0 are arbitrary. In particular, one may always set $t_0 = 0$ in the solutions obtained without losing generality.

All possible Lie reductions can be systematically identified by constructing inequivalent ansätze using the Lie symmetries listed in Case 2 (see Theorem 3). According to the standard algorithm, we take a linear combination of all the Lie symmetries

$$X = \alpha_0 P_t + \alpha_1 D_1 + X^*_\infty = (\alpha_0 + 2\alpha_1 t)\partial_t + \xi(x)\partial_x + \alpha_1 w\partial_w + (\xi_x(x) - \alpha_1)v\partial_v, \tag{27}$$

where α_0 and α_1 are arbitrary parameters. So, the corresponding invariant surface conditions are

$$\begin{aligned}(\alpha_0 + 2\alpha_1 t)w_t + \xi(x)w_x &= \alpha_1 w,\\ (\alpha_0 + 2\alpha_1 t)v_t + \xi(x)v_x &= (\xi_x(x) - \alpha_1)v.\end{aligned} \tag{28}$$

Obviously, (28) is a linear system of two separate first-order PDEs that can be easily integrated. Depending on α_0, α_1, and $\xi(x)$, there are four cases leading to inequivalent ansätze, namely: (I) $\alpha_1\xi(x) \neq 0$; (II) $\alpha_1 \neq 0, \xi(x) = 0$; (III) $\alpha_1 = 0, \alpha_0 \neq 0$, and (IV) $\alpha_1 = \alpha_0 = 0$.

Let us consider Case I. First of all, one may set $\alpha_1 = 1$ without loss of generality. Moreover, the parameter α_0 can be set to zero (see Equation 26). As a result, one needs to solve the system

$$\begin{aligned}tw_t + \xi(x)w_x &= w,\\ tv_t + \xi(x)v_x &= (\xi_x(x) - 1)v,\end{aligned} \tag{29}$$

which is solvable using the method of characteristics. Finally, we arrive at the general solution

$$w(t,x) = \sqrt{t}W(\omega), \quad v(t,x) = \frac{1}{F_x}\mathrm{e}^{-F}V(\omega), \quad \omega = F(x) - \frac{1}{2}\ln t, \quad F(x) \equiv \int \frac{dx}{\xi(x)}. \tag{30}$$

By the definition, formulas (30) form an ansatz that reduces the system in question to ODEs. Thus, substituting (30) into (23), one obtains

$$\begin{aligned}
&\frac{1}{2}e^{2\omega}W(W - W') = VW\big((VW')' - VW'\big) + (m-1)(VW')^2 - \mu_0 e^{2\omega},\\
&\frac{1}{2m}e^{2\omega}WV' = V^2\big((VW')' - VW'\big).
\end{aligned} \tag{31}$$

Hereinafter, primes denote differentiation with respect to ω.

Cases II–IV can be treated in a similar way. Thus, the ansatz

$$w(t,x) = \sqrt{t}W(\omega), \quad v(t,x) = \frac{1}{\sqrt{t}}V(\omega), \quad \omega = x \tag{32}$$

and the reduced system

$$\begin{aligned}
&\frac{1}{2}W^2 = VW(VW')' + (m-1)(VW')^2 - \mu_0\\
&\frac{1}{2m}W = V(VW')'
\end{aligned} \tag{33}$$

is derived in Case II. The ansatz

$$w(t,x) = W(\omega), \quad v(t,x) = \frac{1}{F_x}V(\omega), \quad \omega = F(x) + \alpha t, \quad F(x) \equiv \int \frac{dx}{\xi(x)}, \ \alpha \in \mathbb{R} \tag{34}$$

and the reduced system

$$\begin{aligned}
&\alpha WW' = VW(VW')' + (m-1)(VW')^2 - \mu_0\\
&\alpha WV' = -mV^2(VW')'
\end{aligned} \tag{35}$$

follow in Case *III*. Finally, the ansatz

$$w(t,x) = W(\omega), \quad v(t,x) = \xi(x)V(\omega), \quad \omega = t \tag{36}$$

and the reduced system

$$\begin{aligned}
&WW' = -\mu_0\\
&WV' = 0
\end{aligned} \tag{37}$$

apply in Case IV. Obviously, this ODE system is integrable and the exact solution

$$w(t,x) = \sqrt{-2\mu_0 t}, \quad v(t,x) = \xi(x)$$

of the nonlinear system (23) is immediately obtained. This solution can of course easily be noted from the very beginning. However, we are able also to construct some more complicated solutions of the nonlinear system (23) using the ansätze and the reduced systems derived above.

The nonlinear ODE system (31) is complicated because it involves coefficients explicitly depending on the variable ω. We were able to construct its general solution only under the restrictions $m = 1$ and $\mu_0 = 0$ when the system is reducible to the form

$$\begin{aligned} &WW'' - (W')^2 = \frac{1}{2C^2}W^3(W - W') \\ &VW = Ce^{\omega}. \end{aligned} \tag{38}$$

Hereinafter C is an arbitrary constant. The nonlinear ODE in (38) is solvable in terms of the Lambert function, hence one obtains the solution of (38) in an implicit form as follows:

$$\int \left(W Lambert\left(c_2 \exp\left(-\frac{W^2}{4C^2} \right) \right) + W \right)^{-1} dW = \omega + c_1, \quad V = Ce^{\omega}W^{-1}. \tag{39}$$

Thus, formulas (30) with the above functions $W(\omega)$ and $V(\omega)$ represent a family of exact solutions in implicit forms of the nonlinear system (23) with $m = 1$ and $\mu_0 = 0$.

The ODE system (31) with $m = 1$ and $\mu_0 = 0$ possesses a special solution $W = c_1e^{\omega}$, $V = c_2$ leading to the stationary solution presented below (see Equation 44).

Let us next consider the nonlinear ODE system (33). It can be noted that, using the second ODE, the first equation reduces to the first-order ODE

$$(m-1)(VW')^2 + \frac{1}{2}\left(\frac{1}{m} - 1\right)W^2 = \mu_0. \tag{40}$$

So, two different cases, $m = 1$ and $m \neq 1$, should be examined. Setting $m = 1$, one sees that the above equation is satisfied automatically provided $\mu_0 = 0$ (otherwise a contradiction is obtained). So, solving the second ODE in (33) with respect to $V(x)$ and using ansatz (32), we arrive at the following exact solution of the nonlinear system (23) with $\mu_0 = m - 1 = 0$:

$$w(t,x) = \sqrt{t}F(x), \quad v(t,x) = \frac{1}{\sqrt{t}F_x}\left(C + \frac{1}{2}F^2(x)\right)^{1/2}. \tag{41}$$

Hereinafter, $F(x)$ is an arbitrary differentiable function.

Assuming $m \neq 1$, one may solve (40) with respect to V^2 and substitute the expression obtained into the second ODE in (33). It turns out that the latter is satisfied for an arbitrary function $W(x) = F(x)$. Thus, applying ansatz (32) to the solution of (40) with $m \neq 1$, we obtain the exact solution of the nonlinear system (23) as follows:

$$w(t,x) = \sqrt{t}F(x), \quad v(t,x) = \frac{1}{\sqrt{t}F_x}\left(\frac{\mu_0}{m-1} + \frac{1}{2m}F^2(x)\right)^{1/2}. \tag{42}$$

It should be noted that the exact solution derived in Ref. 3 (see Section 5.1) follows from the exact solution (42) as a particular case by setting $\mu_0 = m - 1$ and applying an appropriate time translation. In fact, instead of the invariant surface condition (47) in Ref. 3, one can simply solve (28) with $\alpha_0 = 0, \alpha_1 = 1, \xi(x) = 0$, as was done here.

We now consider the nonlinear ODE system (35). To the best of our knowledge, this system is not integrable for arbitrary α and μ_0. Assuming $\mu_0 \neq 0$, one may use the ad hoc ansatz

$$W = c_1\omega^{\gamma_1}, \quad V = c_2\omega^{\gamma_2} \tag{43}$$

with to-be-determined constants $c_i \neq 0$ and $\gamma_i,\ i = 1, 2$. Making straightforward calculations and using ansatz (34) the exact solution

$$w(t,x) = c_1\sqrt{F(x)}, \quad v(t,x) = c_2\frac{\sqrt{F(x)}}{F_x},\ (m-1)(c_1c_2)^2 = 4\mu_0$$

can be easily constructed. Because $F(x)$ is arbitrary the above solution can be rewritten as

$$w(t,x) = c_1G(x), \quad v(t,x) = \frac{c_2}{2G_x},\ (m-1)(c_1c_2)^2 = 4\mu_0, \tag{44}$$

where $G(x)$ is still an arbitrary differentiable function.

Ansatz (34) under the assumption $\mu_0 = 0$ leads to the exact solution

$$w(t,x) = c_1(\alpha t + F(x))^{\mp 1/2\sqrt{m}}, \quad v(t,x) = \frac{c_2}{F_x}(\alpha t + F(x))^{1/2},\ c_2^2 = \frac{-2\alpha}{1 \pm \sqrt{m}},\ m \neq 1. \tag{45}$$

In the special case $\mu_0 = m - 1 = 0$, the ODE system (35) reduces to

$$\begin{aligned} \alpha W' &= V(VW')' \\ \alpha WV' &= -V^2(VW')', \end{aligned} \tag{46}$$

which is integrable. In fact, it can be deduced that $WV = C = const$, hence the second-order ODE

$$VV'' - (V')^2 - \frac{\alpha}{C^2}V^3V' = 0$$

is obtained. The latter can be integrated by reducing to a first-order ODE. Finally, the following solution of the nonlinear system (23) with $\mu_0 = m - 1 = 0$ was constructed

$$w(t,x) = C\sqrt{\frac{2c_1c_2}{\alpha}}\frac{e^{c_1\omega}}{\sqrt{1 - c_2e^{2c_1\omega}}}, \quad v(t,x) = \sqrt{\frac{\alpha}{2c_1c_2}}\frac{\sqrt{1 - c_2e^{2c_1\omega}}}{e^{c_1\omega}F_x(x)}, \quad \omega = \alpha t + F(x), \tag{47}$$

where none of the arbitrary parameters $c_1,\ c_2, C$, and α should vanish. Notably, the above nonlinear ODE with $\alpha = 0$ leads to the stationary solution (44).

In conclusion of this section, we present the following observation. By introducing the transformation $v = \frac{1}{\Phi}$, the nonlinear system (23) reduces to the form

$$\begin{aligned} w_t &= \frac{w_{xx}}{\Phi^2} - \frac{w_x\Phi_x}{\Phi^3} + (m-1)\frac{w_x^2}{w\Phi^2} - \mu_0\frac{1}{w}, \\ \Phi_t &= m\left(\frac{w_{xx}}{w\Phi} - \frac{w_x\Phi_x}{w\Phi^2}\right), \end{aligned} \tag{48}$$

which gives exactly the system derived in Ref. 16 (one should additionally set $\mu_0 = m - 1$). In the special case $\mu_0 = m - 1 = 0$, the above system simplifies to the form

$$\begin{aligned} w_t &= \frac{w_{xx}}{\Phi^2} - \frac{w_x \Phi_x}{\Phi^3}, \\ \Phi_t &= \frac{w_{xx}}{w\Phi} - \frac{w_x \Phi_x}{w\Phi^2} \end{aligned} \tag{49}$$

and it easily follows that the relation $\Phi(t, x) = \Omega(x) w(t, x)$ holds (hereafter $\Omega(x)$ is an arbitrary smooth function). As a result, system (49) is reducible to the single equation

$$w_t = \frac{1}{\Omega(x) w} \left(\frac{w_x}{\Omega(x) w} \right)_x . \tag{50}$$

This reduces to the fast diffusion equation

$$u_t = \left(\frac{u_{x^*}}{u} \right)_{x^*} \tag{51}$$

if one applies the transformation

$$w = \pm \sqrt{u(t, x^*)}, \quad x^* = \int \Omega(x) dx. \tag{52}$$

So, it is not surprising that the reduced ODE systems obtained above can be much more easily integrated in the case when the nonlinear system (23) is reducible to the single equation (51).

Finally, we note that any exact solution of the fast diffusion equation (51) generates an exact solution of the nonlinear system (49) by applying the formulas $\Phi(t, x) = \Omega(x) w(t, x)$ and (52). Exact solutions of (51) were constructed in several papers and they are summarized in Section 5.1.10 of Ref. 22 (see also Section 4.2 of Ref. 20 for more references).

5 | DISCUSSION

In this work, a wide class of two-component evolution systems (5) admitting an infinite-dimensional Lie algebra is constructed. Some examples of such systems, in particular the RDC-type equations are highlighted. It is shown that the known nonlinear evolution system (23) related to Ricci flow on warped product manifolds[3,15,16] is a special case of the above-mentioned class of systems. The Lie symmetry properties of this system and its natural generalizations are identified. It is proved that the system and its generalizations admit additionally Lie symmetry operators of scale transformations (see the operators D_1, D_2, and D_3 in Theorem 3).

Several exact solutions are constructed using the Lie symmetry method. It turns out that the reduced systems of ODEs are too complicated to possess exact solutions in explicit form in the general case. Thus, exact solutions are found under coefficient restrictions in the nonlinear system (23). Moreover, a special case is identified in which the system in question is reducible to the fast diffusion equation (51). Thus, known solutions of the 1D equation (1), that is, (51), can be used for construction of exact solutions of (23) with $\mu_0 = m - 1 = 0$.

Interestingly, the evolution systems with infinite-dimensional Lie symmetry obtained in Section 2 (including the nonlinear system (23) related to the Ricci flow) do not possess divergence forms, unlike the fast diffusion equation (1) and many other evolution equations and systems occurring in real-world applications. It turns out, though, that classes of evolution systems admitting an infinite-dimensional Lie symmetry can be constructed if one starts with operator (2) setting $I = -u\partial_u - v\partial_v$, that is,

$$X_\infty = \xi(x)\partial_x - \xi_x(x)(u\partial_u + v\partial_v). \tag{53}$$

Indeed, using the same algorithm as in Section 2, one arrives at new classes of evolution systems in divergence form. In particular, an analog of the class of RDC systems possessing the Lie symmetry (53) reads as

$$\begin{aligned} u_t &= \left[u^{-1}\Phi_2\left(\frac{u}{v}\right)\left(\frac{u}{v}\right)_x\right]_x + \Phi_1\left(\frac{u}{v}\right)\left(\frac{u}{v}\right)_x + u\Phi_0\left(\frac{u}{v}\right), \\ v_t &= \left[v^{-1}\Psi_2\left(\frac{u}{v}\right)\left(\frac{u}{v}\right)_x\right]_x + \Psi_1\left(\frac{u}{v}\right)\left(\frac{u}{v}\right)_x + v\Psi_0\left(\frac{u}{v}\right). \end{aligned} \tag{54}$$

The simplest representative of the class of evolutions systems (54) in divergence form is

$$u_t = \left(u^{-1}\left(\frac{u}{v}\right)_x\right)_x, \quad v_t = \left(v^{-1}\left(\frac{v}{u}\right)_x\right)_x. \tag{55}$$

One notes that (55) is an evolution system involving both fast and cross diffusions.

It should be pointed out that systems admitting the infinite-dimensional Lie symmetry generated by the operator (2) can be constructed by direct calculation using the corresponding transformations with an arbitrary differentiable function $f(x)$, that is, without application of the Lie method. For example, the transformations

$$x^* = f(x),\ u^* = \frac{1}{f'(x)}u,\ v^* = \frac{1}{f'(x)}v \tag{56}$$

can be used for constructing the class of RDC-type systems (54).

All evolution systems obtained, including those related to the Ricci flow, involve cross-diffusion terms. It should be mentioned that cross-diffusion is a distinguished peculiarity in some well-known mathematical models arising in biology, chemistry, ecology, and medicine[23–25] that were extensively studied by different mathematical techniques, including symmetry-based methods,[26–31] during the last decade. From the Lie symmetry point of view, evolution systems with cross-diffusion terms often admit symmetries that are not common for similar systems without cross-diffusion. For example, the class of evolution systems (54) with $\Phi_1 = \Psi_1 = 0$ is an analog of the class of standard reaction–diffusion systems with variable diffusivities

$$\begin{aligned} u_t &= [\Phi_2(u)u_x]_x + \Phi_0(u,v), \\ v_t &= [\Psi_2(v)v_x]_x + \Psi_0(u,v). \end{aligned} \tag{57}$$

As follows from our previous study,[32] there are no systems of the form (57) admitting the infinite-dimensional Lie algebra (2), in contrast to any system of the form (54).

Our final remarks are related to the special status of the differential expression

$$vu_x - uv_x$$

associated with the Lie symmetry (53). This plays a central role in the model of Ref. 23, as well as in PDE systems above. The model in Ref. 23 does not possess an infinite-dimensional symmetry, but the following remarks nevertheless seem in order.

Introducing the Lagrangian

$$\mathfrak{L} = \int l dx$$

with (notations of Ref. 23 are adopted in what follows) $l(c, h)$ being the entropic free energy

$$l(c, h) = c(\ln c - 1) + h(\ln h - 1) + (L - c - h)(\ln(L - c - h) - 1),$$

and with $v \equiv L - c - h$, leads to the chemical potentials

$$\mu_c \equiv \frac{\partial l}{\partial c} = \ln c - \ln v, \quad \mu_h \equiv \frac{\partial l}{\partial h} = \ln h - \ln v.$$

Introducing the constitutive laws

$$u_c = -\frac{D_c v}{v_*}\frac{\partial \mu_c}{\partial x}, \quad u_h = -\frac{D_h v}{v_*}\frac{\partial \mu_h}{\partial x}$$

for the vacancy mediated diffusive velocities implies that the continuity equations

$$c_t + (u_c c)_x = 0, \quad h_t + (u_h h)_x = 0$$

reproduce the model analyzed in Ref. 23, with the associated dissipation

$$\frac{d\mathfrak{L}}{dt} = -\int \left(\frac{D_c vc}{v_*}\left(\frac{\partial \mu_c}{\partial x}\right)^2 + \frac{D_h vh}{v_*}\left(\frac{\partial \mu_h}{\partial x}\right)^2\right) dx$$

(ignoring boundary contributions).

A related viewpoint is relevant here: setting

$$l(u, v) = u(\ln u - 1) + v(\ln v - 1) - (u + v)(\ln(u + v) - 1)$$

(this can be associated with the combinatorial quantities $(u + v)!/u!v!$ under Stirling's approximation) implies

$$\frac{\partial}{\partial x}\left(\frac{\partial l}{\partial u}\right) = \frac{1}{u(u + v)}(vu_x - uv_x), \quad \frac{\partial}{\partial x}\left(\frac{\partial l}{\partial v}\right) = \frac{1}{v(u + v)}(uv_x - vu_x). \tag{58}$$

Obviously, these quantities are related with the absolute differential invariant $I_1 = \frac{1}{uv(u+v)}(vu_x - uv_x)$ of the Lie algebra generated by (53). On the other hand, one may say that the relationship between these quantities is associated via Noether's second theorem applied to the transformations (56). Notably, a straightforward generalization takes place if the above expression for the function $l(u,v)$ is replaced by $vF(\frac{u}{v})$ with an arbitrary function F.

For arbitrary mobilities $M_1(u,v)$ and $M_2(u,v)$, the evolution equations

$$\begin{aligned} u_t &= [M_1 u(v u_x - u v_x)]_x, \\ v_t &= [M_2 v(u v_x - v u_x)]_x \end{aligned} \tag{59}$$

then imply the dissipation

$$\frac{d\mathfrak{L}}{dt} = -\int (u+v)^2 \left(u^3 M_1 \left(\frac{\partial}{\partial x} \left(\frac{\partial l}{\partial u} \right) \right)^2 + v^3 M_2 \left(\frac{\partial}{\partial x} \left(\frac{\partial l}{\partial v} \right) \right)^2 \right) dx.$$

The purely diffusive versions of (54) arise as special cases of (59), being exceptional in possessing an infinite-dimensional symmetry. In particular, the fast diffusion system (55) is obtained when $M_1(u,v) = M_2(u,v) = \frac{1}{u^2v^2}$.

ACKNOWLEDGMENTS

R.Ch. acknowledges that this research was funded by the British Academy's Researchers at Risk Fellowships Program. J.R.K. gratefully acknowledges a Leverhulme Trust Fellowship.

DATA AVAILABILITY STATEMENT

There are no shared data that were used in this manuscript

REFERENCES

1. Nariboli G. Self-similar solutions of some nonlinear equations. *Appl Sci Res*. 1970;22:449-461.
2. Cherniha R, King JR. Lie symmetries and conservation laws of nonlinear multidimensional reaction-diffusion systems with variable diffusivities. *IMA J Appl Math*. 2006;71:391-408.
3. Lopez E, Dimas S, Bozhkov Y. Symmetries of Ricci flows. *Adv Nonlinear Anal*. 2023;12:20230106; arXiv: 2212.13630v2.
4. Hamilton RS. The Ricci flow on surfaces. *Contemp Math*. 1988;71:237-262.
5. Chow B, Knopf D. The Ricci Flow: An Introduction. *Mathematical Surveys and Monographs*, Vol 110. American Mathematical Society; 2004.
6. Daskalopoulos P, Hamilton RS. Geometric estimates for the logarithmic fast diffusion equation. *Commun Anal Geom*. 2004;12:143-164.
7. Fushchych WI, Shtelen WM, Serov MI. *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*. Kluwer Academic; 1993.
8. Basarab-Horwath P, Lahno V, Zhdanov R. The structure of Lie algebras and the classification problem for partial differential equations. *Acta Appl Math*. 2001;69(1):43-94
9. Cherniha R, Henkel M. The exotic conformal Galilei algebra and nonlinear partial differential equations. *J Math Anal Appl*. 2010;369:120-132.
10. Serov MI, Karpaliuk TO, Pliukhin OG, Rassokha IV,. Systems of reaction-convection-diffusion equations invariant under Galilean algebras. *J Math Anal Appl*. 2015;422(1):185-211.
11. Shugrin SM. Conservation laws, invariance, and equations of gas dynamics. *Zh Prikl Mekh Tekh Fiz*. 1989:10-18 (in Russian).
12. Rideau G, Winternitz P. Evolution equations invariant under two-dimensional space-time Schrödinger group. *J Math Phys*. 1993;34(2):558-570.
13. Cherniha RM. Nonlinear Galilei-invariant PDEs with infinite-dimensional Lie symmetry. *J Math Anal Appl*. 2001;253:126-141.

14. Cherniha R, Henkel M. On nonlinear partial differential equations with an infinite-dimensional conditional symmetry. *J Math Anal Appl*. 2004;298:487-500.
15. Angenent SB, Isenberg J, Knopf D. Formal matched asymptotics for degenerate Ricci flow neckpinches. *Nonlinearity*. 2011;24:2265-2280.
16. Angenent SB, Isenberg J, Knopf D. Degenerate neckpinches in Ricci flow. *J Reine Angew Math*. 2015;709:81-117.
17. Lie S. Algemeine Untersuchungen über Differentialgleichungen, die eine continuirliche endliche Gruppe gestatten (in German). *Math Ann*. 1885;25:71-151.
18. Olver P. *Applications of Lie Groups to Differential Equations*. Springer; 1993.
19. Bluman GW, Cheviakov AF, Anco SC. *Applications of Symmetry Methods to Partial Differential Equations*. Springer; 2010.
20. Cherniha R, Serov M, Pliukhin O. *Nonlinear Reaction-Diffusion-Convection Equations: Lie and Conditional Symmetry, Exact Solutions and Their Applications*. Chapman and Hall/CRC; 2018.
21. Courant R, Hilbert D. *Methods of Mathematical Physics: Partial Differential Equations, II*. Wiley-Interscience; 1962.
22. Polyanin AD, Zaitsev VF. *Handbook of Nonlinear Partial Differential Equations*. Chapman and Hall/CRC; 2012.
23. King JR. Mathematical analysis of a model for substitutional diffusion. *Proc R Soc Lond A*. 1990;430:377-404.
24. Keller EF, Segel LA. Traveling bands of chemotactic bacteria: a theoretical analysis. *J Theor Biol*. 1971;30:235-248.
25. Shigesada N, Kawasaki K, Teramoto E. Spatial segregation of interacting species. *J Theor Biol*. 1979;79:83-99.
26. Cherniha R, Didovych M. A (1 + 2)-dimensional simplified Keller-Segel model: Lie symmetry and exact solutions. II. *Symmetry*. 2017;9:13.
27. Zhang L, Xu F. Conservation laws, symmetry reductions, and exact solutions of some Keller-Segel models. *Adv Diff Equ*. 2018;2018:327.
28. de la Rosa R, Garrido TM, Bruzon MS. Symmetry reductions of a (2 + 1)-dimensional Keller-Segel model. *Math Meth Appl Sci*. 2021;1-10.
29. Cherniha R, Davydovych V, Muzyka L. Lie symmetries of the Shigesada–Kawasaki–Teramoto system. *Commun Nonlinear Sci Numer Simul*. 2017;45:81-92.
30. Li J, Zhao J. Symmetry classification and solutions for Shigesada–Kawasaki–Teramoto system. *Ital J Pure Appl Math*. 2022;47:649-663.
31. Cherniha R, Davydovych V, King JR. The Shigesada–Kawasaki–Teramoto model: conditional symmetries, exact solutions and their properties. *Commun Nonlinear Sci Numer Simul*. 2023;124:107313.
32. Cherniha R, King JR. Non-linear reaction-diffusion systems with variable diffusivities: Lie symmetries, ansätze and exact solutions. *J Math Anal Appl*. 2005;308(1):11-35.

How to cite this article: Cherniha R, King JR. Nonlinear systems of PDEs admitting infinite-dimensional Lie algebras and their connection with Ricci flows. *Stud Appl Math*. 2024;153:e12737. https://doi.org/10.1111/sapm.12737