ONE GENERALIZATION OF THE SPECIAL LINEAR GROUP AND MATRIX EQUATIONS

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We generalize the group of unimodular matrices [1] and find its structure. For this goal we propose one extension of the special linear group.

Let $SL_2(\mathbb{F}_p)$ denotes the special linear group of degree 2 over a finite field of order p.

Definition 1. *The set of matrices*

$$\{M_i : Det(M_i) = \pm 1, M_i \in GL_2(\mathbb{F}_p)\}$$

forms ***extended special linear group*** *in* $GL_2(\mathbb{F}_p)$ *and is denoted by* $ESL_2(\mathbb{F}_p)$.

As it is studied by us $ESL_2(\mathbb{F}_p)$ has a structure of semidirect product $SL_2(\mathbb{F}_p) \rtimes \mathbb{C}_2$, where $\mathbb{C}_2 \simeq \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$.

Theorem 1. *Let A be a simple matrix and $A \in SL_2(\mathbb{F})$ [2], then for A there is a solution $B \in SL_2(\mathbb{F})$ of the matrix equation*

$$X^2 = A \quad (1)$$

if and only if

$$trA + 2 \quad (2)$$

is quadratic element in $\mathbb{F}$ or 0, where $\mathbb{F}$ is a field.

If $X \in ESL_2(\mathbb{F})$ then the matrix equation (1) has a solutions iff

$$trA \pm 2 \quad (3)$$

is a quadratic element in $\mathbb{F}$ or 0. This solution $X \in ESL_2(\mathbb{F}) \setminus SL_2(\mathbb{F})$ iff $(trA - 2)$ is quadratic element or 0 in $\mathbb{F}$ but $(trA + 2)$ is not. Conversely $X \in SL_2(\mathbb{F})$ iff $(trA + 2)$ is quadratic element. Solutions belong to $ESL_2(\mathbb{F})$ and $SL_2(\mathbb{F})$ iff $(trA + 2)$ and $(trA - 2)$ are quadratic elements. In the case $A \in GL_2(\mathbb{F})$ this condition (2) takes form:

$$trA \pm 2\sqrt{detA} \quad (4)$$

is quadratic element in $\mathbb{F}$ or 0 and $detA$ is quadratic too.

Theorem 2. *If a matrix $A \in GL_2(\mathbb{F}_p)$ is semisimple* [2] *with different eigenvalues and at least one an eigenvalue $\lambda_i \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, $i \in \{1,2\}$, $p > 2$, then $\sqrt{A} \in GL_2(\mathbb{F}_p)$ iff of A satisfies:*

$$(\frac{\lambda_i}{p}) = 1 \textit{ in the square extention that is } \mathbb{F}_{p^2}.$$

Matrices with a determinant -1 correspond to the elements changing Euclidean space orientation.

Corollary 1. *Let A be simple matrix and $A \in SL_2(\mathbb{F}_p)$* [2], *then for matrix $A \in SL_2(\mathbb{F}_p)$ there is a solution $B \in SL_2(\mathbb{F}_p)$ of the matrix equation*

$$X^2 = A \tag{5}$$

if and only if

$$\left(\frac{\operatorname{tr} A + 2}{p}\right) \in \{0, 1\}. \tag{6}$$

If $X \in ESL_2(\mathbb{F}_p)$ then the matrix equation (5) *has a solution iff*

$$\left(\frac{\operatorname{tr} A \pm 2}{p}\right) \in \{0, 1\}. \tag{7}$$

This solution $X \in ESL_2(\mathbb{F}_p) \setminus SL_2(\mathbb{F}_p)$ iff $\left(\frac{trA-2}{p}\right) = 1$ or 0, but $\left(\frac{\operatorname{tr} A+2}{p}\right) = -1$. Conversely $X \in SL_2(\mathbb{F}_p)$ iff $\left(\frac{\operatorname{tr} A+2}{p}\right) = 1$. Solutions $X_i \in ESL_2(\mathbb{F})$ and $SL_2(\mathbb{F})$ iff $\left(\frac{\operatorname{tr} A+2}{p}\right) = 1$ and $(\operatorname{tr} A - 2) = 1$. In the case $A \in GL_2(\mathbb{F}_p)$ this condition (2) *takes form:*

$$\left(\frac{\operatorname{tr} A \pm 2\sqrt{detA}}{p}\right) \in \{0, 1\}. \tag{8}$$

Corollary 2. *If $A \in GL(F_2)$ the condition 2 takes the form:* $\left(\frac{\operatorname{tr} A}{p}\right) \in \{0, 1\}$.

References

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