Ministry of Education and Science of Ukraine
National University of Kyiv-Mohyla Academy
Faculty of Computer Sciences
Department of Mathematics

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Contents

Abstract

Let X be a nonempty set. $\mathrm{Bin}(X)$ is the collection of all groupoids defined on X. Let $\circ, * \in \mathrm{Bin}(X)$. We define a binary operation $\square$ on $\mathrm{Bin}(X)$ as follows: $\forall x, y \in X : \ x[\circ\square*]y = (x \circ y) * (y \circ x)$. In fact, $(\mathrm{Bin}(X), \square)$ is a monoid with left-zero operation $\circ_{lz}$ being its identity, where $\forall x, y \in X : x \circ_{lz} y = x$. $\mathrm{ZBin}(X)$ is the set of all elemets of $\mathrm{Bin}(X)$ that commute with every other elements under $\square$.

In this thesis, we study the left-distributivity relation on the semigroup $\mathrm{Bin}(X)$ and the group $\mathrm{ZBin}(X)$. We research the question of trivial left-distributivity neighborhoods in $\mathrm{Bin}(X)$. Furthermore, we give a criterion, which characterizes those elements of $\mathrm{ZBin}(X)$, which the given element distributes from the left with.

Keywords: groupoid, locally-zero, left-distributivity

Анотація

Нехай X – непорожня множина. $\mathrm{Bin}(X)$ – це родина всіх групоїдів, визначених на X. Нехай $\circ, * \in \mathrm{Bin}(X)$. Визначимо бінарну операцію $\square$ на X таким чином: $\forall x, y \in X : \ x[\circ\square*]y = (x \circ y) * (y \circ x)$. $(\mathrm{Bin}(X), \square)$ – це моноїд, а операція лівих нулів $\circ_{lz}$ – його нейтральний елемент, де $\forall x, y \in X : x \circ_{lz} y = x$. $\mathrm{ZBin}(X)$ – це множина всіх елементів з $\mathrm{Bin}(X)$, які комутують з усіма іншими елементами з $\mathrm{Bin}(X)$ відносно $\square$.

Кваліфікаційну роботу присвячено вивченню відношення лівої дистрибутивності на напівгрупі $\mathrm{Bin}(X)$ і групі $\mathrm{ZBin}(X)$. Ми досліджуємо питання тривіальних околів відносно лівої дистрибутивності в $\mathrm{Bin}(X)$. Також ми надаємо критерій, що дозволяє описати ті елементи $\mathrm{ZBin}(X)$, з якими даний елемент дистрибуює зліва.

Ключові слова: групоїд, локальнонульова операція, ліва дистрибутивність

Introduction

The notion of $\mathrm{Bin}(X)$ was introduced by H.S. Kim and J. Neggers in [5] as a collection of all groupoids on a non-empty set X. They also defined an operation $\square$ on $\mathrm{Bin}(X)$, such that a $\square$-product of two groupoids $(X, \circ)$ and $(X, *)$ yields a new groupoid $(X, \circ\square*)$, where $\forall x, y \in X : \ x[\circ\square*]y = (x \circ y) * (y \circ x)$. In [5], some basic properties of $(\mathrm{Bin}(X), \square)$ were studied. In particular, it was proved to be a monoid with left-zero operation $\circ_{lz}$ being its identity, where $\forall x, y \in X : x \circ_{lz} y = x$.

Other area of research related to our topic is the study of center of $\mathrm{Bin}(X)$, denoted as $\mathrm{ZBin}(X)$, i.e. the set of all elemets of $\mathrm{Bin}(X)$ that commute with every other elements under $\square$. In [6], locally-zero operation was first defined by H. Fayoumi as such an operation that its restriction to the two-points set is either left- or right-zero operation for an arbitrary pair of points. H. Fayoumi characterized the center of $\mathrm{Bin}(X)$ as the set of all locally-zero operations. She also studied the question of associativity of locally-zero operations and their invertibility with respect to $\square$.

In [8], S.S. Ahn, H.S. Kim and J. Neggers proposed to represent locally-zero operations on the set X with undirected graphs. We let the set of vertices of the graph associated with an operation to be equal to X. Then, two vertices are adjecent, if the restriction of the operation to the two-points set, that consists of these vertices, is a left-zero. Such a representation opens a new perspective on the topic, i.e. the possibility to look at the center of $\mathrm{Bin}(X)$ through the lence of graph theory.

I. Bilyi and S. Kozerenko carried on in [9], where they described the $\square$-ptoduct of elements of $\mathrm{ZBin}(X)$ in terms of graphs. They also studied the associativity of locally-zero operations using graph theory. Invertibility of locally-zero operations was also researched in [9]. It was shown that they are the only invertible graph operations, where graph operation is such an operation that the product of two arbitrary points is equal to one of them.

In this thesis, we study the left-distributivity relation on $\mathrm{Bin}(X)$. As the set X is predifined, we may refer to a groupoid as to the intended binary operation. We define out-neighborhood of an operation as the set of all the operations, which it distributes from the left with. Respectively, in-neighborhood of an operation is the set of all the operations, that distribute from the left with the given one. We show that the intersection of out-neighborhoods of all operations defined on the given set is equal to the set of left- and right-zero operations. Furthermore, the intersection of in-neighborhoods of all operations on a given set is equal to the set that only contains right-zero operation. In this thesis, we give the neccessary conditions for an operation to have a trivial in- and out-neighborhoods. We also give an example of operation, that only distributes with left- and right-zero operations. Finding an example of an operation, which in-neighborhood only contains right-zero operation, remains an open question. We proceed with studying left-distributivity in $\mathrm{ZBin}(X)$ and propose a criterion, which characterizes those locally-zero operations that belong to the out-neighborhood of the given one in terms of graphs.

1 Definitions

Definition 1.1. A *binary operation* $\circ$ defined on a nonempty set X is a function $\circ : X \times X \to X$.

Definition 1.2. The binary operation $\circ$ defined on X is said to be *closed* over $X' \subset X$ if
$$\forall x, y \in X' : x \circ y \in X'.$$

Definition 1.3. A *groupoid* is a pair $(X, \circ)$, where X is a nonempty set and $\circ$ is a binary operation closed over X.

Definition 1.4. The binary operation $\circ$ is said to be *associative* over X if
$$\forall x, y, z \in X : x \circ (y \circ z) = (x \circ y) \circ z.$$

Definition 1.5. A *semigroup* is a groupoid $(X, \circ)$ where $\circ$ is associative over X.

Definition 1.6. A *left identity (left neutral element)* of $(X, \circ)$ is $e \in X$ such that
$$\forall x \in X : \ e \circ x = x.$$

Definition 1.7. A *right identity (right neutral element)* of $(X, \circ)$ is $e \in X$ such that
$$\forall x \in X : \ x \circ e = x.$$

Definition 1.8. An *identity (neutral element)* of $(X, \circ)$ is $e \in X$ that is both left and right identity, i.e.
$$\forall x \in X : \ e \circ x = x = x \circ e.$$

Remark 1.9. One should notice, that *if an identity element exists, it is unique.* On the contrary, suppose that there are two distinct identity elements $e_1 \neq e_2$ of $(X, \circ)$. Then we have $e_1 = e_1 \circ e_2 = e_2$, a contradiction.

Definition 1.10. A *monoid* is a semigroup equipped with an identity.

Definition 1.11. Let $(X, \circ)$ be a groupoid with an identity e and $x \in X$. An element $y \in X$ is said to be a *left inverse* of x if
$$y \circ x = e.$$

The element $x \in X$ that has a left inverse is called *left-invertible.*

Definition 1.12. Let $(X, \circ)$ be a groupoid with an identity e and $x \in X$. An element $y \in X$ is said to be a *right inverse* of x if

$$x \circ y = e.$$

The element $x \in X$ that has a right inverse is called *right-invertible.*

Definition 1.13. Let $(X, \circ)$ be a groupoid with an identity e and $x \in X$. An element $y \in X$ is said to be a *(two-sided) inverse* of x (and denoted $y = x^{-1}$) if it is both left and right inverse, i.e.

$$y \circ x = e = x \circ y.$$

The element $x \in X$ that has an inverse is called *invertible.*

Remark 1.14. Note, that *if* $(X, \circ)$ *is a monoid, then any of its elements may have at most one two-sided inverse.* To prove that fact, suppose that $x_1^{-1} \neq x_2^{-1}$ are two distinct inverses of an element $x \in X$. Then we have $x_1^{-1} = x_1^{-1} \circ e = x_1^{-1} \circ (x \circ x_2^{-1}) = (x_1^{-1} \circ x) \circ x_2^{-1} = e \circ x_2^{-1} = x_2^{-1}$, a contradiction. Yet, that is not always the case for a groupoid, i.e. *an element of a groupoid may have many inverses.*

Definition 1.15. A *group* is a monoid $(X, \circ)$ where every element has its inverse, i.e.

$$\forall x \in X \; \exists x^{-1} \in X : \; x^{-1} \circ x = e = x \circ x^{-1}.$$

Definition 1.16. A *left zero element* of $(X, \circ)$ is $0 \in X$ such that

$$\forall x \in X : \; 0 \circ x = 0.$$

Definition 1.17. A *right zero element* of $(X, \circ)$ is $0 \in X$ such that

$$\forall x \in X : \; x \circ 0 = 0.$$

Definition 1.18. A *zero element* of $(X, \circ)$ is $0 \in X$ that is both left and right zero, i.e.

$$\forall x \in X : \; 0 \circ x = 0 = x \circ 0.$$

Remark 1.19. Similarly to identity, *if a zero element exists, it is unique.* On the contrary, suppose that there are two distinct zero elements $0_1 \neq 0_2$ of $(X, \circ)$. Then we have $0_1 = 0_1 \circ 0_2 = 0_2$, a contradiction.

Definition 1.20. A binary operation $\circ_{lz}$ defined on X is called a *left-zero operation*, if every element is a left zero in $(X, \circ_{lz})$, i.e.

$$\forall x, y \in X : \ x \circ_{lz} y = x.$$

Definition 1.21. A binary operation $\circ_{rz}$ defined on X is called a *right-zero operation* , if every element is a right zero in $(X, \circ_{rz})$, i.e.

$$\forall x, y \in X : \ x \circ_{rz} y = y.$$

Definition 1.22. Let $\circ$ be a binary operation defined on X. The element $x \in X$ is said to be *idempotent* with respect to $\circ$ if

$$x \circ x = x.$$

Definition 1.23. The operation $\circ$ is *idempotent* over X if all elements of X are idempotent with respect to $\circ$, i.e.:

$$\forall x \in X : \ x \circ x = x.$$

Definition 1.24. The operation $\circ$ is *left cancellative* if

$$\forall a, b, c \in X : \ a \circ b = a \circ c \Longrightarrow b = c.$$

Definition 1.25. The operation $\circ$ is *right cancellative* if

$$\forall a, b, c \in X : \ b \circ a = c \circ a \Longrightarrow b = c.$$

Definition 1.26. The operation $\circ$ is *cancellative* if it is both left and right cancellative.

Definition 1.27. A binary operation $\circ$ defined on X is called a *constant* at $c \in X$ *operation* and denoted ⓒ if

$$\forall x, y \in X : \ x \circ y = c.$$

Definition 1.28. A binary operation $\circ$ defined on X is called a *graph operation* if

$$\forall x, y \in X : \ x \circ y \in \{x, y\}.$$

Definition 1.29. The binary operation $\circ$ is said to be *commutative* over X if

$$\forall x, y \in X : x \circ y = y \circ x.$$

Definition 1.30. A *center* $Z_\circ(X)$ of groupoid $(X, \circ)$ is a collection of all elements $x \in X$ that commute with every other element under $\circ$, i.e.

$$Z_\circ(X) = \{x \in X \mid \forall y \in X : \ x \circ y = y \circ x\}.$$

2 Bin(X)

In this thesis we study the left-distributivity relation on the semigroup $(\mathrm{Bin}(X), \Box)$, initially introduced by H.S.Kim in [5]. Firstly, let us define this semigroup and recall some its properties.

Definition 2.1. Let X be a nonempty set. $\mathrm{Bin}(X)$ is the collection of all groupoids defined on X, i.e.

$$\mathrm{Bin}(X) = \{(X, \circ) \mid \forall x, y \in X:\ x \circ y \in X\}.$$

From now on, instead of $(X, \circ) \in \mathrm{Bin}(X)$ we will simply write $\circ \in \mathrm{Bin}(X)$, as the set X is predefined.

Definition 2.2. Let $\circ, * \in \mathrm{Bin}(X)$. We define a binary operation $\Box$ on $\mathrm{Bin}(X)$ as follows:

$$\forall x, y \in X:\ x[\circ\Box*]y = (x \circ y) * (y \circ x).$$

2.1 Preliminaries

It was shown in [5] that $\mathrm{Bin}(X)$ is a monoid with $\circ_{lz}$ being its identity. Now, let us discuss zeroes of $\mathrm{Bin}(X)$.

Proposition 2.3. *The semigroup* $(\mathrm{Bin}(X), \Box)$ *does not contain any left zeroes, whenever* $|X| \geqslant 2$.

Proof. Suppose that there exists an operation $\circ$ that is a left zero of $\mathrm{Bin}(X)$. Then for any operation $* \in \mathrm{Bin}(X)$ we have $\circ\Box* = \circ$. In particular, consider an operation ⓒ constant at $c \in X$. Then

$$\forall x, y \in X:\ x \circ y = x[\circ\Box ⓒ]y = (x \circ y) ⓒ (y \circ x) = c.$$

As it holds for any $c \in X$, choosing two distinct $c_1, c_2 \in X$ leads us to

$$\forall x, y \in X:\ c_2 = x \circ y = c_1,$$

a contadiction. $\Box$

As $\mathrm{Bin}(X)$ does not contain any left zeroes, it follows directly that it cannot contain a two-sided zero as well. Nevertheless, it might be of interest for us to have a look at right zeroes of $\mathrm{Bin}(X)$.

Proposition 2.4. *An operation is a right zero of* $(\mathrm{Bin}(X), \Box)$ *iff it is constant.*

Proof.
$\longrightarrow$ Suppose that an operation $\circ$ is a right zero of $\mathrm{Bin}(X)$. Than for any operation $* \in \mathrm{Bin}(X)$ we have $*\Box\circ = \circ$. In particular, consider an operation ⓒ constant at $c \in X$. Then

$$\forall x, y \in X: \ x \circ y = x[ⓒ\Box\circ]y = (xⓒy) \circ (yⓒx) = c \circ c,$$

which means that $\circ$ is a constant at $c \circ c$ operation.
$\longleftarrow$ Now, for any constant at $c \in X$ operation ⓒ we have

$$\forall * \in \mathrm{Bin}(X), \forall x, y \in X: \ x[*\Boxⓒ]y = (x * y)ⓒ(y * x) = c = xⓒy,$$

which means that ⓒ is a right zero of $\mathrm{Bin}(X)$. $\Box$

3 Main results

Now, let us introduce the notion of left-distributivity, which is the main subject of this thesis.

Definition 3.1. Let $\circ, *$ be binary operations defined on X. We say that operation $\circ$ is *left-distributive* over $*$ if

$$\forall x, y, z \in X: \; x \circ (y * z) = (x \circ y) * (x \circ z).$$

In this case, we write $\circ \hookrightarrow *$.

Definition 3.2. Let $\circ \in \mathrm{Bin}(X)$. The *out-neighborhood* of $\circ$ is the set $N^+[\circ]$ defined as follows:

$$N^+[\circ] = \{* \in \mathrm{Bin}(X) | \circ \hookrightarrow *\}.$$

Consequently, the *in-neighborhood* of $\circ$ is the set $N^-[\circ]$ defined as

$$N^-[\circ] = \{* \in \mathrm{Bin}(X) | * \hookrightarrow \circ\}.$$

Example 3.3. Consider $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$, i.e. the set of real numbers with usual addition and multiplication on it. Then $+ \in N^+[\cdot]$ and $\cdot \in N^-[+]$.

3.1 Left-distributivity in Bin(X)

At first, it is worthwhile to have a look at in- and out-neighborhoods of some frequently used operations in $\mathrm{Bin}(X)$.

Proposition 3.4. *For the right-zero operation, we have*

$$N^+[\circ_{rz}] = N^-[\circ_{rz}] = \mathrm{Bin}(X).$$

Proof. Let $* \in \mathrm{Bin}(X)$. Then for all $a, b, c \in X$ we have

$$a \circ_{rz} (b * c) = b * c = (a \circ_{rz} b) * (a \circ_{rz} c),$$

which means that $* \in N^+[\circ_{rz}]$. Similarly, for all $a, b, c \in X$

$$a * (b \circ_{rz} c) = a * c = (a * b) \circ_{rz} (a * c),$$

which means that $* \in N^-[\circ_{rz}]$. □

Proposition 3.5. *For the left-zero operation, we have*

$$N^+[\circ_{lz}] = \{* \in \mathrm{Bin}(X) |\ * \textit{ is idempotent}\},$$
$$N^-[\circ_{lz}] = \mathrm{Bin}(X).$$

Proof. Let $* \in \mathrm{Bin}(X)$. Then for all $a, b, c \in X$ we have

$$a \circ_{lz} (b * c) = a,$$
$$(a \circ_{lz} b) * (a \circ_{lz} c) = a * a,$$

which means that the equality holds and so $* \in N^+[\circ_{lz}]$ iff

$$\forall a \in X : \ a * a = a.$$

Now, for all $a, b, c \in X$ we have

$$a * (b \circ_{lz} c) = a * b = (a * b) \circ_{lz} (a * c),$$

which means that $* \in N^-[\circ_{lz}]$. □

The above two propositions show that $\{\circ_{lz}, \circ_{rz}\} \subset \bigcap_{\circ \in \mathrm{Bin}(X)} N^+[\circ]$ and $\{\circ_{rz}\} \subset \bigcap_{\circ \in \mathrm{Bin}(X)} N^-[\circ]$. In fact, reverse inclusions also hold.

Theorem 3.6.

$$\bigcap_{\circ \in \mathrm{Bin}(X)} N^+[\circ] = \{\circ_{lz}, \circ_{rz}\}.$$

Proof.
$- \supset -$ Follows directly from Propositions 3.4 and 3.5 .
$- \subset -$ Suppose that there exists an operation $\bullet \in \bigcap_{\circ \in \mathrm{Bin}(X)} N^+[\circ]$ such that $\bullet \notin \{\circ_{lz}, \circ_{rz}\}$. Then one can find $a, b, c, d \in X$ with

$$\begin{cases} a \bullet b \neq a, \\ c \bullet d \neq d. \end{cases}$$

Assume $c \neq d$. Consider an operation $* \in \mathrm{Bin}(X)$ with

$$\begin{cases} a * c = a, \\ a * d = b, \\ a * (c \bullet d) = a. \end{cases}$$

Supposing that $\bullet \in N^+[*]$, we have

$$a = a * (c \bullet d) = (a * c) \bullet (a * d) = a \bullet b,$$

which is a contradiction.
Now, assume $c = d$. In this case, consider an operation $* \in \mathrm{Bin}(X)$ with c being a left zero element for $*$. Supposing that $\bullet \in N^+[*]$, $\forall x, y \in X$ we have

$$c = c * (x \bullet y) = (c * x) \bullet (c * y) = c \bullet c,$$

which is a contradiction.

$\square$

Theorem 3.7.

$$\bigcap_{\circ \in \mathrm{Bin}(X)} N^-[\circ] = \{\circ_{rz}\}.$$

Proof.
$- \supset -$ Follows directly from Proposition 3.4.
$- \subset -$ Suppose that there exists an operation $\bullet \in \bigcap_{\circ \in \mathrm{Bin}(X)} N^-[\circ]$ such that $\bullet \neq \circ_{rz}$. Then one can find $a, b \in X$ with

$$a \bullet b \neq b.$$

Consider an operation $* \in \mathrm{Bin}(X)$ with

$$\begin{cases} a * a = b, \\ (a \bullet a) * (a \bullet a) = b. \end{cases}$$

Supposing that $\bullet \in N^-[*]$, we have

$$a \bullet b = a \bullet (a * a) = (a \bullet a) * (a \bullet a) = b,$$

which is a contradiction. $\square$

Now, it is of interest for us to study the properties of those $\circ \in \mathrm{Bin}(X)$ whose out- and in-neighborhoods only contain $\{\circ_{lz}, \circ_{rz}\}$ and $\{\circ_{rz}\}$ respectively.

Proposition 3.8. *Let $\circ \in \mathrm{Bin}(X)$, $|X| \geqslant 2$.*

$$\textit{If } N^+[\circ] = \{\circ_{lz}, \circ_{rz}\}, \textit{ then } \circ \textit{ does not have right zeroes.}$$

Proof. Let $\circ \in \mathrm{Bin}(X)$ and $N^{+}[\circ] = \{\circ_{lz}, \circ_{rz}\}$. Assume that there exists $c \in X$, which is a right zero for $\circ$, i.e.

$$\forall x \in X: \ x \circ c = c.$$

Consider the constant at c operation $\copyright$. Then for all $x, y, z \in X$ it holds

$$x \circ (y \copyright z) = x \circ c = c = (x \circ y) \copyright (x \circ z),$$

implying that $\copyright \in N^{+}[\circ]$. Since $|X| \geqslant 2$, $\copyright \notin \{\circ_{lz}, \circ_{rz}\}$, which is a contadiction. □

Proposition 3.9. *Let* $\circ \in \mathrm{Bin}(X),\ |X| \geqslant 2$.

If $N^{-}[\circ] = \{\circ_{rz}\}$, *then* $\circ$ *does not have idempotents.*

Proof. Let $\circ \in \mathrm{Bin}(X)$ and $N^{-}[\circ] = \{\circ_{rz}\}$. Assume that there exists $c \in X$ that is an idempotent for $\circ$, i.e.

$$c \circ c = c.$$

Consider the constant on c operation $\copyright$. Then for all $x, y, z \in X$ it holds

$$x \copyright (y \circ z) = c = c \circ c = (x \copyright y) \circ (x \copyright z),$$

implying that $\copyright \in N^{-}[\circ]$. Since $|X| \geqslant 2$, $\copyright \neq \circ_{rz}$, which is a contadiction. □

Remark 3.10. Note that Proposition 3.9 implies that if $N^{-}[\circ] = \{\circ_{rz}\}$ then $\circ$ does not have any left or right zeroes as well. That is the case, as any left or right zero would also be an idempotent element.

Now, let us present a non-trivial example of a binary operation $\circ \in \mathrm{Bin}(X)$ with $N^{+}[\circ] = \{\circ_{lz}, \circ_{rz}\}$.

Definition 3.11. Let $\circ \in \mathrm{Bin}(X)$ and $f : X \to X$. A *diagonally twisted* $\circ$*-operation* for f is an operation $\circ^f$ defined as follows:

$$\forall a, b \in X: \ a \circ^f b = \begin{cases} a \circ b, & a \neq b \\ f(a), & a = b \end{cases}$$

Definition 3.12. A *fixed point* of a map $f : X \to X$ is such point $x \in X$ that $f(x) = x$.

Theorem 3.13. *Let $|X| \geqslant 3$ and $f : X \to X$ be a map without fixed points. Then, for the diagonally f-twisted $\circ_{lz}$-operation $\circ_{lz}^f$, it holds*

$$N^+[\circ_{lz}^f] = \{\circ_{lz}, \circ_{rz}\}.$$

Proof. Let $* \in N^+[\circ_{lz}^f]$. At first, we will show that $*$ is idempotent. Fix an arbitrary element $a \in X$. Since $|X| \geqslant 3$, there exists an element $b \notin \{a, a * a\}$. Keeping in mind that $* \in N^+[\circ_{lz}^f]$, we have

$$b = b \circ_{lz}^f (a * a) = (b \circ_{lz}^f a) * (b \circ_{lz}^f a) = b * b.$$

So, $b = b * b \notin \{a, a * a\}$. Therefore,

$$a = a \circ_{lz}^f (b * b) = (a \circ_{lz}^f b) * (a \circ_{lz}^f b) = a * a.$$

Now, we show that $*$ is a graph operation. On the contrary, assume there exist $a, b \in X$ with $a * b \notin \{a, b\}$. Keeping in mind that $* \in N^+[\circ_{lz}^f]$ and $*$ is idempotent, we have

$$f(a*b) = (a*b) \circ_{lz}^f (a*b) = ((a*b) \circ_{lz}^f a) * ((a*b) \circ_{lz}^f b) = (a*b) * (a*b) = a*b,$$

implying that $a*b$ is a fixed point for f. The obtained contradiction shows that $*$ is a graph operation.

Now, we are ready to show that $* \in \{\circ_{lz}, \circ_{rz}\}$. Fix an element $a \in X$. As $*$ is a graph operation, we have $a * f(a) \in \{a, f(a)\}$.

Suppose that $a * f(a) = a$. As f does not have fixed points, $f(a) \neq a$. Again, keeping in mind that $* \in N^+[\circ_{lz}^f]$, we have

$$f(a) = a \circ_{lz}^f a = a \circ_{lz}^f (a * f(a)) = (a \circ_{lz}^f a) * (a \circ_{lz}^f f(a)) = f(a) * a.$$

This means that for any $b \neq a$ it holds

$$a \circ_{lz}^f (a * b) = (a \circ_{lz}^f a) * (a \circ_{lz}^f b) = f(a) * a = f(a).$$

So, by the definition of $\circ_{lz}^f$, $a * b = a$. As it holds for any $b \neq a$ and $a * a = a$ (because $*$ is idempotent), we conclude that a is a left zero for $*$.

Now, suppose that $a * f(a) = f(a)$. Then for any $b \neq a$ it holds

$$a \circ_{lz}^f (b * a) = (a \circ_{lz}^f b) * (a \circ_{lz}^f a) = a * f(a) = f(a).$$

Again, by the definition of $\circ_{lz}^{f}$, $b * a = a$. As it holds for any $b \neq a$ and $a * a = a$ (because $*$ is idempotent), we conclude that a is a right zero for $*$.

This result implies that for an arbitrary $a \in X$ it is either left or right zero for $*$. Now, if $* \notin \{\circ_{lz}, \circ_{rz}\}$, there must be a left zero $a \in X$ and a right zero $b \in X \setminus \{a\}$ for $*$. Hence,

$$a = a * b = b,$$

which is a contradiction. $\square$

We might also want to describe the in-neighborhood of $\circ_{lz}^{f}$ for this case. Firstly, let us define the following constructions.

Definition 3.14. A map $f : X \to X$ is called a *homomorphism* between $\circ, * \in \mathrm{Bin}(X)$ if

$$\forall a, b \in X: \ f(a \circ b) = f(a) * f(b).$$

If $* = \circ$, we say that $f : X \to X$ is a homomorphism for $*$.

Definition 3.15. A map $f : X \to X$ is called a *right semi-homomorphism* between $\circ, * \in \mathrm{Bin}(X)$ if

$$\forall a, b \in X: \ f(a \circ b) = a * f(b).$$

If $* = \circ$, we say that $f : X \to X$ is a right semi-homomorphism for $*$.

Proposition 3.16. *Let $|X| \geqslant 2$ and $f : X \to X$ be a map without fixed points. Then for the diagonally f-twisted $\circ_{lz}$-operation $\circ_{lz}^{f}$ it holds*

$$N^{-}[\circ_{lz}^{f}] = \{* \in \mathrm{Bin}(X) \mid * \textit{ is left cancellative and } f \textit{ is a right semi-homomorphism for } *\}.$$

Proof.
$- \subset -$ Assume $* \in N^{-}[\circ_{lz}^{f}]$. Then for any $a, b \in X$ it holds

$$a * f(b) = a * (b \circ_{lz}^{f} b) = (a * b) \circ_{lz}^{f} (a * b) = f(a * b).$$

Therefore, f is a right semi-homomophism for $*$. Further, if $*$ is not left cancellative, then there exist $a, b, c \in X$ with $a * b = a * c$ and $b \neq c$. We have

$$a * b = a * (b \circ_{lz}^{f} c) = (a * b) \circ_{lz}^{f} (a * c) = (a * b) \circ_{lz}^{f} (a * b) = f(a * b),$$

which means that f has a fixed point, a contradiction.
$- \supset -$ Let $* \in \mathrm{Bin}(X)$ be a left-cancellative operation such that f is a right

semi-homomorphism for $*$. Fix a triplet of elements $a, b, c \in X$. Suppose that $b = c$, then

$$a*(b\circ_{lz}^f c) = a*(b\circ_{lz}^f b) = a*f(b) = f(a*b) = (a*b)\circ_{lz}^f(a*b) = (a*b)\circ_{lz}^f(a*c).$$

If $b \neq c$, then $a * b \neq a * c$, because $*$ is left-cancellative. So, we have

$$a * (b \circ_{lz}^f c) = a * b = (a * b) \circ_{lz}^f (a * c).$$

Therefore, for both cases, we have $* \in N^-[\circ_{lz}^f]$. □

In the example above we restricted f to the diagonal. Now, let us generalize the obtained results to the case when f acts regardless of the choice of a pair (a, b).

Definition 3.17. Let $\circ \in \mathrm{Bin}(X)$, $f : X \to X$. A *left-twisted* $\circ$*-operation* for f is defined as follows:

$$\forall a, b \in X : \; a \circ b = f(a).$$

Consequently, A *right-twisted* $\circ$*-operation* for f is defined as follows:

$$\forall a, b \in X : \; a \circ b = f(b).$$

Proposition 3.18. *For any map $f : X \to X$ and the corresponding left-twisted operation $\circ$ we have*

$$N^+[\circ] = \{* \in \mathrm{Bin}(X) | \; * \textit{ is idempotent over } \mathrm{Im}f\}.$$

Proof.
$- \subset -$ Let $* \in N^+[\circ]$. Then for any $a \in X$ we have

$$f(a) * f(a) = (a \circ a) * (a \circ a) = a \circ (a * a) = f(a),$$

which is exactly what we wanted to show.
$- \supset -$ Assume that $\forall y \in Imf : \; y * y = y$. Then for all $a, b, c \in X$ it holds

$$a \circ (b * c) = f(a) = f(a) * f(a) = (a \circ b) * (a \circ c).$$

Therefore, $* \in N^+[\circ]$. □

Proposition 3.19. *For any map $f : X \to X$ and the corresponding left-twisted operation $\circ$ we have*

$$N^-[\circ] = \{* \in \mathrm{Bin}(X) | \; f \textit{ is a right semi-homomorphism for } *\}.$$

Proof.
$-\subset-$ Let $* \in N^-[\circ]$. Then for any $a, b \in X$ we have

$$f(a * b) = (a * b) \circ (a * b) = a * (b \circ b) = a * f(b).$$

Thus, f is a right semi-homomorphism for $*$.
$-\supset-$ Now, let f be a right semi-homomorphism for $*$. Then for all $a, b, c \in X$ it holds

$$a * (b \circ c) = a * f(b) = f(a * b) = (a * b) \circ (a * c)$$

implying that $* \in N^-[\circ]$. □

Proposition 3.20. *For any map $f : X \to X$ and the corresponding right-twisted operation $\circ$ we have*

$$N^+[\circ] = \{* \in \mathrm{Bin}(X) |\ f \textit{ is a homomorphism for } *\}.$$

Proof.
$-\subset-$ Let $* \in N^+[\circ]$. Then for any $a, b \in X$ we have

$$f(a) * f(b) = (a \circ a) * (a \circ b) = a \circ (a * b) = f(a * b),$$

which is exactly what we wanted to show.
$-\supset-$ Assume that $* \in \mathrm{Bin}(X)$ and f is a homomorphism for $*$. Then for all $a, b, c \in X$ it holds

$$a \circ (b * c) = f(b * c) = f(b) * f(c) = (a \circ b) * (a \circ c).$$

Therefore, $* \in N^+[\circ]$. □

Proposition 3.21. *For any map $f : X \to X$ and the corresponding right-twisted operation $\circ$ we have*

$$N^-[\circ] = \{* \in \mathrm{Bin}(X) |\ f \textit{ is a right semi-homomorphism for } *\}.$$

Proof.
$-\subset-$ Let $* \in N^-[\circ]$. Then for any $a, b \in X$ we have

$$f(a * b) = (a * b) \circ (a * b) = a * (b \circ b) = a * f(b).$$

Thus, f is a right semi-homomorphism for $*$.
$-\supset-$ Now, let f be a right semi-homomorphism for $*$. Then for all $a, b, c \in X$ it holds

$$a * (b \circ c) = a * f(c) = f(a * c) = (a * b) \circ (a * c)$$

implying that $* \in N^-[\circ]$. □

3.2 Left-distributivity in ZBin(X)

In [6], H. Fayoumi introduced the notion of locally-zero operation.

Definition 3.22. A binary operation $\bullet$ defined on X is called a *locally-zero* operation if for any pair $x, y \in X$ the restriction of $\bullet$ to the subset $\{x, y\}$ equals to $\circ_{lz}$ or $\circ_{rz}$, i.e.

$$\forall x, y \in X : \ \bullet|_{\{x,y\}} \in \{\circ_{lz}, \circ_{rz}\}.$$

Remark 3.23. Notice that in case $x = y$ both $\circ_{lz}$ and $\circ_{rz}$ give the same result $x \circ x = x$. So, the locally-zero operation is idempotent.

Recall that the center of a groupoid is the set of all its elements that commute with every other element under the given operation. In case of $\mathrm{Bin}(X)$, we have the following structure.

Definition 3.24. The center of $\mathrm{Bin}(X)$ is denoted $\mathrm{ZBin}(X)$ and defined as

$$Z\mathrm{Bin}(X) = \{\bullet \in \mathrm{Bin}(X) |\ \forall \circ \in \mathrm{Bin}(X) : \ \bullet \Box \circ = \circ \Box \bullet\}.$$

One should mention, that as $\mathrm{ZBin}(X)$ is a subset of $\mathrm{Bin}(X)$ and also $\circ_{lz}$, the identity element of $\mathrm{Bin}(X)$, lies in $\mathrm{ZBin}(X)$, $(\mathrm{ZBin}(X), \Box)$ is a monoid as well.

In [6], Fayoumi has also characterized the center of $\mathrm{Bin}(X)$ in terms of locally zero operations.

Theorem 3.25. *A groupoid $\bullet \in \mathrm{ZBin}(X)$ iff $\bullet$ is locally-zero operation on X.*

So, we will refer to locally-zero operations as to the elements of $\mathrm{ZBin}(X)$.

Observation 3.26. *There are exactly $2^{\binom{n}{2}}$ locally-zero operations on an n-element set X.*

In fact, defining a locally-zero operation we only need to choose the pairs of distinct elements that would form left-zeroes, consequently, all other pairs would form right-zeroes. Moreover, as the operation must be idempotent, the case of two identical elements can only be defined in one way. So, as there are $\binom{n}{2}$ two-element subsets of X and $2^{\binom{n}{2}}$ ways to choose which of them would form left-zeroes, the observation holds. Recall that the number of undirected graphs on n vertices is exactly $2^{\binom{n}{2}}$. Therefore, we could naturally associate an undirected graph on n vertices with locally-zero operation.

Definition 3.27. Let $\bullet \in \mathrm{ZBin}(X)$. A graph $G(\bullet) = (V, E)$ is defined as follows:

$$\begin{aligned} &V(G(\bullet)) = X; \\ &E(G(\bullet)) = \{\{x, y\} | \bullet|_{\{x,y\}} = \circ_{lz}\}. \end{aligned}$$

Remark 3.28. Let $\bullet \in \mathrm{ZBin}(X)$. By Remark 3.23, we have $x \bullet x = x$ for all $x \in X$. This implies that we would have a loop on each vertex of $G(\bullet)$. Therefore, we treat loops as redundant.

Example 3.29. Consider $\circ_{lz} \in Z\mathrm{Bin}(X)$. Obviously, this operation's graph is complete, as the restriction of $\circ_{lz}$ to any two-points set yields $\circ_{lz}$ again. Now, consider $\circ_{rz} \in Z\mathrm{Bin}(X)$. This operation's graph is, in its turn, empty, as the restriction of $\circ_{lz}$ to any two-points set yields $\circ_{rz}$.

Remark 3.30. Let $\bullet \in \mathrm{ZBin}(X)$. From now on, instead of $\{x, y\} \in E(G(\bullet))$, we will simply write $xy \in E(G(\bullet))$.

In [9] I. Bilyi and S. Kozerenko have proposed the following theorem.

Theorem 3.31. *Let $\circ, * \in \mathrm{ZBin}(X)$. Then $G(\circ\Box*) = \overline{G(\circ)\triangle G(*)}$.*

Proof. For all $x, y \in X$, consider the product:

$$x[\circ\Box*]y = (x \circ y) * (y \circ x) = \begin{cases} x, & \circ|_{\{x,y\}} = \circ_{lz} \text{ and } *|_{\{x,y\}} = \circ_{lz}; \\ x, & \circ|_{\{x,y\}} = \circ_{rz} \text{ and } *|_{\{x,y\}} = \circ_{rz}; \\ y, & \circ|_{\{x,y\}} = \circ_{lz} \text{ and } *|_{\{x,y\}} = \circ_{rz}; \\ y, & \circ|_{\{x,y\}} = \circ_{rz} \text{ and } *|_{\{x,y\}} = \circ_{lz}. \end{cases}$$

We conclude, on the one hand, $xy \in E(G(\circ\Box*))$ if it is in both $E(G(\circ))$ and $E(G(*))$. On the other hand, $xy \in E(G(\circ\Box*))$ if it is neither in $E(G(\circ))$ nor in $E(G(*))$. So it follows:

$$G(\circ\Box*) = (G(\circ) \cap G(*)) \cup (\overline{G(\circ)} \cap \overline{G(*)}) = \overline{G(\circ) \triangle G(*)},$$

which proves the theorem. $\square$

This result lets us easily multiply locally-zero operations. Moreover, a precise look at the proof leads to the following observation.

Corollary 3.32. *Let $\bullet \in \mathrm{ZBin}(X)$. Then it is self-invertible, i.e. $\bullet\Box\bullet = \circ_{lz}$.*

Proof. For all $x, y \in X$, we have

$$x[\circ\Box*]y = (x \circ y) * (y \circ x) = x = x \circ_{lz} y \Longleftrightarrow$$
$$\Longleftrightarrow \circ|_{\{x,y\}} = *|_{\{x,y\}} = \circ_{lz} \ \vee \ \circ|_{\{x,y\}} = *|_{\{x,y\}} = \circ_{rz}.$$

Therefore, on same pairs of elements $\{x, y\}$ $\circ$ and $*$ must be defined in the same way, which leads us to the conclusion that $\circ\Box* = \circ_{lz}$ iff $\circ = *$. $\Box$

By Corollary 3.32, every locally-zero operation is self-invertible, which leads us to the fact that $\mathrm{ZBin}(X)$ is actually a group. However, locally-zero operations are in fact the only invertible graph operations.

Theorem 3.33. *For a graph operation $\circ \in \mathrm{Bin}(X)$ the next conditions are equivalent:*

1. *$\circ$ is left-invertible under $\Box$;*
2. *$\circ$ is right-invertible under $\Box$;*
3. *$\circ$ is a locally-zero.*

Proof.
[*3* $\longrightarrow$ *1*] Follows directly from Corollary 3.32.
[*3* $\longrightarrow$ *2*] Follows directly from Corollary 3.32.

Before we prove the next two implications, notice that $\circ \in \mathrm{Bin}(X)$ is locally-zero iff

$$\forall x, y \in X : \ \{x \circ y, y \circ x\} = \{x, y\}.$$

[*1* $\longrightarrow$ *3*] Assume that $\circ$ is right-invertible under $\Box$. Then

$$\exists * \in \mathrm{Bin}(X) : \ \circ\Box* = \circ_{lz}.$$

Since $\circ$ is a graph operation, for all $x, y \in X$, we already have $x \circ y \in \{x, y\}$. Now, suppose that $x \circ y = y \circ x$. Then

$$x = x \circ_{lz} y = x[\circ\Box*]y = (x \circ y) * (y \circ x) = (y \circ x) * (x \circ y) = y[\circ\Box*]x = y.$$

Therefore, $x \circ y = y \circ x$ could only be the case for $x = y$, which means that $\circ$ is a locally-zero operation.
[*2* $\longrightarrow$ *3*] Assume that $\circ$ is left-invertible under $\Box$. Then

$$\exists * \in \mathrm{Bin}(X) : \ *\Box\circ = \circ_{lz}.$$

Again, since $\circ$ is a graph operation, for all $x, y \in X$, we have $x \circ y \in \{x, y\}$. Therefore, we obtain

$$x = x \circ_{lz} y = x[*\Box\circ]y = (x * y) \circ (y * x) \in \{x * y, y * x\},$$
$$y = y \circ_{lz} x = y[*\Box\circ]x = (y * x) \circ (x * y) \in \{x * y, y * x\}.$$

Note, that if $x \neq y$, we have the inclusion of two-element set in another two-element set, which results into them being equal. In turn, if $x = y$, we have $x * y = x * x = y * x$, so again we have the inclusion of one-element set in one-element set. So, in any case, we have $\{x, y\} = \{x * y, y * x\}$, leading us to the conclusion that $*$ is locally-zero. This fact together with the assumption, Corollary 3.32 and Remark 1.14 (any elemet of the monoid may have at most one two-sided inverse) imply that $* = \circ$, which ends the proof. $\Box$

Having studied the properties of locally-zero operations as elements of center $(Z\mathrm{Bin}(X), \Box)$, let us now discuss their properties simply as operations on X.

Definition 3.34. Let $\circ \in \mathrm{Bin}(X)$ and $x, y, z \in X$. Then (x, y, z) is said to be a *triplet, associative under* $\circ$ if

$$x \circ (y \circ z) = (x \circ y) \circ z.$$

Proposition 3.35. *Let $\circ \in \mathrm{ZBin}(X)$ and $a, b, c \in X$. Then the triplet (a, b, c) is not associative iff a, b, c are pairwise distinct and $E(G(\circ)[\{a, b, c\}]) = \{ac\}$ or $E(G(\circ)[\{a, b, c\}]) = \{ab, bc\}$.*

Proof. At first, let us show that if some of the elements in the triplet are equal, the triplet can only be associative. Consider three following cases.

1. If $a = b$, then

$$a \circ (a \circ c) = \begin{cases} a \circ a = a, & a \circ c = a \\ a \circ c = c, & a \circ c = c \end{cases} = a \circ c = (a \circ a) \circ c.$$

2. If $b = c$, then

$$a \circ (b \circ b) = a \circ b = \begin{cases} a \circ b = a, & a \circ b = a \\ b \circ b = b, & a \circ b = b \end{cases} = (a \circ b) \circ b.$$

3. If $a = c$, then

$$a \circ (b \circ a) = \begin{cases} a \circ b, & b \circ a = b \\ a \circ a, & b \circ a = a \end{cases} = a = \begin{cases} a \circ a, & a \circ b = a \\ b \circ a, & a \circ b = b \end{cases} = (a \circ b) \circ a.$$

Now, let a, b, c be pairwise distinct and consider eight following cases.

1. If $E(G(\circ)[\{a,b,c\}]) = \{ac\}$, then
$$a \circ (b \circ c) = a \circ c = a \neq c = b \circ c = (a \circ b) \circ c.$$

2. If $E(G(\circ)[\{a,b,c\}]) = \{ab, bc\}$, then
$$a \circ (b \circ c) = a \circ b = a \neq c = a \circ c = (a \circ b) \circ c.$$

3. If $E(G(\circ)[\{a,b,c\}]) = \{ab\}$, then
$$a \circ (b \circ c) = a \circ c = c = a \circ c = (a \circ b) \circ c.$$

4. If $E(G(\circ)[\{a,b,c\}]) = \{ac, bc\}$, then
$$a \circ (b \circ c) = a \circ b = b = b \circ c = (a \circ b) \circ c.$$

5. If $E(G(\circ)[\{a,b,c\}]) = \{bc\}$, then
$$a \circ (b \circ c) = a \circ b = b = b \circ c = (a \circ b) \circ c.$$

6. If $E(G(\circ)[\{a,b,c\}]) = \{ab, ac\}$, then
$$a \circ (b \circ c) = a \circ c = a = a \circ c = (a \circ b) \circ c.$$

7. If $E(G(\circ)[\{a,b,c\}]) = \emptyset$, then
$$a \circ (b \circ c) = a \circ b = a = a \circ c = (a \circ b) \circ c.$$

8. If $E(G(\circ)[\{a,b,c\}]) = \{ab, ac, bc\}$, then
$$a \circ (b \circ c) = a \circ c = c = b \circ c = (a \circ b) \circ c.$$

As those are all possible cases, the proposition holds. □

Theorem 3.36. $\circ \in \mathrm{ZBin}(X)$ *is a semigroup iff it is either left- or right-zero operation.*

Proof.
$\longleftarrow$ It is easy to see that both $\circ_{lz}$ and $\circ_{rz}$ form a semigroup on X.
$\longrightarrow$ On the contrary, suppose that $\circ \in Z\mathrm{Bin}(X)$ is a semigroup (i.e. it is associative), but it is neither $\circ_{lz}$ nor $\circ_{rz}$. Then there exist such elements $a, b, c \in X$, that $ab \in E(G(\circ))$ and $bc \notin E(G(\circ))$. Now, if $ac \in E(G(\circ))$, then triplet (b, a, c) is not associative by Proposition 3.35. Besides, if $ac \notin E(G(\circ))$, then triplet (a, c, b) is not associative by Proposition 3.35. Both cases are a contradiction, which proves the theorem. □

Proposition 3.37. *Let $\circ \in \mathrm{ZBin}(X)$ and $|X| \geqslant 2$. Then $\circ$ has no two-sided identity.*

Proof. Suppose that the left identity for $\circ$ exists. Let us denote it as $e \in X$. Fix an element $x \in X \setminus \{e\}$. We have

$$e \circ x = x.$$

By the definition of a locally-zero operation, we conclude that $\circ|_{\{x,e\}} = \circ_{rz}$. So we obtain

$$x \circ e = e \neq x,$$

which means that if e is a left identity, it cannot be a right one and so a two-sided identity as well. □

Proposition 3.38. *Let $\circ \in \mathrm{ZBin}(X)$ and $|X| \geqslant 2$. Then $\circ$ has no two-sided zero.*

Proof. Suppose that the left zero for $\circ$ exists. Let us denote it as $0 \in X$. Fix an element $x \in X$. We have

$$0 \circ x = 0.$$

By the definition of a locally-zero operation, we conclude that $\circ|_{\{x,0\}} = \circ_{lz}$. So we obtain

$$x \circ 0 = x \neq 0,$$

which means that if 0 is a left zero, it cannot be a right one and so a two-sided zero as well. □

Proposition 3.39. *Let $\circ \in \mathrm{ZBin}(X)$. Then for all $x, y \in X$ they commute under $\circ$ iff $x = y$.*

Proof. Fix a pair of elements $x, y \in X$. By the definution of a locally-zero operation, $\circ|_{\{x,y\}} \in \{\circ_{lz}, \circ_{rz}\}$, which means that $\{x \circ y, y \circ x\} = \{x, y\}$. So the proof is straightforward. □

Now, having studied some general properties of $\mathrm{ZBin}(X)$, let us finally have a look at the left-distributivity in $\mathrm{ZBin}(X)$.

Proposition 3.40. *Let $\circ, * \in \mathrm{ZBin}(X)$. Then $\circ \hookrightarrow *$ iff the left-distributivity property holds for all pairwise distinct triplets $a, b, c \in X$, i.e.*

$$\circ \hookrightarrow * \iff \forall a, b, c \in X, a \neq b \neq c : \ a \circ (b * c) = (a \circ b) * (a \circ c).$$

Proof.
$\longrightarrow$ It follows directly from the definition of left-distributivity.
$\longleftarrow$ We want to show that for all $\circ, * \in \mathrm{ZBin}(X)$ and triplets $a, b, c \in X$ the left-distributivity property holds automatically, if some of the elements are equal. Put $\mathcal{A} := a \circ (b * c)$ and $\mathcal{B} := (a \circ b) * (a \circ c)$.

Let $a = b$. Then $\mathcal{A} = a \circ (a * c)$ and $\mathcal{B} = (a \circ a) * (a \circ c) = a * (a \circ c)$. Consider the following four cases:

$$\begin{aligned}
ac \in E(G(\circ)) \cap E(G(*)) &\Longrightarrow a \circ (a * c) = a \circ a = a = a * a = a * (a \circ c),\\
ac \notin E(G(\circ)) \cup E(G(*)) &\Longrightarrow a \circ (a * c) = a \circ c = c = a * c = a * (a \circ c),\\
ac \in E(G(\circ)) \setminus E(G(*)) &\Longrightarrow a \circ (a * c) = a \circ c = a = a * a = a * (a \circ c),\\
ac \in E(G(*)) \setminus E(G(\circ)) &\Longrightarrow a \circ (a * c) = a \circ a = a = a * c = a * (a \circ c).
\end{aligned}$$

In any case, $\mathcal{A} = \mathcal{B}$ holds.

Let $a = c$. Then $\mathcal{A} = a \circ (b * a)$ and $\mathcal{B} = (a \circ b) * (a \circ a) = (a \circ b) * a$. Again, consider the following four cases:

$$\begin{aligned}
ab \in E(G(\circ)) \cap E(G(*)) &\Longrightarrow a \circ (b * a) = a \circ b = a = a * a = (a \circ b) * a,\\
ab \notin E(G(\circ)) \cup E(G(*)) &\Longrightarrow a \circ (b * a) = a \circ a = a = b * a = (a \circ b) * a,\\
ab \in E(G(\circ)) \setminus E(G(*)) &\Longrightarrow a \circ (b * a) = a \circ a = a = a * a = (a \circ b) * a,\\
ab \in E(G(*)) \setminus E(G(\circ)) &\Longrightarrow a \circ (b * a) = a \circ b = b = b * a = (a \circ b) * a.
\end{aligned}$$

In any case, $\mathcal{A} = \mathcal{B}$ holds.

Let $b = c$. Then

$$\mathcal{A} = a \circ (b * b) = a \circ b = (a \circ b) * (a \circ b) = \mathcal{B}.$$

So, in any case $\mathcal{A} = \mathcal{B}$, which is exactly what we wanted to show. $\square$

Now, we are finally ready to propose the following theorem, which lets us characterize $N^+[\circ]$ for $\circ \in Z\mathrm{Bin}(X)$ in terms of graphs.

Theorem 3.41. *Let $\circ, * \in \mathrm{ZBin}(X)$. Then $\circ \hookrightarrow *$ iff for all triplets of pairwise distinct elements $a, b, c \in X$ such that $ab \in E(G(\circ))$ and $ac \notin E(G(\circ))$ it holds*

$$ac \in E(G(*)) \iff bc \in E(G(*)).$$

Proof.
$\longrightarrow$ Assume that $\circ \hookrightarrow *$ and for the triplet of pairwise distinct elements $a, b, c \in X$ holds $ab \in E(G(\circ))$ and $ac \notin E(G(\circ))$.

Firstly, assume $ac \in E(G(*))$ and $bc \notin E(G(*))$. Then

$$c = a \circ c = a \circ (b * c) = (a \circ b) * (a \circ c) = a * c = a,$$

which is a contradiction. Therefore, the left-to-right implication holds.

Now, assume $bc \in E(G(*))$. Then

$$a = a \circ b = a \circ (b * c) = (a \circ b) * (a \circ c) = a * c,$$

implying that $ac \in E(G(*))$. So, the right-to-left implication also holds.

$\longleftarrow$ By Proposition 3.40, we only need to check the case, when $a, b, c \in X$ are pairwise distinct. So, put $\mathcal{A} := a \circ (b * c)$ and $\mathcal{B} := (a \circ b) * (a \circ c)$ and consider four following cases.

Let $ab, ac \in E(G(\circ))$. Then

$$\mathcal{A} = a \circ (b * c) = \begin{cases} a \circ b, & b * c = b \\ a \circ c, & b * c = c \end{cases} = a = a * a = (a \circ b) * (a \circ c) = \mathcal{B}.$$

Let $ab, ac \notin E(G(\circ))$. Then

$$\mathcal{A} = a \circ (b * c) = \begin{cases} a \circ b = b, & b * c = b \\ a \circ c = c, & b * c = c \end{cases} = b * c = (a \circ b) * (a \circ c) = \mathcal{B}.$$

Let $ab \in E(G(\circ))$ and $ac \notin E(G(\circ))$. Keeping in mind the condition in the statement, i.e. $ac \in E(G(*)) \iff bc \in E(G(*))$, consider the following two cases:

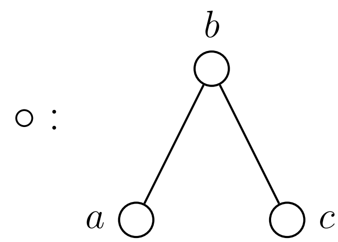
$$ac \in E(G(*)) \Longrightarrow \mathcal{A} = a \circ (b * c) = a \circ b = a = a * c = (a \circ b) * (a \circ c) = \mathcal{B},$$
$$ac \notin E(G(*)) \Longrightarrow \mathcal{A} = a \circ (b * c) = a \circ c = c = a * c = (a \circ b) * (a \circ c) = \mathcal{B}.$$

Finally, let $ab \notin E(G(\circ))$ and $ac \in E(G(\circ))$. Using the condition in the statement of the theorem for the ordered triplet (a, c, b), we obtain that in this case $ab \in E(G(*)) \iff bc \in E(G(*))$. So, again, the following two cases are possible:

$$ab \in E(G(*)) \Longrightarrow \mathcal{A} = a \circ (b * c) = a \circ b = b = b * a = (a \circ b) * (a \circ c) = \mathcal{B},$$
$$ab \notin E(G(*)) \Longrightarrow \mathcal{A} = a \circ (b * c) = a \circ c = a = b * a = (a \circ b) * (a \circ c) = \mathcal{B}.$$

So, in any possible case $\mathcal{A} = \mathcal{B}$, leading us to the conclusion that $\circ \hookrightarrow *$. $\square$

Example 3.42. Let $X = \{a, b, c\}$. Consider an operation $\circ \in \mathrm{ZBin}(X)$ represented by the following graph

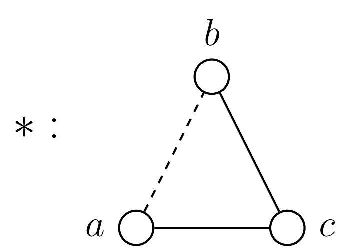


By Theorem 3.41, $\circ \hookrightarrow *$ for some $* \in \mathrm{ZBin}(X)$ iff for all triplets of pairwise distinct elements $a, b, c \in X$ such that $ab \in E(G(\circ))$ and $ac \notin E(G(\circ))$ it holds

$$ac \in E(G(*)) \iff bc \in E(G(*)).$$

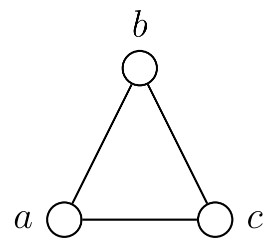
We have only two possible triplets that satisfy the condition, namely (a, b, c) and (c, b, a). Keeping that in mind, let us construct all possible cases for the operation $*$.

First, suppose $ac \in E(G(*))$. Considering the triplet (a, b, c), we immediately obtain $bc \in E(G(*))$, so we have the following picture, where the existence of the edge ba is still in question:



Now, let us have a look at the tripple (c, b, a). Similarly, $ca \in E(G(*))$ results in $ba \in E(G(*))$, so we have

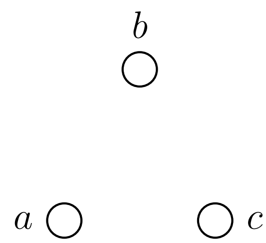
$*:$



That means that $*$ is actually a left-zero operation.

Now, suppose $ac \notin E(G(*))$. Then, taking triplet (a, b, c) into account, we have $bc \notin E(G(*))$. Likewise, considering the tripple (c, b, a), we obtain $ba \notin E(G(*))$. Therefore, we have the following graph for that case

$*:$



So, $*$ is a right-zero operation. Thus, we conclude that $N^+[\circ] = \{\circ_{lz}, \circ_{rz}\}$.

Summary

The main objective of this thesis was to study the left-distributivity relation on the semigroup $(\mathrm{Bin}(X), \square)$.

In Section 1, we gave the basic definitions, which are important for understanding the topic.

In Section 2, we introduced the notion of $\mathrm{Bin}(X)$ together with the $\square$-product, firstly proposed by H.S. Kim and J. Neggers in [5], and recalled some of its properties, such as its algebraic structure, the existence of identity element and the existance of left and right zeros.

We switched to the main topic in Section 3, where we finally introduced the notion of left-distributivity, as well as in- and out-neighborhoods.

Subsection 3.1 was devoted to researching the left-distributivity relation on $\mathrm{Bin}(X)$. At first, we had a look at in- and out-neighborhoods of such frequently used operations, as left- and right-zero. It turned out that all the operations in $\mathrm{Bin}(X)$ distribute from the left with both of them. Moreover, the right-zero operation distributes from the left with all operations in $\mathrm{Bin}(X)$. That observation led us to obtaining the following result: in fact, the intersection of out-neighborhoods of all operations in $\mathrm{Bin}(X)$ is equal to the set only containing left- and right-zero operations; furthermore, the intersection of in-neighborhoods of all operations in $\mathrm{Bin}(X)$ is equal to the set only containing right-zero operation. It was of interest for us to give the example of operation, that has trivial in- and out-neighborhoods, so we proceeded with proving the necessary conditions for that. Then, we proposed the example of operation with trivial out-neighborhood. However, the same question for in-neighborhood still remains open.

In Subsection 3.2, we studied the left-distributivity relation on center of $\mathrm{Bin}(X)$, denoted as $\mathrm{ZBin}(X)$. We began by recalling the notion of $\mathrm{ZBin}(X)$ and some of its properties. The way to represent elements of $\mathrm{ZBin}(X)$ (which one can also call locally-zero operations) with undirected graphs, proposed by S.S. Ahn, H.S. Kim and J. Neggers in [8], was helpful to obtain some interesting results. We were able to efficiently $\square$-multiply locally-zero operations, as the $\square$-product of two elements of $\mathrm{ZBin}(X)$ can be represented as nothing other than

the graph, which is the complement to a simmetric difference of these operations‘ graphs. $(Z\mathrm{Bin}(X), \Box)$ turned out to be a group, with every element being self-invertible. Furthermore, we showed that locally-zero operations are the only graph operations that have an inverse. The algebraic structure of elements of $\mathrm{ZBin}(X)$ themselves was also of interest for us. We proved that only a few of them are semigroups, namely left- and right-zero operations. We also showed that none of them has a two-sided identity or two-sided zero element. Moreover, while all of them commute under $\Box$ with all the operations in $\mathrm{Bin}(X)$ by definition, none of them is commutative itself. Finally, being quite familiar with $\mathrm{ZBin}(X)$, we were able to propose the criterion, which characterizes those locally-zero operations that belong to the out-neigborhood of the given one in terms of graphs.

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