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CATEGORY OF A_∞-CATEGORIES AND DERIVED CATEGORIES

We define natural A_∞-transformations and construct A_∞-category of A_∞-functors. The notion of strict units in an A_∞-category is made weaker. The 2-category of A_∞-categories, functors and transformations is described. We study quotient of an A_∞-category over a full subcategory. The conventional derived category is obtained as the 0-th cohomology of the quotient of differential graded category of complexes over acyclic complexes.

The study of higher homotopy associativity conditions for topological spaces began with Stasheff's article [14, I]. In a sequel to this paper [14, II] Stasheff defines also A_∞-algebras and their homotopy-bar constructions. These algebras and their applications to topology were actively studied, for instance, by Smirnov [12] and Kadeishvili [6, 7]. We adopt some notations of Getzler and Jones [5], which reduce the number of signs in formulas. The notion of an A_∞-category is a natural generalization of A_∞-algebras. It arose in connection with Floer homology in Fukaya's work [2,3] and was related by Kontsevich to mirror symmetry [10]. See Keller [9] for a survey on A_∞-algebras and categories.

In the present article we show that given two A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, one can construct a third A_∞-category $A_\infty(\mathcal{A},\mathcal{B})$, whose objects are A_∞-functors $f : \mathcal{A} \to \mathcal{B}$, and morphisms are natural A_∞-transformations between such functors. This result was also obtained by Fukaya [3] and by Kontsevich and Soibelman [11], independently and, apparently, earlier. We describe compositions between such categories of A_∞-functors, which allow to construct a 2-category of iso-strictly unital A_∞-categories. The latter notion is a generalization of strictly unital A_∞-categories (cf. Keller [9]).

We study properties of Drinfeld's categories or quotients of an A_∞-category over a full subcategory. Originally they were defined by Drinfeld for differential graded categories (private communication). In fact, already Bondal and Kapranov [1] proposed to produce triangulated categories out of differential graded categories. The usefulness of A_∞-approach is explained by our construction of A_∞-functor, which assigns to a complex its K-injective resolution.

Conventions

We assume that all classes and sets are small sets with respect to some universe, $\Bbbk$ denotes a unital associative commutative ring.

It is easy to understand the line

$$\mathcal{A}(X_0,X_1)\otimes_\Bbbk \mathcal{A}(X_1,X_2)\otimes_\Bbbk\cdots\otimes_\Bbbk \mathcal{A}(X_{n-1},X_n),$$

and it is much harder to understand the order in

$$\mathcal{A}(X_{n-1},X_n)\otimes_\Bbbk\cdots\otimes_\Bbbk \mathcal{A}(X_1,X_2)\otimes_\Bbbk \mathcal{A}(X_0,X_1).$$

That is why we use the right operators: the composition of two maps (or morphisms) $f : X \to Y$ and $g : Y \to Z$ is denoted $fg : X \to Z$; a map is written on elements as $f : x \mapsto xf = (x)f$. However, these conventions are not used systematically, and $f(x)$ might be used instead.

If C is a $\mathbb{Z}$-graded $\Bbbk$-module, then sC denotes the same $\Bbbk$-module with the shifted grading $(sC)^d = C^{d+1}$. The "identity" map $C \to sC$ of degree -1 is also denoted s. We follow Getzler–Jones sign convention [5] (Koszul's rule):

$$(x\otimes y)(f\otimes g) = (-)^{yf}xf\otimes yg = (-1)^{\deg y\cdot\deg f}xf\otimes yg.$$

A_∞-categories, A_∞-functors and A_∞-transformations

2.1. Definition. A graded $\Bbbk$-quiver $\mathcal{A}$ is the following data: a class of objects $\operatorname{Ob}\mathcal{A}$; a $\mathbb{Z}$-graded $\Bbbk$-module $\mathcal{A}(X,Y) = \operatorname{Hom}_{\mathcal{A}}(X,Y)$ for each pair of objects X, Y. A morphism of $\Bbbk$-quivers $f : \mathcal{A} \to \mathcal{B}$ is given by a map $f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, $X \mapsto Xf$ and by a $\Bbbk$-linear map $\mathcal{A}(X,Y) \to \mathcal{B}(Xf, Xg)$ for each pair of objects X, Y of $\mathcal{A}$.

To a given graded $\Bbbk$-quiver $\mathcal{A}$ we associate another graded $\Bbbk$-quiver – its tensor coalgebra $T\mathcal{A}$, which has the same class of objects as $\mathcal{A}$. For each sequence $(X_0, X_1, X_2, \dots, X_n)$ of objects of $\mathcal{A}$ there is the $\mathbb{Z}$-graded $\Bbbk$-module $\mathcal{A}(X_0,X_1) \otimes_\Bbbk \mathcal{A}(X_1,X_2) \otimes_\Bbbk \cdots \otimes_\Bbbk \mathcal{A}(X_{n-1},X_n)$. For $n = 0$ it reduces to $\Bbbk$ in degree 0. The graded $\Bbbk$-module $T\mathcal{A}(X,Y) = \oplus_{n\geqslant 0} T^n\mathcal{A}(X,Y)$ is the sum of the above modules over all sequences which start with X and finish with Y. The quiver $T\mathcal{A}$ is equipped with the cut comultiplication

$$\Delta : T\mathcal{A}(X,Y) \to \bigoplus_{Z\in\operatorname{Ob}\mathcal{A}} T\mathcal{A}(X,Z) \bigotimes_{\Bbbk} T\mathcal{A}(Z,Y),$$
$$h_1 \otimes h_2 \otimes \cdots \otimes h_n \mapsto$$
$$\mapsto \sum_{k=0}^{n} h_1 \otimes \cdots \otimes h_k \bigotimes h_{k+1} \otimes \cdots \otimes h_n.$$

2.2. Definition. A_∞-category $\mathcal{A}$ is the following data: a graded $\Bbbk$-quiver $\mathcal{A}$; a differential $b : Ts\mathcal{A} \to Ts\mathcal{A}$ of degree 1, which is a (1,1)-coderivation (we say that a system of $\Bbbk$-linear maps $r : Ts\mathcal{A}(X,Y) \to Ts\mathcal{B}(Xf, Yg)$ is a (f,g)-coderivation, if $f,g : Ts\mathcal{A} \to Ts\mathcal{B}$ are cocategory homomorphisms and equation

$$r\Delta = \Delta(f \otimes r + r \otimes g)$$

holds). It means that a $\Bbbk$-quiver morphism b, determined by a system of $\Bbbk$-linear maps $b\operatorname{pr}_1 : Ts\mathcal{A} \to s\mathcal{A}$ with components of degree 1

$$b_n : s\mathcal{A}(X_0,X_1) \otimes s\mathcal{A}(X_1,X_2) \otimes \cdots \otimes$$
$$\otimes\, s\mathcal{A}(X_{n-1},X_n) \to s\mathcal{A}(X_0,X_n),$$

$n \geqslant 1$, (note that $b_0 = 0$) via formula

$$b_{kl} = (b|_{T^k s\mathcal{A}})\operatorname{pr}_l : T^k s\mathcal{A} \to T^l s\mathcal{A},$$
$$b_{kl} = \sum_{\substack{r+n+t=k\\ r+1+t=l}} 1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}, \qquad (2.1)$$

satisfies equation $b^2 = 0$.

In particular, $b|_{T^0} = 0$, $b_{k0} = 0$, and $k < l$ implies $b_{kl} = 0$. Since b^2 is a (1,1)-coderivation of degree 2, the equation $b^2 = 0$ is equivalent to its particular case $b^2\operatorname{pr}_1 = 0$, that is, for all $k > 0$

$$\sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^k s\mathcal{A} \to s\mathcal{A}. \qquad (2.2)$$

The definition of a coderivation $b\Delta = \Delta(1\otimes b + b\otimes 1)$ implies (2.1).

Using other, more traditional, form of components of b:

$$m_n = (\mathcal{A}^{\otimes n} \xrightarrow{s^{\otimes n}} (s\mathcal{A})^{\otimes n} \xrightarrow{b_n} s\mathcal{A} \xrightarrow{s^{-1}} \mathcal{A})$$

we rewrite (2.2) as follows:

$$\sum_{r+n+t=k} (-)^{t+rn} (1^{\otimes r} \otimes m_n \otimes 1^{\otimes t}) m_{r+1+t} = 0 : T^k\mathcal{A} \to \mathcal{A}.$$

Notice that this equation differs in sign from [9], because we are using right operators!

2.3. Definition. A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is the following data: A_∞-categories $\mathcal{A}$ and $\mathcal{B}$; a map $f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, $X \mapsto Xf$; a cocategory homomorphism $f : Ts\mathcal{A} \to Ts\mathcal{B}$ of degree 0, which commutes with the differential b, that is, a $\Bbbk$-quiver morphism f, determined by a system of $\Bbbk$-linear maps $f\operatorname{pr}_1 : Ts\mathcal{A} \to s\mathcal{B}$ with components of degree 0

$$f_n : s\mathcal{A}(X_0,X_1) \otimes s\mathcal{A}(X_1,X_2) \otimes \cdots \otimes$$
$$\otimes\, s\mathcal{A}(X_{n-1},X_n) \to s\mathcal{B}(X_0f, X_nf),$$

$n \geqslant 1$, (note that $f_0 = 0$) via formula

$$f_{kl} = (f|_{T^k s\mathcal{A}})\operatorname{pr}_l : T^k s\mathcal{A} \to T^l s\mathcal{B},$$
$$f_{kl} = \sum_{i_1+\dots+i_l=k} f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}, \qquad (2.3)$$

such that equation $fb = bf$ holds.

In particular, $f_{00} = \mathrm{id} : \Bbbk_X \to \Bbbk_{Xf}$, and $k < l$ implies $f_{kl} = 0$. Since fb and bf are both (f,f)-coderivations of degree 1, the equation $fb = bf$ is equivalent to its particular case $fb\operatorname{pr}_1 = bf\operatorname{pr}_1$, that is, for all $k > 0$

$$\sum_{l>0;\, i_1+\dots+i_l=k} (f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}) b_l =$$
$$= \sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) f_{r+1+t} : T^k s\mathcal{A} \to s\mathcal{B}. \qquad (2.4)$$

The definition of a cocategory homomorphism $f\Delta = \Delta(f \otimes f)$, $f\varepsilon = \varepsilon$ implies (2.3).

Using m_n we rewrite (2.4) as follows:

$$\sum_{i_1+\dots+i_l=k}^{l>0} (-)^\sigma (f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_l}) m_l = \sum_{r+n+t=k} (-)^{t+rn} (1^{\otimes r} \otimes m_n \otimes 1^{\otimes t}) f_{r+1+t} : T^k \mathcal{A} \to \mathcal{B},$$

$$\sigma = (i_2 - 1) + 2(i_3 - 1) + \dots + (l-2)(i_{l-1} - 1) + (l-1)(i_l - 1).$$

Notice that this equation differs in sign from [9], because we are using right operators.

2.4. Definition. A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ of degree d (pre natural transformation in terms of [3]) is the following data: A_∞-categories $\mathcal{A}$ and $\mathcal{B}$; A_∞-functors $f, g : \mathcal{A} \to \mathcal{B}$; a (f,g)-coderivation $r : Ts\mathcal{A} \to Ts\mathcal{B}$ of degree d. It is determined by a system of $\Bbbk$-linear maps $r\,\mathrm{pr}_1 : Ts\mathcal{A} \to s\mathcal{B}$ with components of degree d

$$r_n : s\mathcal{A}(X_0, X_1) \otimes s\mathcal{A}(X_1, X_2) \otimes \dots \otimes s\mathcal{A}(X_{n-1}, X_n) \to s\mathcal{B}(X_0 f, X_n g),$$

$n \geqslant 0$, via formula

$$r_{kl} = (r|_{T^k s\mathcal{A}})\,\mathrm{pr}_l : T^k s\mathcal{A} \to T^l s\mathcal{B},$$

$$r_{kl} = \sum_{\substack{q+1+t=l \\ i_1+\dots+i_q+n+j_1+\dots+j_t=k}} (f_{i_1} \otimes \dots \otimes f_{i_q} \otimes r_n \otimes g_{j_1} \otimes \dots \otimes g_{j_t}). \tag{2.5}$$

Note that r_0 is a system of $\Bbbk$-linear maps ${}_X r_0 : \Bbbk \to s\mathcal{B}(Xf, Xg)$, $X \in \mathrm{Ob}\,\mathcal{A}$. In fact, the terms '$A_\infty$-transformation' and 'coderivation' are synonyms.

In particular, r_{0l} vanishes unless $l = 1$, and $r_{01} = r_0$. The component r_{kl} vanishes unless $1 \leqslant l \leqslant k + 1$. The definition of a (f,g)-coderivation $r\Delta = \Delta(f \otimes r + r \otimes g)$ implies (2.5).

2.5. Example. The restriction of an A_∞-transformation r to T^1 is

$$r|_{T^1 s\mathcal{A}} = r_1 \oplus [(f_1 \otimes r_0) + (r_0 \otimes g_1)],$$

where $r_1 : s\mathcal{A}(X,Y) \to s\mathcal{B}(Xf, Yg)$,

$$f_1 \otimes r_0 : s\mathcal{A}(X,Y) = s\mathcal{A}(X,Y) \otimes \Bbbk \xrightarrow{f_1 \otimes r_0} s\mathcal{B}(Xf, Yf) \otimes s\mathcal{B}(Yf, Yg),$$

$$r_0 \otimes g_1 : s\mathcal{A}(X,Y) = \Bbbk \otimes s\mathcal{A}(X,Y) \xrightarrow{r_0 \otimes g_1} s\mathcal{B}(Xf, Xg) \otimes s\mathcal{B}(Xg, Yg).$$

Cocategory of coderivations

Let $\mathcal{A}$, $\mathcal{B}$ be graded $\Bbbk$-quivers, and let $f^0, f^1, \dots, f^n : T\mathcal{A} \to T\mathcal{B}$ be cocategory homomorphisms. Consider n coderivations $r_1, \dots, r_n$ as in

$$f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{n-1} \xrightarrow{r^n} f^n : T\mathcal{A} \to T\mathcal{B}.$$

We construct from these data the following system of $\Bbbk$-linear maps $\theta = (r^1 \otimes \dots \otimes r^n)\theta : T\mathcal{A}(X,Y) \to T\mathcal{B}(Xf^0, Yf^n)$ of degree $\deg r^1 + \dots + \deg r^n$. Its components $\theta_{kl} = \theta|_{T^k\mathcal{A}}\,\mathrm{pr}_l : T^k\mathcal{A} \to T^l\mathcal{B}$ are given by the following formula

$$\theta_{kl} = \sum f^0_{i^0_1} \otimes \dots \otimes f^0_{i^0_{m_0}} \otimes r^1_{j_1} \otimes f^1_{i^1_1} \otimes \dots \otimes f^1_{i^1_{m_1}} \otimes \dots \otimes r^n_{j_n} \otimes f^n_{i^n_1} \otimes \dots \otimes f^n_{i^n_{m_n}},$$

where summation is taken over all terms with $m_0 + m_1 + \dots + m_n + n = l$,

$$i^0_1 + \dots + i^0_{m_0} + j_1 + i^1_1 + \dots + i^1_{m_1} + \dots + j_n + i^n_1 + \dots + i^n_{m_n} = k.$$

The component θ_{kl} vanishes unless $n \leqslant l \leqslant k + n$. If $n = 0$, we set $()\theta = f^0$. If $n = 1$, the formula gives $(r^1)\theta = r^1$.

3.1. Proposition. *For each $n \geqslant 0$ the map θ satisfies the equation*

$$(p^1 \otimes p^2 \otimes \dots \otimes p^n)\theta\Delta = \Delta \sum_{k=0}^{n} (p^1 \otimes \dots \otimes p^k)\theta \otimes (p^{k+1} \otimes \dots \otimes p^n)\theta. \tag{3.1}$$

Cocategory homomorphisms. Graded cocategories form a symmetric monoidal category gCoCat. If $\mathcal{C}$ and $\mathcal{D}$ are cocategories, there is a new cocategory $\mathcal{C} \otimes \mathcal{D}$, whose class of objects is $\mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{D}$, and $\mathcal{C} \otimes \mathcal{D}(X \times U, Y \times W) = \mathcal{C}(X,Y) \otimes_\Bbbk \mathcal{D}(U,W)$. It is equipped with tensor product of comultiplications, using the symmetry in graded $\Bbbk$-modules.

Let $\phi : T\mathcal{A} \otimes T\mathcal{C} \to T\mathcal{B}$ be a cocategory homomorphism. It is determined uniquely by its composition with pr_1, that is, by a family $\phi\,\mathrm{pr}_1 = (\phi_{nm})_{n,m \geqslant 0}$, $\phi_{nm} : T^n\mathcal{A} \otimes T^m\mathcal{C} \to \mathcal{B}$, $\phi_{00} = 0$. Indeed, for given families of composable arrows $f^0 \xrightarrow{p^1} f^1 \xrightarrow{p^2} \dots f^{n-1} \xrightarrow{p^n} f^n$ of $\mathcal{A}$ and $g^0 \xrightarrow{t^1} g^1 \xrightarrow{t^2} \dots \to g^{m-1} \xrightarrow{t^m} g^m$ of $\mathcal{C}$ we have

$$(p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\phi =$$
$$= \sum_{\substack{i_1+\dots+i_k=n\\ j_1+\dots+j_k=m}} (-)^\sigma (p^1 \otimes \dots \otimes p^{i_1} \otimes t^1 \otimes \dots \otimes t^{j_1})\phi_{i_1 j_1} \otimes$$
$$\otimes (p^{i_1+1} \otimes \dots \otimes p^{i_1+i_2} \otimes t^{j_1+1} \otimes \dots \otimes t^{j_1+j_2})\phi_{i_2 j_2}$$
$$\otimes \dots \otimes (p^{i_1+\dots+i_{k-1}+1} \otimes \dots \otimes$$
$$\otimes p^{i_1+\dots+i_k} \otimes t^{j_1+\dots+j_{k-1}+1} \otimes \dots \otimes t^{j_1+\dots+j_k})\phi_{i_k j_k}.$$

The sign depends on parity of

$$\sigma = (t^1 + \dots + t^{j_1})(p^{i_1+1} + \dots + p^{i_1+\dots+i_k}) +$$
$$+ (t^{j_1+1} + \dots + t^{j_1+j_2})(p^{i_1+i_2+1} + \dots + p^{i_1+\dots+i_k})$$
$$+ \dots + (t^{j_1+\dots+j_{k-2}+1} + \dots + t^{j_1+\dots+j_{k-1}}) \times$$
$$\times (p^{i_1+\dots+i_{k-1}+1} + \dots + p^{i_1+\dots+i_k}).$$

We abbreviate $(-1)^{(\deg r)(\deg p)}$ to $(-)^{rp}$. By definition homomorphism ϕ satisfies equation

$$\begin{array}{ccccc} T\mathcal{A}\otimes T\mathcal{C} & \xrightarrow{\phi} & T\mathcal{B} & \xrightarrow{\Delta} & T\mathcal{B}\otimes T\mathcal{B} \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \uparrow{\scriptstyle \phi\otimes\phi} \\ T\mathcal{A}\otimes T\mathcal{A}\otimes T\mathcal{C}\otimes T\mathcal{C} & & \xrightarrow{1\otimes c\otimes 1} & & T\mathcal{A}\otimes T\mathcal{C}\otimes T\mathcal{A}\otimes T\mathcal{C} \end{array}$$

where $(f \otimes g)c = (-)^{fg} g \otimes f$ is the 'signed' symmetry.

Introduce $\Bbbk$-linear maps $(p^1 \otimes \dots \otimes p^n)\chi : T\mathcal{A} \to T\mathcal{B}$ by the formula $a[(p^1 \otimes \dots \otimes p^n)\chi] = (a \otimes p^1 \otimes \dots \otimes p^n)\phi$, $a \in T\mathcal{A}$. Then the above equation is equivalent to

$$(p^1 \otimes p^2 \otimes \dots \otimes p^n)\chi\Delta =$$
$$= \Delta \sum_{k=0}^{n} (p^1 \otimes \dots \otimes p^k)\chi \otimes (p^{k+1} \otimes \dots \otimes p^n)\chi.$$

for all $n \geqslant 0$.

3.2. Proposition. *The map* $\alpha : T\mathcal{A} \otimes T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \to T\mathcal{B}$, $a \otimes p^1 \otimes \dots \otimes p^n \mapsto a[(p^1 \otimes \dots \otimes p^n)\theta]$, *is a cocategory homomorphism of degree 0. For any cocategory homomorphism* $\phi : T\mathcal{A} \otimes T\mathcal{C}^1 \otimes T\mathcal{C}^2 \otimes \dots \otimes T\mathcal{C}^q \to T\mathcal{B}$ *of degree 0 there is a unique cocategory homomorphism* $\psi : T\mathcal{C}^1 \otimes T\mathcal{C}^2 \otimes \dots \otimes T\mathcal{C}^q \to T\operatorname{Coder}(\mathcal{A},\mathcal{B})$ *of degree 0, such that*

$$\phi = \bigl(T\mathcal{A} \otimes T\mathcal{C}^1 \otimes T\mathcal{C}^2 \otimes \dots \otimes T\mathcal{C}^q \xrightarrow{1\otimes\psi}$$
$$\xrightarrow{1\otimes\psi} T\mathcal{A} \otimes T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \xrightarrow{\alpha} T\mathcal{B}\bigr).$$

We interpret the above proposition as existence of internal hom-objects $\mathsf{Hom}(T\mathcal{A}, T\mathcal{B}) = T\operatorname{Coder}(\mathcal{A},\mathcal{B})$ in the monoidal category of cocategories of the form $T\mathcal{C}^1 \otimes T\mathcal{C}^2 \otimes \dots \otimes T\mathcal{C}^r$.

Enriched category of graded $\Bbbk$-quivers

Let us show that category of graded $\Bbbk$-quivers is enriched in gCoCat. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be graded $\Bbbk$-quivers. Consider the cocategory homomorphism given by the upper right path in the diagram

$$\begin{array}{ccc} T\mathcal{A}\otimes T\operatorname{Coder}(\mathcal{A},\mathcal{B})\otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) & \xrightarrow{\alpha\otimes 1} & T\mathcal{B}\otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) \\ \downarrow{\scriptstyle 1\otimes M} & = & \downarrow{\scriptstyle \alpha} \\ T\mathcal{A}\otimes T\operatorname{Coder}(\mathcal{A},\mathcal{C}) & \xrightarrow{\alpha} & T\mathcal{C} \end{array} \qquad (4.1)$$

By Proposition 3.2 there is a graded cocategory morphism of degree 0

$$M : T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) \to$$
$$\to T\operatorname{Coder}(\mathcal{A},\mathcal{C}).$$

Denote by $\mathbb{I}$ a graded 1-object-0-morphisms $\Bbbk$-quiver, that is, $\operatorname{Ob}\mathbb{I} = \{1\}$, $\mathbb{I}(1,1) = 0$. Then $T\mathbb{I} = \Bbbk$ is a unit object of the monoidal category of graded cocategories. Denote by $\mathbf{r} : T\mathcal{A} \otimes T\mathbb{I} \to T\mathcal{A}$ and $\mathbf{l} : T\mathbb{I} \otimes T\mathcal{A} \to T\mathcal{A}$ the corresponding natural cocategory isomorphisms. By Proposition 3.2 there exists a unique cocategory morphism $\eta_{\mathcal{A}} : T\mathbb{I} \to T\operatorname{Coder}(\mathcal{A},\mathcal{A})$, such that

$$\mathbf{r} = \bigl(T\mathcal{A} \otimes T\mathbb{I} \xrightarrow{1\otimes\eta_{\mathcal{A}}}$$
$$\xrightarrow{1\otimes\eta_{\mathcal{A}}} T\mathcal{A} \otimes T\operatorname{Coder}(\mathcal{A},\mathcal{A}) \xrightarrow{\alpha} T\mathcal{A}\bigr).$$

Namely, the object $1 \in \operatorname{Ob}\mathbb{I}$ goes to the identity homomorphism $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, which acts as identity map on objects, and has only one non-vanishing component

$$(\mathrm{id}_{\mathcal{A}})_1 = \mathrm{id} : \mathcal{A}(X,Y) \to \mathcal{A}(X,Y).$$

4.1. Proposition. *(See also Kontsevich and Soibelman [11]) Multiplication* M *is associative and* η *is its two-sided unit:*

$$\begin{array}{ccc} T\operatorname{Coder}(\mathcal{A},\mathcal{B})\otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C})\otimes T\operatorname{Coder}(\mathcal{C},\mathcal{D}) & \xrightarrow{M\otimes 1} & T\operatorname{Coder}(\mathcal{A},\mathcal{C})\otimes T\operatorname{Coder}(\mathcal{C},\mathcal{D}) \\ \downarrow{\scriptstyle 1\otimes M} & & \downarrow{\scriptstyle M} \\ T\operatorname{Coder}(\mathcal{A},\mathcal{B})\otimes T\operatorname{Coder}(\mathcal{B},\mathcal{D}) & \xrightarrow{M} & T\operatorname{Coder}(\mathcal{A},\mathcal{D}) \end{array}$$

Let us find explicit formulas for M. It is defined on objects as composition: if $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ are cocategory morphisms, then $(f, g)M = fg : \mathcal{A} \to \mathcal{C}$. On coderivations M is specified by its composition with $\mathrm{pr}_1 : T\,\mathrm{Coder}(\mathcal{A}, \mathcal{C}) \to \mathrm{Coder}(\mathcal{A}, \mathcal{C})$. Let

$$M_{nm} = M\big|_{T^n \otimes T^m} \mathrm{pr}_1 :$$

$$T^n\,\mathrm{Coder}(\mathcal{A}, \mathcal{B}) \otimes T^m\,\mathrm{Coder}(\mathcal{B}, \mathcal{C}) \to \mathrm{Coder}(\mathcal{A}, \mathcal{C}),$$

$$M_{nm} : (f^0, f^1) \otimes \cdots \otimes (f^{n-1}, f^n) \otimes \otimes (g^0, g^1) \otimes \cdots \otimes (g^{m-1}, g^m) \to (f^0 g^0, f^n g^m)$$

be its components, where $f^0, \dots, f^n : \mathcal{A} \to \mathcal{B}$ and $g^0, \dots, g^m : \mathcal{B} \to \mathcal{C}$ are cocategory morphisms. Then $M_{00} = 0$.

The component M_{nm} is determined recursively from the following equation:

$$\begin{aligned}(p^1 \otimes \cdots \otimes p^n)\theta(t^1 \otimes \cdots \otimes t^m)\theta = \\ = (p^1 \otimes \cdots \otimes p^n \otimes t^1 \otimes \cdots \otimes t^m) M_{nm} \\ + \sum_{\substack{i_1+\cdots+i_k=n \\ j_1+\cdots+j_k=m}}^{k>1} (-)^\sigma \big[(p^1 \otimes \cdots \otimes p^{i_1} \otimes t^1 \otimes \cdots \otimes t^{j_1}) M_{i_1 j_1} \otimes \\ \otimes (p^{i_1+1} \otimes \cdots \otimes p^{i_1+i_2} \otimes t^{j_1+1} \otimes \cdots \otimes t^{j_1+j_2}) M_{i_2 j_2} \otimes \\ \otimes \cdots \otimes (p^{i_1+\cdots+i_{k-1}+1} \otimes \cdots \otimes p^{i_1+\cdots+i_k} \otimes \\ \otimes t^{j_1+\cdots+j_{k-1}+1} \otimes \cdots \otimes t^{j_1+\cdots+j_k}) M_{i_k j_k}\big]\theta. \quad (4.2)\end{aligned}$$

Since $(p^1 \otimes \cdots \otimes p^n \otimes t^1 \otimes \cdots \otimes t^m) M_{nm}$ is a coderivation, it is determined by its composition with projection pr_1. Composing (4.2) with pr_1 we get

$$\begin{aligned}(p^1 \otimes \cdots \otimes p^n)\theta(t^1 \otimes \cdots \otimes t^m)\theta\,\mathrm{pr}_1 = \\ = (p^1 \otimes \cdots \otimes p^n \otimes t^1 \otimes \cdots \otimes t^m) M_{nm}\,\mathrm{pr}_1. \quad (4.3)\end{aligned}$$

Therefore, if $m = 0$ and n is positive, M_{n0} is given by the formula:

$$M_{n0} : (f^0, f^1) \otimes \ldots \otimes (f^{n-1}, f^n) \otimes \Bbbk_{g^0} \to (f^0 g^0, f^n g^0),$$

$$(r^1 \otimes \cdots \otimes r^n) \mapsto (r^1 \otimes \cdots \otimes r^n \otimes g^0) M_{n0},$$

$$[(r^1 \otimes \cdots \otimes r^n \otimes g^0) M_{n0}]\,\mathrm{pr}_1 = (r^1 \otimes \cdots \otimes r^n)\theta g^0\,\mathrm{pr}_1.$$

If $m = 1$, then M_{n1} is given by the formula:

$$M_{n1} : (f^0, f^1) \otimes \ldots \otimes (f^{n-1}, f^n) \otimes \otimes (g^0, g^1) \to (f^0 g^0, f^n g^1),$$

$$(r^1 \otimes \cdots \otimes r^n \otimes t^1) \mapsto (r^1 \otimes \cdots \otimes r^n \otimes t^1) M_{n1},$$

$$[(r^1 \otimes \cdots \otimes r^n \otimes t^1) M_{n1}]\,\mathrm{pr}_1 = (r^1 \otimes \cdots \otimes r^n)\theta t^1\,\mathrm{pr}_1.$$

Explicitly we write

$$[(r^1 \otimes \cdots \otimes r^n \otimes g^0) M_{n0}]_k = \sum_l (r^1 \otimes \cdots \otimes r^n)\theta_{kl} g^0_l, \quad (4.4)$$

$$[(r^1 \otimes \cdots \otimes r^n \otimes t^1) M_{n1}]_k = \sum_l (r^1 \otimes \cdots \otimes r^n)\theta_{kl} t^1_l. \quad (4.5)$$

Finally, $M_{nm} = 0$ for $m > 1$, since the left hand side of (4.3) vanishes.

4.2. Examples. 1) The component M_{01} is the composition: $(f^0 \otimes t^1) M_{01} = f^0 t^1$.

2) The component M_{10} is the composition: $(r^1 \otimes g^0) M_{10} = r^1 g^0$.

3) If $r : f \to g : \mathcal{A} \to \mathcal{B}$ and $p : h \to k : \mathcal{B} \to \mathcal{C}$ are A_∞-transformations, then $(r \otimes p) M_{11}$ has the following components:

$$[(r \otimes p) M_{11}]_0 = r_0 p_1,$$
$$[(r \otimes p) M_{11}]_1 = r_1 p_1 + (f_1 \otimes r_0) p_2 + + (r_0 \otimes g_1) p_2, \text{ etc.}$$

4) If $f \xrightarrow{r} g \xrightarrow{p} h : \mathcal{A} \to \mathcal{B}$ are A_∞-transformations, and $k : \mathcal{B} \to \mathcal{C}$ is an A_∞-functor, then $(r \otimes p \otimes k) M_{20}$ has the following components:

$$[(r \otimes p \otimes k) M_{20}]_0 = (r_0 \otimes p_0) k_2,$$
$$\begin{aligned}[(r \otimes p \otimes k) M_{20}]_1 = (r_1 \otimes p_0) k_2 + (r_0 \otimes p_1) k_2 + \\ +(r_0 \otimes p_0 \otimes h_1) k_3 + (r_0 \otimes g_1 \otimes p_0) k_3 + \\ +(f_1 \otimes r_0 \otimes p_0) k_3.\end{aligned}$$

5) If $f \xrightarrow{r} g \xrightarrow{p} h : \mathcal{A} \to \mathcal{B}$ and $t : k \to l : \mathcal{B} \to \mathcal{C}$ are A_∞-transformations, then $(r \otimes p \otimes t) M_{21}$ has the following components:

$$[(r \otimes p \otimes t) M_{21}]_0 = (r_0 \otimes p_0) t_2,$$
$$\begin{aligned}[(r \otimes p \otimes t) M_{21}]_1 = (r_1 \otimes p_0) t_2 + (r_0 \otimes p_1) t_2 + \\ +(r_0 \otimes p_0 \otimes h_1) t_3 + (r_0 \otimes g_1 \otimes p_0) t_3 + \\ +(f_1 \otimes r_0 \otimes p_0) t_3.\end{aligned}$$

A_∞-category of A_∞-functors

Let us construct a new A_∞-category $A_\infty(\mathcal{A}, \mathcal{B})$ out of given two $\mathcal{A}$ and $\mathcal{B}$. First we describe its underlying graded $\Bbbk$-quiver. Its objects are A_∞-functors $f : \mathcal{A} \to \mathcal{B}$. Given two such functors $f, g : \mathcal{A} \to \mathcal{B}$ we define the graded $\Bbbk$-module $\mathrm{Hom}_{A_\infty(\mathcal{A},\mathcal{B})}(f,g)$ as the space of all A_∞-transformations $r : f \to g$, namely,

$$[\mathrm{Hom}_{A_\infty(\mathcal{A},\mathcal{B})}(f,g)]^{d+1} = \{r : f \to g \mid \text{transformation } r : Ts\mathcal{A} \to Ts\mathcal{B} \text{ has degree } d\}.$$

Denote $(f,g) = s\,\mathrm{Hom}_{A_\infty(\mathcal{A},\mathcal{B})}(f,g) = \mathrm{Hom}_{\mathrm{Coder}(s\mathcal{A},s\mathcal{B})}(f,g)$ for the sake of brevity. The degree of r as an element of (f,g) will be exactly d:

$$(f,g)^d = \{r : f \to g \mid \text{transformation } r : Ts\mathcal{A} \to Ts\mathcal{B} \text{ has degree } d\}.$$

We will use only this (natural) degree of r in order to permute it with other things by Koszul's rule.

5.1. Proposition. *(See also Fukaya [3] and Kontsevich and Soibelman [11]) There exists a unique (1,1)-coderivation $B : TsA_\infty(\mathcal{A},\mathcal{B}) \to TsA_\infty(\mathcal{A},\mathcal{B})$ of degree 1, such that $B_0 = 0$ and*

$$(r^1\otimes\cdots\otimes r^n)\theta b = [(r^1\otimes\cdots\otimes r^n)B]\theta + (-)^{r^1+\cdots+r^n} b(r^1\otimes\cdots\otimes r^n)\theta \quad (5.1)$$

for all $n \geqslant 0$, $r^1\otimes\cdots\otimes r^n \in (f^0,f^1)\otimes\cdots\otimes(f^{n-1},f^n)$. It satisfies $B^2=0$, thus, it gives an A_∞-structure of $A_\infty(\mathcal{A},\mathcal{B})$.

Let us find explicitly the components of B, composing (5.1) with $\mathrm{pr}_1 : Ts\mathcal{B} \to s\mathcal{B}$:

$$B_1 : (f,g) \to (f,g), \quad r \mapsto (r)B_1 = [r,b] = rb - (-)^r br,$$
$$B_n : (f^0,f^1)\otimes\cdots\otimes(f^{n-1},f^n) \to (f^0,f^n),$$
$$r^1\otimes\cdots\otimes r^n \mapsto (r^1\otimes\cdots\otimes r^n)B_n, \text{ for } n>1,$$

where the last transformation is defined by its composition with pr_1:

$$[(r^1\otimes\cdots\otimes r^n)B_n]\,\mathrm{pr}_1 = [(r^1\otimes\cdots\otimes r^n)\theta]b\,\mathrm{pr}_1,$$

in other terms, for $n>1$

$$[(r^1\otimes\cdots\otimes r^n)B_n]_k = \sum_l (r^1\otimes\cdots\otimes r^n)\theta_{kl}b_l. \quad (5.2)$$

Since $B^2=0$, we have, in particular,

$$\sum_{r+n+t=k} (1^{\otimes r}\otimes B_n\otimes 1^{\otimes t})B_{r+1+t} = 0 : T^k sA_\infty(\mathcal{A},\mathcal{B}) \to sA_\infty(\mathcal{A},\mathcal{B}).$$

5.2. Examples. 1) When $n=1$, $r : f \to g : \mathcal{A} \to \mathcal{B}$, we find components of A_∞-transformation $(r)B_1$ as follows (see Examples 2.5):

$$[(r)B_1]_0 = r_0b_1,$$
$$[(r)B_1]_1 = r_1b_1 + (f_1\otimes r_0)b_2 + (r_0\otimes g_1)b_2 - (-)^r b_1r_1,$$
$$[(r)B_1]_2 = r_2b_1 + (f_2\otimes r_0)b_2 + (f_1\otimes r_1)b_2 + (r_1\otimes g_1)b_2 + (r_0\otimes g_2)b_2 + (f_1\otimes f_1\otimes r_0)b_3 + (f_1\otimes r_0\otimes g_1)b_3 + (r_0\otimes g_1\otimes g_1)b_3 - (-)^r b_2r_1 - (-)^r(1\otimes b_1)r_2 - (-)^r(b_1\otimes 1)r_2.$$

2) When $n=2$, $f \xrightarrow{r} g \xrightarrow{p} h : \mathcal{A} \to \mathcal{B}$, we find components of A_∞-transformation $(r\otimes p)B_2$ as follows:

$$[(r\otimes p)B_2]_0 = (r_0\otimes p_0)b_2,$$
$$[(r\otimes p)B_2]_1 = (r_1\otimes p_0)b_2 + (r_0\otimes p_1)b_2 + (r_0\otimes p_0\otimes h_1)b_3 + (r_0\otimes g_1\otimes p_0)b_3 + (f_1\otimes r_0\otimes p_0)b_3.$$

Differentials. Let $\mathcal{A}$, $\mathcal{C}^1$, $\mathcal{C}^2$, $\dots$, $\mathcal{C}^q$, $\mathcal{B}$ be A_∞-categories. Let $\phi : Ts\mathcal{A}\otimes Ts\mathcal{C}^1\otimes Ts\mathcal{C}^2\otimes\cdots\otimes Ts\mathcal{C}^q \to Ts\mathcal{B}$, $(a\otimes c^1\otimes c^2\otimes\cdots\otimes c^q) \mapsto a.(c^1\otimes c^2\otimes\cdots\otimes c^q)\chi$ be a cocategory homomorphism of degree 0. The homomorphism ϕ commutes with the differential:

$$\phi b = \Bigl(\sum_{r+t=q} 1^{\otimes r}\otimes b\otimes 1^{\otimes t}\Bigr)\phi$$

if and only if χ does:

$$(c^1\otimes c^2\otimes\cdots\otimes c^q)\chi b = \sum_{k=1}^q (-)^{c^{k+1}+\cdots+c^q}(c^1\otimes\cdots\otimes c^kb\otimes\cdots\otimes c^q)\chi + (-)^{c^1+\cdots+c^q} b(c^1\otimes c^2\otimes\cdots\otimes c^q)\chi. \quad (5.3)$$

In particular, for $q=1$ we get an equation $(c)\chi b = (cb)\chi + (-)^c b(c)\chi$.

5.3. Corollary. *There is a unique A_∞-category structure for $A_\infty(\mathcal{A},\mathcal{B})$, such that the action homomorphism $\alpha : Ts\mathcal{A}\otimes TsA_\infty(\mathcal{A},\mathcal{B}) \to Ts\mathcal{B}$ is an A_∞-functor.*

5.4. Proposition. *For any A_∞-functor $\phi : Ts\mathcal{A}\otimes Ts\mathcal{C}^1\otimes Ts\mathcal{C}^2\otimes\cdots\otimes Ts\mathcal{C}^q \to Ts\mathcal{B}$ there is a unique A_∞-functor $\psi : Ts\mathcal{C}^1\otimes Ts\mathcal{C}^2\otimes\cdots\otimes Ts\mathcal{C}^q \to TsA_\infty(\mathcal{A},\mathcal{B})$, such that*

$$\phi = \bigl(Ts\mathcal{A}\otimes Ts\mathcal{C}^1\otimes Ts\mathcal{C}^2\otimes\cdots\otimes Ts\mathcal{C}^q \xrightarrow{1\otimes\psi} Ts\mathcal{A}\otimes TsA_\infty(\mathcal{A},\mathcal{B}) \xrightarrow{\alpha} Ts\mathcal{B}\bigr). \quad (5.4)$$

Enriched category of A_∞-categories

Graded differential cocategories form a symmetric monoidal category dgCoCat. If $\mathcal{C}$ and $\mathcal{D}$ are cocategories, there is a new cocategory $\mathcal{C}\otimes\mathcal{D}$, whose class of objects is $\operatorname{Ob}\mathcal{C}\times\operatorname{Ob}\mathcal{D}$ and $\mathcal{C}\otimes\mathcal{D}(X\times U, Y\times W)=\mathcal{C}(X,Y)\otimes_{\Bbbk}\mathcal{D}(U,W)$. It is equipped with tensor product of comultiplications, using the symmetry in dg-$\Bbbk$-modules. Therefore, there might be categories enriched in dgCoCat. It turns out that category of A_∞-categories is one of them.

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be A_∞-categories. There is a graded cocategory morphism of degree 0

$$M : TsA_\infty(\mathcal{A},\mathcal{B})\otimes TsA_\infty(\mathcal{B},\mathcal{C})\to TsA_\infty(\mathcal{A},\mathcal{C}),$$

defined in Section 4 via diagram (4.1). Since all cocategory morphisms α, $\alpha\otimes 1$ in this diagram commute with the differential by Corollary 5.3, the cocategory morphism M also commutes with the differential:

$$(1\otimes B+B\otimes 1)M=MB \tag{6.1}$$

by Proposition 5.4. Therefore, M is an A_∞-functor. The unit $\eta_{\mathcal{A}} : T\mathbb{I}\to T\operatorname{Coder}(\mathcal{A},\mathcal{A})$, $1\mapsto \mathrm{id}_{\mathcal{A}}$ also is an A_∞-functor for trivial reasons. We summarize the above statements as follows: category A_∞ of A_∞-categories is enriched in dgCoCat. Moreover, it is enriched in monoidal subcategory of dgCoCat generated by $Ts\mathcal{C}$, where $\mathcal{C}$ are A_∞-categories.

6.1. Definition (ω-globular set of A_∞-categories). Natural A_∞-transformation $r : f\to g : \mathcal{A}\to\mathcal{B}$ (natural transformation in terms of [3]) is an A_∞-transformation of degree -1 such that $rb+br=0$ (that is, $(r)B_1=0$). The ω-globular set A_ω of A_∞-categories is defined as follows: objects (0-morphisms) are A_∞-categories $\mathcal{A}$; 1-morphisms are A_∞-functors $f : \mathcal{A}\to\mathcal{B}$; 2-morphisms are natural A_∞-transformations $r : f\to g$ $\mathcal{A}\to\mathcal{B}$; 3-morphisms $\lambda : r\to s : f\to g : \mathcal{A}\to\mathcal{B}$ are (f,g)-coderivations of degree -2, such that $r-s=[\lambda,b]$; for $n\geqslant 3$ n-morphisms $\lambda_n : \lambda_{n-1}\to\mu_{n-1} : \dots : r\to s : f\to g : \mathcal{A}\to\mathcal{B}$ are (f,g)-coderivations of degree $1-n$, such that $\lambda_{n-1}-\mu_{n-1}=[\lambda_n,b]$ (notice that both sides are (f,g)-coderivations of degree $2-n$).

Below we describe a truncation of this ω-globular set which is a 2-category.

A 2-category of A_∞-categories

Its objects are A_∞-categories, 1-morphisms are A_∞-functors, and 2-morphisms are equivalence classes of natural A_∞-transformations. Natural A_∞-transformations $r,s : f\to g : \mathcal{A}\to\mathcal{B}$ are declared equivalent, if they are connected by a 3-morphism $\lambda : r\to s$. Clearly, this is an equivalence relation. Compositions are defined as follows.

Composition of 1-morphisms $\operatorname{Mor}_1(\mathcal{A},\mathcal{B})\times\operatorname{Mor}_1(\mathcal{B},\mathcal{C})\to\operatorname{Mor}_1(\mathcal{A},\mathcal{C})$, $(f,h)\mapsto fh$ is the composition of A_∞-functors. Both compositions of 1-morphisms with 2-morphisms $\operatorname{Mor}_2(\mathcal{A},\mathcal{B})\times\operatorname{Mor}_1(\mathcal{B},\mathcal{C})\to\operatorname{Mor}_2(\mathcal{A},\mathcal{C})$, $(r,h)\mapsto rh$ and $\operatorname{Mor}_1(\mathcal{A},\mathcal{B})\times\operatorname{Mor}_2(\mathcal{B},\mathcal{C})\to\operatorname{Mor}_2(\mathcal{A},\mathcal{C})$, $(f,p)\mapsto fp$ are compositions of $\Bbbk$-linear maps $Ts\mathcal{A}\to Ts\mathcal{B}\to Ts\mathcal{C}$. Compatibility with the equivalence relation is immediate: $rh-sh=[\lambda h,b]$, $fp-ft=[f\lambda,b]$.

For $f,g,h : \mathcal{A}\to\mathcal{B}$ we define vertical composition of 2-morphisms $\operatorname{Mor}_2(f,g)\otimes\operatorname{Mor}_2(g,h)\to\operatorname{Mor}_2(f,h)$, $r\otimes p\mapsto r\cdot p$ as the quotient map of $B_2 : (f,g)\otimes(g,h)\to(f,h)$ modulo equivalence relation. We have to prove that $r\cdot p$, the equivalence class of $(r\otimes p)B_2$, is well-defined. Considering the case $f\overset{r\Rrightarrow s}{\xrightarrow{\lambda}} g\xrightarrow{p} h : \mathcal{A}\to\mathcal{B}$ we find that

$$r\cdot p-s\cdot p=([\lambda,b]\otimes p)B_2=$$
$$=-(\lambda\otimes p)(1\otimes B_1+B_1\otimes 1)B_2=(\lambda\otimes p)B_2B_1,$$

thus, $r\cdot p$ and $s\cdot p$ are equivalent. Considering the case $f\xrightarrow{r} g\overset{p\Rrightarrow t}{\xrightarrow{\mu}} h : \mathcal{A}\to\mathcal{B}$ we find that

$$r\cdot p-r\cdot t=(r\otimes[\mu,b])B_2=$$
$$=(r\otimes\mu)(1\otimes B_1+B_1\otimes 1)B_2=-(r\otimes\mu)B_2B_1,$$

thus, $r\cdot p$ and $r\cdot t$ are equivalent. The vertical composition of 2-morphisms is associative, since for all $f\xrightarrow{r} g\xrightarrow{p} h\xrightarrow{t} k : \mathcal{A}\to\mathcal{B}$ we have

$$r\cdot(p\cdot t)-(r\cdot p)\cdot t=(r\otimes p\otimes t)(1\otimes B_2)B_2+$$
$$+(r\otimes p\otimes t)(B_2\otimes 1)B_2=-(r\otimes p\otimes t)B_3B_1,$$

thus, $r\cdot(p\cdot t)$ and $(r\cdot p)\cdot t$ are equivalent.

Let us check compatibility of compositions in the case $f\xrightarrow{r} g\xrightarrow{p} h : \mathcal{A}\to\mathcal{B}$, $k : \mathcal{B}\to\mathcal{C}$. Applying equation (6.1) to $T^2sA_\infty(\mathcal{A},\mathcal{B})\otimes T^0sA_\infty(\mathcal{B},\mathcal{C})$ we find that

$$(r\otimes p)M_{20}B_1+(rk\otimes pk)B_2=(r\otimes[p,b])M_{20}-$$
$$-([r,b]\otimes p)M_{20}+(r\otimes p)B_2k=(r\otimes p)B_2k.$$

Therefore, $rk\cdot pk$ and $(r\cdot p)k$ are equivalent. Similarly we check compatibility of compositions in the case $k : \mathcal{C}\to\mathcal{A}$, $f\xrightarrow{r} g\xrightarrow{p} h : \mathcal{A}\to\mathcal{B}$, which gives that $kr\cdot kp$ and $k(r\cdot p)$ are equivalent.

Now let us prove distributivity. Consider $r : f\to g : \mathcal{A}\to\mathcal{B}$, $p : h\to k : \mathcal{B}\to\mathcal{C}$. Applying equation (6.1) to $T^1sA_\infty(\mathcal{A},\mathcal{B})\otimes T^1sA_\infty(\mathcal{B},\mathcal{C})$ we find that

$$(rh\otimes gp)B_2-(fp\otimes rk)B_2+(r\otimes p)M_{11}B_1=$$
$$=(r\otimes[p,b])M_{11}-([r,b]\otimes p)M_{11}=0.$$

Therefore, $rh \cdot gp$ and $fp \cdot rk$ are equivalent and define the horizontal composition $r \circ_h p$.

Clearly, strict unit 1-morphisms exist – the identity A_∞-functors $\mathrm{id}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$. However, in general, unit 2-morphisms 1_f of A_∞-functors $f : \mathcal{A} \to \mathcal{B}$ are missing. Summing up, we have described a 1-unital (but not 2-unital) 2-category A_∞ of A_∞-categories.

Strictly unital A_∞-categories. A *strict unit* of an object X of an A_∞-category $\mathcal{A}$ is an element $1_X \in \mathcal{A}^0(X,X)$, such that $(f \otimes 1_X)m_2 = f$, $(1_X \otimes g)m_2 = g$, whenever these make sense, and $(\dots \otimes 1_X \otimes \dots)m_n = 0$ if $n \neq 2$ (see e.g. [3, 4, 9]). We may write it as a map $1_X : \Bbbk \to \mathcal{A}(X,X)$, $1 \mapsto 1_X$. Assume that $\mathcal{A}$ has a strict unit for each object X. For example, a differential graded category $\mathcal{A}$ has strict units. Then we introduce a coderivation $\mathbf{i}^\mathcal{A} : \mathrm{id}_\mathcal{A} \to \mathrm{id}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$, whose components are $\mathbf{i}_0^\mathcal{A} : \Bbbk \to s\mathcal{A}(X,X)$, $1 \mapsto 1_X s = {}_X\mathbf{i}_0^\mathcal{A}$, and $\mathbf{i}_k^\mathcal{A} = 0$ for $k > 0$. The conditions on 1_X imply that $(1 \otimes \mathbf{i}_0^\mathcal{A})b_2 = 1 : s\mathcal{A}(Y,X) \to s\mathcal{A}(Y,X)$ and $(\mathbf{i}_0^\mathcal{A} \otimes 1)b_2 = -1 : s\mathcal{A}(X,Z) \to s\mathcal{A}(X,Z)$. One deduces that $\mathbf{i}^\mathcal{A}$ is a natural A_∞-transformation. If an A_∞-category $\mathcal{A}$ has two such transformations – strict units $\mathbf{i}$ and $\mathbf{i}'$, then due to the above equations they coincide. We call $\mathcal{A}$ *strictly unital* if it has a strict unit $\mathbf{i}^\mathcal{A}$.

For any A_∞-functor $f : \mathcal{C} \to \mathcal{A}$ the natural A_∞-transformation $1_f s = f\mathbf{i}^\mathcal{A} : f \to f : \mathcal{C} \to \mathcal{A}$ has the components ${}_X(f\mathbf{i}^\mathcal{A})_0 = {}_{Xf}\mathbf{i}_0^\mathcal{A} : \Bbbk \to s\mathcal{A}(Xf, Xf)$ and $(f\mathbf{i}^\mathcal{A})_k = 0$ for $k > 0$. It is the unit 2-endomorphism of f. Indeed, for $g \xrightarrow{r} f \xrightarrow{f\mathbf{i}^\mathcal{A}} f : \mathcal{C} \to \mathcal{A}$ we have $(r \cdot f\mathbf{i}^\mathcal{A})_k = (r_k \otimes (f\mathbf{i}^\mathcal{A})_0)b_2 = r_k$. For $f \xrightarrow{f\mathbf{i}^\mathcal{A}} f \xrightarrow{p} g : \mathcal{C} \to \mathcal{A}$, where p is a natural A_∞-transformation we have $(f\mathbf{i}^\mathcal{A} \cdot p)_k = ((f\mathbf{i}^\mathcal{A})_0 \otimes p_k)b_2 = -p_k((f\mathbf{i}^\mathcal{A})_0 \otimes 1)b_2 = p_k$.

7.1. Definition. Let $\mathcal{A}$, $\mathcal{B}$ be strictly unital A_∞-categories. An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is called unital if $f\mathbf{i}^\mathcal{B}$ is equivalent to $\mathbf{i}^\mathcal{A} f$.

The product of two unital A_∞-functors is also unital. The 2-subcategory A_∞^{su} of A_∞ consisting of strictly unital A_∞-categories and unital A_∞-functors is indeed a 2-category. If $r : f \to g : \mathcal{B} \to \mathcal{C}$ is an isomorphism of A_∞-functors and f is unital, then so is g. Indeed, distributivity law in A_∞ implies

$$r \cdot (\mathbf{i}^\mathcal{B} g) = \mathbf{i}^\mathcal{B} \circ_h r = (\mathbf{i}^\mathcal{B} f) \cdot r = = (f\mathbf{i}^\mathcal{C}) \cdot r = r \circ_h \mathbf{i}^\mathcal{C} = r \cdot (g\mathbf{i}^\mathcal{C}).$$

If A_∞-category $\mathcal{B}$ is strictly unital, then so is $\mathcal{C} = A_\infty(\mathcal{A}, \mathcal{B})$. Indeed, for an arbitrary A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ there is the unit 2-endomorphism $1_f s = f\mathbf{i}^\mathcal{B} : f \to f$. We set $\mathbf{i}_0^\mathcal{C} : \Bbbk \to [sA_\infty(\mathcal{A}, \mathcal{B})]^{-1}(f,f)$, $1 \mapsto 1_f s$, and $\mathbf{i}_k^\mathcal{C} = 0$ for $k > 0$. For any element $r \in \mathcal{C}(g,f)$ we have $(r \otimes 1_f s)B_2 = r$. For any element $p \in \mathcal{C}(f,h)$ we have $p(1_f s \otimes 1)B_2 = p((f\mathbf{i}^\mathcal{B})_0 \otimes 1)b_2 = -p$. We have also $\mathbf{i}^\mathcal{C} B_1 = 0$ and $(\dots \otimes \mathbf{i}^\mathcal{C} \otimes \dots)B_n = 0$ if $n > 2$, due to (5.2). Therefore, $\mathbf{i}^\mathcal{C}$ satisfies the required conditions.

Another approach to $\mathbf{i}^\mathcal{C}$ uses the A_∞-functor $M : TsA_\infty(\mathcal{A}, \mathcal{B}) \otimes TsA_\infty(\mathcal{B}, \mathcal{B}) \to TsA_\infty(\mathcal{A}, \mathcal{B}) = \mathcal{C}$. We have $(1 \otimes \mathrm{id}_\mathcal{B})M = \mathrm{id}_\mathcal{C}$ by (4.4), and the natural A_∞-transformations $(1 \otimes \mathbf{i}^\mathcal{B})M$ and $\mathbf{i}^\mathcal{C}$ of $\mathrm{id}_\mathcal{C}$ coincide. Indeed, $[(1 \otimes \mathbf{i}^\mathcal{B})M]_0 : \Bbbk \to (s\mathcal{C})^{-1}(f,f)$, $1 \mapsto (f \otimes \mathbf{i}^\mathcal{B})M_{01} = f\mathbf{i}^\mathcal{B} = \mathbf{i}^\mathcal{C}_{0\,f}$. For all $n \geqslant 0$ we have $[(1 \otimes \mathbf{i}^\mathcal{B})M]_n : r^1 \otimes \dots \otimes r^n \mapsto (r^1 \otimes \dots \otimes r^n \otimes \mathbf{i}^\mathcal{B})M_{n1}$. By (4.5) the components

$$[(r^1 \otimes \dots \otimes r^n \otimes \mathbf{i}^\mathcal{B})M_{n1}]_k = = \sum_l (r^1 \otimes \dots \otimes r^n)\theta_{kl}\mathbf{i}_l^\mathcal{B} = (r^1 \otimes \dots \otimes r^n)\theta_{k0}\mathbf{i}_0^\mathcal{B}$$

vanish for $n > 0$.

A homotopy version of unit in an A_∞ is introduced by Fukaya, Oh, Ohta and Ono [4] and summarized in [3]. We use another generalization of a strict unit, which suits better to our purposes.

Iso-strictly unital A_∞-categories. Let $\phi : \mathcal{B} \to \mathcal{C}$ be an invertible A_∞-functor, that is, there exists an A_∞-functor $\phi^{-1} : \mathcal{C} \to \mathcal{B}$ such that $\phi\phi^{-1} = \mathrm{id}_\mathcal{B}$ and $\phi^{-1}\phi = \mathrm{id}_\mathcal{C}$. The A_∞-functor ϕ is invertible if and only if $\phi : \mathrm{Ob}\,\mathcal{B} \to \mathrm{Ob}\,\mathcal{C}$ is bijective, and $\phi_1 : \mathcal{B}(X,Y) \to \mathcal{C}(X\phi, Y\phi)$ is bijective for all pairs X, Y of objects of $\mathcal{A}$. If both $\mathcal{B}$ and $\mathcal{C}$ are strictly unital, then any invertible A_∞-functor $\phi : \mathcal{B} \to \mathcal{C}$ is unital.

7.2. Definition. An A_∞-category $\mathcal{A}$ is *iso-strictly unital* if there is an invertible A_∞-functor $\psi : \mathcal{A} \to \mathcal{B}$ with a strictly unital A_∞-category $\mathcal{B}$. The 2-endomorphism $\mathbf{i}^\mathcal{A} = \psi\mathbf{i}^\mathcal{B}\psi^{-1} : \mathrm{id}_\mathcal{A} \to \mathrm{id}_\mathcal{A}$ is called the *unit* of $\mathcal{A}$.

Notice that the equivalence class of $\mathbf{i}^\mathcal{A}$ does not depend on the choice of ψ. Indeed, if $\xi : \mathcal{A} \to \mathcal{C}$ is another invertible A_∞-functor with a strictly unital $\mathcal{C}$, then the unitality of $\psi^{-1}\xi : \mathcal{B} \to \mathcal{C}$ implies $\mathbf{i}^\mathcal{B}\psi^{-1}\xi \equiv \psi^{-1}\xi\mathbf{i}^\mathcal{C}$, whence $\psi\mathbf{i}^\mathcal{B}\psi^{-1} \equiv \xi\mathbf{i}^\mathcal{C}\xi^{-1}$.

7.3. Definition. Let $\mathcal{A}$, $\mathcal{B}$ be iso-strictly unital A_∞-categories. An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is called unital if $f\mathbf{i}^\mathcal{B}$ is equivalent to $\mathbf{i}^\mathcal{A} f$.

Again we have that the product of two unital A_∞-functors is unital. If $r : f \to g : \mathcal{B} \to \mathcal{C}$ is an isomorphism of A_∞-functors and f is unital, then so is g.

7.4. Proposition. *If A_∞-category $\mathcal{B}$ is iso-strictly unital, then for an arbitrary A_∞-category $\mathcal{A}$ the A_∞-category $\mathcal{C} = A_\infty(\mathcal{A}, \mathcal{B})$ is iso-strictly unital. Its unit is $\mathbf{i}^\mathcal{C} = (1 \otimes \mathbf{i}^\mathcal{B})M$.*

7.5. Proposition. *The full 2-subcategory of A_∞ consisting of iso-strictly unital A_∞-categories is 2-unital.*

We introduce the (unital) 2-subcategory A_∞^{isu} of A_∞ consisting of iso-strictly unital A_∞-categories and unital A_∞-functors.

Cohomology of A_∞-categories. There is a 2-functor $H^\bullet : A_\infty^{isu} \to \mathbb{Z}\text{-grad-}\Bbbk\text{-}\mathcal{C}at$. It maps an iso-strictly unital A_∞-category $\mathcal{A}$ into a $\mathbb{Z}$-graded $\Bbbk$-linear category $H^\bullet(\mathcal{A})$, whose class of objects is $\operatorname{Ob}\mathcal{A}$, and morphism space between objects X and Y is $H^\bullet(\mathcal{A})(X,Y) = H^\bullet(\mathcal{A}(X,Y), m_1)$, the cohomology with respect to the differential $m_1 = sb_1s^{-1}$. Composition of morphisms $H^\bullet(m_2)$ is determined by the chain map

$$m_2 = (s\otimes s)b_2s^{-1} : \mathcal{A}(X,Y)\otimes\mathcal{A}(Y,Z)\to\mathcal{A}(X,Z).$$

The unit of an object $X \in \operatorname{Ob} H^\bullet(\mathcal{A})$ is the cohomology class of an element ${}_X1 \in \mathcal{A}^0(X,X)$ such that ${}_X\mathbf{i}_0^{\mathcal{A}} : \Bbbk \to (s\mathcal{A})^{-1}(X,X)$, $1 \mapsto {}_X1s$. To a unital A_∞-functor $f : \mathcal{A}\to\mathcal{B}$ is assigned functor $H^\bullet(f_1) : H^\bullet(\mathcal{A}) \to H^\bullet(\mathcal{B})$. It preserves composition due to property $H^\bullet((fg)_1) = H^\bullet(f_1g_1) = H^\bullet(f_1)H^\bullet(g_1)$. Since $f\mathbf{i}^{\mathcal{B}} \equiv \mathbf{i}^{\mathcal{A}}f$ implies ${}_{Xf}\mathbf{i}_0^{\mathcal{B}} = {}_X\mathbf{i}_0^{\mathcal{A}}f_1 + {}_Xv_0b_1$, or ${}_{Xf}1 = {}_X1f_1 + ({}_Xv_0s^{-1})m_1$, the map $H^\bullet(f_1)$ preserves units.

For each natural A_∞-transformation $r : f \to g : \mathcal{A}\to\mathcal{B}$ and each $X \in \operatorname{Ob}\mathcal{A}$ there is an element ${}_Xr \in \mathcal{B}^0(Xf, Xg)$ such that ${}_Xr_0 : \Bbbk \to (s\mathcal{B})^{-1}(Xf,Xg)$, $1 \mapsto {}_Xrs$. To a 2-morphism $r : f \to g$ (equivalence class of natural A_∞-transformation) is assigned a natural transformation $X \mapsto [{}_Xr] \in H^0(\mathcal{B}(Xf,Xg), m_1)$. The natural transformation property follows from the identity for $a \in \mathcal{A}(X,Y)$

$$as[(f_1\otimes{}_Yr_0)b_2 + ({}_Xr_0\otimes g_1)b_2 + r_1b_1 + b_1r_1] = 0,$$

$$a[(f_1\otimes{}_Yr)m_2 - ({}_Xr\otimes g_1)m_2 + (sr_1s^{-1})m_1 + m_1(sr_1s^{-1})] = 0,$$

which implies

$$(H(f_1)\otimes[{}_Yr])H(m_2) = ([{}_Xr]\otimes H(g_1))H(m_2)$$

in cohomology.

One check. easily that both compositions of 1-morphisms and 2-morphisms are preserved. Finally, the vertical composition of 2-morphisms is preserved due to the property

$$x[(r\otimes p)B_2] = x[(r\otimes p)B_2]_0s^{-1} = ({}_Xr_0\otimes{}_Xp_0)b_2s^{-1} = ({}_Xr\otimes{}_Xp)m_2.$$

In practice we use the 2-subfunctor $H^0 : A_\infty^{isu} \to \Bbbk\text{-}\mathcal{C}at$, which takes $\mathcal{A}$ into a $\Bbbk$-linear category, $\operatorname{Ob}H^0(\mathcal{A}) = \operatorname{Ob}\mathcal{A}$, $H^0(\mathcal{A})(X,Y) = H^0(\mathcal{A}(X,Y), m_1)$. For example, homotopy category of complexes is H^0 of the differential graded category of complexes.

A_∞-quotient over a full subcategory

Let $\mathcal{B}\hookrightarrow\mathcal{C}$ be a full A_∞-subcategory. It means that $\operatorname{Ob}\mathcal{B}\subset\operatorname{Ob}\mathcal{C}$, $\mathcal{B}(X,Y) = \mathcal{C}(X,Y)$ for all $X, Y \in \operatorname{Ob}\mathcal{B}$, and the operations for $\mathcal{B}$ coincide with those for $\mathcal{C}$. Let us define a kind of quotient category for the embedding $\mathcal{B}\hookrightarrow\mathcal{C}$.

8.1. Definition. Let $T^+s\mathcal{C} = \oplus_{n>0}T^ns\mathcal{C}$ and $\mathcal{E} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ be the following graded $\Bbbk$-quivers: the class of objects is $\operatorname{Ob}T^+s\mathcal{C} = \operatorname{Ob}\mathcal{E} = \operatorname{Ob}\mathcal{C}$, the morphisms for $X, Y \in \operatorname{Ob}\mathcal{E}$ are

$$T^+s\mathcal{C}(X,Y) = \oplus_{C_1,\dots,C_{n-1}\in\mathcal{C}}s\mathcal{C}(X,C_1)\otimes \otimes s\mathcal{C}(C_1,C_2)\otimes\cdots\otimes s\mathcal{C}(C_{n-2},C_{n-1})\otimes \otimes s\mathcal{C}(C_{n-1},Y),$$

$$s\mathcal{E}(X,Y) = \oplus_{C_1,\dots,C_{n-1}\in\mathcal{B}}s\mathcal{C}(X,C_1)\otimes \otimes s\mathcal{C}(C_1,C_2)\otimes\cdots\otimes s\mathcal{C}(C_{n-2},C_{n-1})\otimes \otimes s\mathcal{C}(C_{n-1},Y),$$

where in the second case summation extends over all sequences of objects $(C_1,\dots,C_{n-1})$ of $\mathcal{B}$. To the empty sequence ($n = 1$) corresponds the summand $s\mathcal{C}(X,Y)$.

Let us endow $s^{-1}T^+s\mathcal{C}$ with a structure of A_∞-category, given by $\underline{b} : T(T^+s\mathcal{C}) \to T(T^+s\mathcal{C})$, with the components $\underline{b}_0 = 0$, $\underline{b}_1 = b : T^+s\mathcal{C} \to T^+s\mathcal{C}$, $\underline{b}_k = 0$ for $k > 1$. This A_∞-category is denoted $\underline{\mathcal{C}} = (s^{-1}T^+s\mathcal{C}, \underline{b})$. There is an A_∞-functor $\underline{j} : \mathcal{C} \to (s^{-1}T^+s\mathcal{C}, \underline{b})$, specified by its components $\underline{j}_k : T^ks\mathcal{C} \to T^+s\mathcal{C}$, $k \geqslant 1$, where $\underline{j}_k$ is the canonical embedding of the direct summand. The property $b\underline{j} = \underline{j}\underline{b}$, or

$$\sum_{r+k+t=n}(1^{\otimes r}\otimes b_k\otimes 1^{\otimes t})\underline{j}_{r+1+t} = \underline{j}_n b : T^ns\mathcal{C} \to T^+s\mathcal{C},$$

is clear – it is just the expression of b in terms of its components.

There is a cocategory automorphism $\mu : TT^+s\mathcal{C} \to TT^+s\mathcal{C}$, specified by its components $\mu_k = \mu^{(k)} : T^kT^+s\mathcal{C} \to T^+s\mathcal{C}$, $k \geqslant 1$, where $\mu : T^+s\mathcal{C}\otimes T^+s\mathcal{C} \to T^+s\mathcal{C}$ is the multiplication in tensor algebra, $\mu^{(k)} = 0$ for $k \leqslant 0$, $\mu^{(1)} = 1 : T^+s\mathcal{C} \to T^+s\mathcal{C}$, $\mu^{(2)} = \mu$, $\mu^{(3)} = (\mu\otimes 1)\mu : (T^+s\mathcal{C})^{\otimes 3} \to T^+s\mathcal{C}$ and so on. Its inverse is the cocategory automorphism

$\mu^{-1} = \mu^{-} : TT^+s\mathcal{C} \to TT^+s\mathcal{C}$, specified by its components $\mu^{-}{}_k = (-)^{k-1}\mu^{(k)} : T^kT^+s\mathcal{C} \to T^+s\mathcal{C}$.

8.2. Proposition. *The conjugate codifferential $\bar{b} = \mu \underline{b} \mu^{-1} : T(T^+s\mathcal{C}) \to T(T^+s\mathcal{C})$ has the following components: $\bar{b}_0 = 0$, $\bar{b}_1 = b$ and for $n \geqslant 2$*

$$\bar{b}_n = \mu^{(n)} \sum_{m;q<k;t<l} 1^{\otimes q} \otimes b_m \otimes 1^{\otimes t} :$$
$$T^k s\mathcal{C} \otimes (T^+s\mathcal{C})^{\otimes n-2} \otimes T^l s\mathcal{C} \to T^+s\mathcal{C}, \quad (8.1)$$
$$\bar{b}_n = \mu^{(n)}b - (1\otimes \mu^{(n-1)}b)\mu - (\mu^{(n-1)}b\otimes 1)\mu + +(1\otimes \mu^{(n-2)}b\otimes 1)\mu^{(3)} : (T^+s\mathcal{C})^{\otimes n} \to T^+s\mathcal{C}, \quad (8.2)$$

for all $n \geqslant 0$. The operations $\bar{b}_n$ restrict to maps $s\mathcal{E}^{\otimes n} \to s\mathcal{E}$ via the natural embedding $s\mathcal{E} \subset T^+s\mathcal{C}$ of graded $\Bbbk$-quivers. Hence, $\bar{b}$ turns $s\mathcal{E}$ and $\overline{\mathcal{C}} \overset{def}{=} (s^{-1}T^+s\mathcal{C}, \bar{b})$ into an A_∞-category.

In particular, (8.2) gives

$$\begin{aligned}\bar{b}_2 &= \mu b - (1\otimes b + b\otimes 1)\mu,\\ \bar{b}_3 &= \mu^{(3)}b - (1\otimes \mu b)\mu - (\mu b\otimes 1)\mu + \\ &\quad + (1\otimes b\otimes 1)\mu^{(3)},\\ \bar{b}_4 &= \mu^{(4)}b - (1\otimes \mu^{(3)}b)\mu - (\mu^{(3)}b\otimes 1)\mu + \\ &\quad + (1\otimes \mu b\otimes 1)\mu^{(3)}.\end{aligned}$$

8.3. Corollary. *The cocategory isomorphism $\mu^{-1} : \underline{\mathcal{C}} = (s^{-1}T^+s\mathcal{C}, \underline{b}) \to \overline{\mathcal{C}} = (s^{-1}T^+s\mathcal{C}, \bar{b})$ is an A_∞-functor. Its composition with $\underline{\jmath}$ is a strict A_∞-functor $\bar{\jmath} = \underline{\jmath}\mu^{-1} : \mathcal{C} \to \mathsf{D}(\mathcal{C}|\mathcal{B})$, $X \mapsto X$, whose components are the direct summand embedding $\bar{\jmath}_1 : s\mathcal{C}(X,Y) = T^1s\mathcal{C}(X,Y) \hookrightarrow s\mathcal{E}(X,Y)$ and $\bar{\jmath}_n = 0$ for $n > 1$.*

Strict unitality. Assume that A_∞-category $\mathcal{C}$ is strictly unital. Its arbitrary full A_∞-subcategory $\mathcal{B}$ is strictly unital as well. Let us show that in these assumptions $\mathcal{E} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ is also strictly unital. We take the same elements $1_X \in \mathcal{C}^0(X,X) \subset \mathcal{E}^0(X,X)$ as strict units of $\mathcal{E}$. We have $1_X s\bar{b}_1 = 1_X sb = 1_X sb_1 = 0$. Explicit formulas give $(\cdots \otimes 1_X s \otimes \dots)\bar{b}_n = 0$ for $n > 2$. The map $\bar{b}_2 : T^ks\mathcal{C}(Y,X) \otimes s\mathcal{C}(X,X) \to T^+s\mathcal{C}(Y,X)$ is the sum of maps

$$1^{\otimes k-t} \otimes b_{t+1} : s\mathcal{C}(Y,C_1)\otimes\cdots\otimes \otimes s\mathcal{C}(C_{k-t},C_{k-t+1})\otimes\cdots\otimes s\mathcal{C}(C_{k-1},X)\otimes s\mathcal{C}(X,X) \to s\mathcal{C}(Y,C_1)\otimes\cdots\otimes s\mathcal{C}(C_{k-t},X)$$

over $t > 0$. Therefore, the map $\mathbf{i}_0^{\mathcal{E}} : \Bbbk_X \to (s\mathcal{E})^{-1}(X,X)$, $1 \mapsto 1_X s$ satisfies equations $(1 \otimes \mathbf{i}_0^{\mathcal{E}})\bar{b}_2 = (1^{\otimes k} \otimes \mathbf{i}_0^{\mathcal{C}})(1^{\otimes k-1} \otimes b_2) = 1^{\otimes k-1} \otimes 1 = 1$. Similarly, for $\bar{b}_2 : s\mathcal{C}(X,X) \otimes T^ks\mathcal{C}(X,Z) \to T^+s\mathcal{C}(X,Z)$ we have $(\mathbf{i}_0^{\mathcal{E}} \otimes 1)\bar{b}_2 = (\mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes k})(b_2 \otimes 1^{\otimes k-1}) = -1 \otimes 1^{\otimes k-1} = -1$. Therefore, $\mathcal{E}$ and $\overline{\mathcal{C}}$ are strictly unital with the unit $\mathbf{i}^{\mathcal{E}}$.

Differential graded categories. If $b_k = 0$ for $k > 2$, then explicit formulae in the case of $\mathcal{E}$ show that we also have $\bar{b}_k = 0$ for $k > 2$. Combining this fact with the above unitality considerations we see that if $\mathcal{C}$ is a differential graded category, then so is $\mathsf{D}(\mathcal{C}|\mathcal{B})$. The differential graded category $\mathcal{E} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ was constructed by V. G. Drinfeld in the following terms[1]. This construction was the starting point of present article.

Write down elements of $\mathcal{E}(X,Y)$ as sequences $f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_{n-1}}f_n$, where $f_i \in \mathcal{C}(C_{i-1},C_i)$, $C_0 = X$, $C_n = Y$, and $C_i \in \operatorname{Ob}\mathcal{B}$ for $0 < i < n$. The symbol ε_C for $C \in \operatorname{Ob}\mathcal{B}$ is assigned degree -1. Its differential is set equal to $\varepsilon_C d = 1_C$. The graded Leibniz rule gives

$$\begin{aligned}(f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_{n-1}}f_n)d &= \\ &= \sum_{q+1+t=n} (-)^{f_{n-t+1}+\dots+f_n-t} f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_q} \\ &\qquad (f_{q+1}m_1)\varepsilon_{C_{q+1}}f_{q+2}\dots\varepsilon_{C_{n-1}}f_n \\ &+ \sum_{q+2+t=n} (-)^{f_{n-t}+\dots+f_n-t} f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_q}(f_{q+1} \\ &\qquad \cdot f_{q+2})\varepsilon_{C_{q+2}}f_{q+3}\dots\varepsilon_{C_{n-1}}f_n,\end{aligned}$$

where $f_{q+1}\cdot f_{q+2} = (f_{q+1}\otimes f_{q+2})m_2$ is the composition. Introduce a degree -1 map

$$\begin{aligned} s &: \mathcal{E} \to s\mathcal{E} \subset T^+s\mathcal{C}, \\ & f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_{n-1}}f_n \mapsto f_1s\otimes f_2s\otimes\cdots\otimes f_ns.\end{aligned}$$

One can check that $ds = s\bar{b}_1$, where, naturally, $\bar{b}_1 = b = \sum_{q+1+t=n} 1^{\otimes q}\otimes b_1\otimes 1^{\otimes t} + \sum_{q+2+t=n} 1^{\otimes q}\otimes b_2\otimes 1^{\otimes t}$.

Composition $\overline{m}_2$ in $\mathcal{E}$ consists of concatenation and composition m_2 in $\mathcal{C}$:

$$\begin{aligned}(f_1\varepsilon_{C_1}\cdots f_{n-1}\varepsilon_{C_{n-1}}f_n \otimes g_1\varepsilon_{D_1}g_2\cdots \\ \cdots\varepsilon_{D_{m-1}}g_m)\overline{m}_2 = \\ = f_1\varepsilon_{C_1}\cdots f_{n-1}\varepsilon_{C_{n-1}}(f_n\cdot g_1)\varepsilon_{D_1}g_2\cdots\varepsilon_{D_{m-1}}g_m.\end{aligned}$$

One can check that $\overline{m}_2 s = (s\otimes s)\bar{b}_2$; here $\bar{b}_2 = 1^{\otimes n-1}\otimes b_2\otimes 1^{\otimes m-1}$.

Specifically this construction applies to the case of differential graded category $\mathcal{C} = \mathsf{C}(\mathcal{A})$ of complexes of objects of an abelian category $\mathcal{A}$. One may take for $\mathcal{B}$ the subcategory of acyclic complexes $\mathcal{B} = \mathsf{A}(\mathcal{A})$.

[1] Communicated to us by B. L. Tsygan

A_∞-quotient functor

Let $\mathcal{B} \hookrightarrow \mathcal{C}$, $\mathcal{J} \hookrightarrow \mathcal{I}$ be full A_∞-subcategories. Let $i : \mathcal{C} \to \mathcal{I}$ be an A_∞-functor, such that $Xi \in \mathrm{Ob}\,\mathcal{J}$ for $X \in \mathrm{Ob}\,\mathcal{B}$. Then it restricts to an A_∞-functor $\mathcal{B} \to \mathcal{J}$, denoted by i'. We are going to construct an extension of this functor to the A_∞-quotient categories $\mathcal{E} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ and $\mathcal{F} = \mathsf{D}(\mathcal{I}|\mathcal{J})$.

Let us begin with a strict A_∞-functor $\underline{i} : \underline{\mathcal{C}} \to \underline{\mathcal{I}}$, given by its components $\underline{i}_1 = i : T^+s\mathcal{C} \to T^+s\mathcal{I}$ and $\underline{i}_k = 0$ for $k > 1$. The equation $\underline{i}\underline{b} = \underline{b}\underline{i}$ reduces to familiar $ib = bi$. Therefore, $\bar{i} \overset{\text{def}}{=} \mu\underline{i}\mu^{-1} : \bar{\mathcal{C}} \to \bar{\mathcal{I}}$ is an A_∞-functor as well.

The following diagram of A_∞-functors commutes

$$\begin{array}{ccccccc}
\mathcal{B} & \hookrightarrow & \mathcal{C} & \xrightarrow{\underline{j}^{\mathcal{C}}} & \underline{\mathcal{C}} & \xrightarrow{\mu^{-1}} & \bar{\mathcal{C}} \\
{\scriptstyle i'}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle \underline{i}}\downarrow & & {\scriptstyle \bar{i}}\downarrow \\
\mathcal{J} & \hookrightarrow & \mathcal{I} & \xrightarrow{\underline{j}^{\mathcal{I}}} & \underline{\mathcal{I}} & \xrightarrow{\mu^{-1}} & \bar{\mathcal{I}}
\end{array}$$

9.1. Proposition. *The A_∞-functor $\bar{i}$ has the following components:*

$$\bar{i}_n = \sum_{l_1+\dots+l_k=n} (-)^{k-1}(\mu^{(l_1)}\otimes\dots\otimes\mu^{(l_k)})i^{\otimes k}\mu^{(k)} : (T^+s\mathcal{C})^{\otimes n} \to T^+s\mathcal{I}. \quad (9.1)$$

Restriction of this map to $T^{k_1}s\mathcal{C}\otimes\dots\otimes T^{k_n}s\mathcal{C}$ is

$$\bar{i}_n = \mu^{(n)} \sum_{(l_1,\dots,l_t)\in L(k_1,\dots,k_n)} (i_{l_1}\otimes\dots\otimes i_{l_t}) : T^{k_1}s\mathcal{C}\otimes\dots\otimes T^{k_n}s\mathcal{C} \to T^+s\mathcal{I}, \quad (9.2)$$

$$L(k_1,\dots,k_n) = \cup_{t>0}\{(l_1,\dots,l_t)\in\mathbb{Z}^t_{>0} \mid \forall q,s\in\mathbb{Z}_{>0}, q\leqslant t, s\leqslant n \quad l_1+\dots+l_q = k_1+\dots+k_s \iff q=t, s=n\}.$$

These maps restrict to maps $\bar{i}_n : T^n s\mathsf{D}(\mathcal{C}|\mathcal{B}) \to s\mathsf{D}(\mathcal{I}|\mathcal{J})$, which are components of an A_∞-functor $\mathsf{D}(i|i') = \bar{i} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{I}|\mathcal{J})$. The restriction of $\bar{i}_n$ to $T^n s\mathcal{C} \xhookrightarrow{\bar{j}_1^{\otimes n}} T^n s\mathsf{D}(\mathcal{C}|\mathcal{B})$ equals $T^n s\mathcal{C} \xrightarrow{i_n} s\mathcal{I} \xhookrightarrow{\bar{j}_1} s\mathsf{D}(\mathcal{I}|\mathcal{J})$.

For example,

$$\bar{i}_1 = i,$$
$$\bar{i}_2 = \mu i - (i\otimes i)\mu,$$
$$\bar{i}_3 = \mu^{(3)}i - (i\otimes\mu i)\mu - (\mu i\otimes i)\mu + (i\otimes i\otimes i)\mu^{(3)}.$$

9.2. Corollary. *We have a commutative diagram of A_∞-functors*

$$\begin{array}{ccccc}
\mathcal{B} & \hookrightarrow & \mathcal{C} & \xrightarrow{\bar{j}^{\mathcal{C}}} & \mathsf{D}(\mathcal{C}|\mathcal{B}) \\
{\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle \bar{i}}\downarrow \\
\mathcal{J} & \hookrightarrow & \mathcal{I} & \xrightarrow{\bar{j}^{\mathcal{I}}} & \mathsf{D}(\mathcal{I}|\mathcal{J})
\end{array}$$

When $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ are A_∞-functors, then $\underline{f}\,\underline{g} = \underline{fg} : \underline{\mathcal{A}} \to \underline{\mathcal{C}}$. This implies $\bar{f}\bar{g} = \overline{fg} : \bar{\mathcal{A}} \to \bar{\mathcal{C}}$. Assume that $\mathcal{A}' \hookrightarrow \mathcal{A}$, $\mathcal{B}' \hookrightarrow \mathcal{B}$, $\mathcal{C}' \hookrightarrow \mathcal{C}$ are full A_∞-subcategories such that $(\mathrm{Ob}\,\mathcal{A}')f \subset \mathrm{Ob}\,\mathcal{B}'$, $(\mathrm{Ob}\,\mathcal{B}')g \subset \mathrm{Ob}\,\mathcal{C}'$. Denote $f' = f|_{\mathcal{A}'}$, $g' = g|_{\mathcal{B}'}$. Since $\mathsf{D}(f|f')$ and $\mathsf{D}(g|g')$ are just the restrictions of $\bar{f}$ and $\bar{g}$, we conclude that

$$\mathsf{D}(f|f')\mathsf{D}(g|g') = \mathsf{D}(fg|f'g') : \mathsf{D}(\mathcal{A}|\mathcal{A}') \to \mathsf{D}(\mathcal{C}|\mathcal{C}'). \quad (9.3)$$

Iso-strict unitality. Assume that A_∞-category $\mathcal{C}$ is strictly unital. As we know from Section 8.1 $\mathsf{D}(\mathcal{C}|\mathcal{B})$ and $\bar{\mathcal{C}}$ are strictly unital with the unit $\mathbf{i}^{\bar{\mathcal{C}}}$. Since $\mu^{-1} : \underline{\mathcal{C}} \to \bar{\mathcal{C}}$ is an invertible A_∞-functor, $\underline{\mathcal{C}}$ is iso-strictly unital (see Section 7.2). Its unit is $\mathbf{i}^{\underline{\mathcal{C}}} = \mu^{-1}\mathbf{i}^{\bar{\mathcal{C}}}\mu : \mathrm{id}_{\underline{\mathcal{C}}} \to \mathrm{id}_{\underline{\mathcal{C}}} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$. Its components are

$$\mathbf{i}_0^{\underline{\mathcal{C}}} = \mathbf{i}_0^{\bar{\mathcal{C}}},$$
$$\mathbf{i}_1^{\underline{\mathcal{C}}} = (\mathbf{i}_0^{\bar{\mathcal{C}}}\otimes 1 + 1\otimes\mathbf{i}_0^{\bar{\mathcal{C}}})\mu,$$
$$\mathbf{i}_2^{\underline{\mathcal{C}}} = (1\otimes\mathbf{i}_0^{\bar{\mathcal{C}}}\otimes 1)\mu^{(3)},$$
$$\mathbf{i}_k^{\underline{\mathcal{C}}} = 0 \qquad \text{for } k > 2.$$

Let $\mathcal{D} \hookrightarrow \mathcal{A}$ be a full subcategory of an iso-strictly unital A_∞-category $\mathcal{A}$. Let $\phi : \mathcal{A} \to \mathcal{C}$ be an invertible A_∞-functor with strictly unital $\mathcal{C}$. Then $\underline{\phi} : \underline{\mathcal{A}} \to \underline{\mathcal{C}}$ and $\bar{\phi} : \bar{\mathcal{A}} \to \bar{\mathcal{C}}$ are also invertible A_∞-functors. Denote by $\mathcal{B}$ the full subcategory of $\mathcal{C}$ with $\mathrm{Ob}\,\mathcal{B} = (\mathrm{Ob}\,\mathcal{D})\phi$. Denote $\phi' = \phi|_{\mathcal{D}} : \mathcal{D} \to \mathcal{B}$. Due to (9.3) $\mathsf{D}(\phi|\phi') : \mathsf{D}(\mathcal{A}|\mathcal{D}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$ is invertible. Hence, $\mathsf{D}(\mathcal{A}|\mathcal{D})$ is iso-strictly unital.

The A_∞-functor $\bar{j}^{\mathcal{C}} : \mathcal{C} \to \mathsf{D}(\mathcal{C}|\mathcal{B})$ is unital. Therefore, A_∞-functor $\bar{j}^{\mathcal{A}} = \phi\bar{j}^{\mathcal{C}}\bar{\phi}^{-1} : \mathcal{A} \to \mathsf{D}(\mathcal{A}|\mathcal{D})$ is also unital.

A_∞-quotient transformation

Let $\mathcal{B} \hookrightarrow \mathcal{C}$ and $\mathcal{J} \hookrightarrow \mathcal{I}$ be full A_∞-subcategories. Let $f, g : \mathcal{C} \to \mathcal{I}$ be two A_∞-functors such that $(\mathrm{Ob}\,\mathcal{B})f \subset \mathrm{Ob}\,\mathcal{J}$, $(\mathrm{Ob}\,\mathcal{B})g \subset \mathrm{Ob}\,\mathcal{J}$, and let $r : f \to g : \mathcal{C} \to \mathcal{I}$ be an A_∞-transformation. Denote by $r' : f' \to g' : \mathcal{B} \to \mathcal{J}$ the restriction of r to $\mathcal{B}$. Let us define an A_∞-transformation $\underline{r} : \underline{f} \to \underline{g} : \underline{\mathcal{C}} \to \underline{\mathcal{I}}$ via its components

$$\underline{r}_0 = r_0\underline{j}_1,$$
$$\underline{r}_0 = [\Bbbk \xrightarrow{r_0} (s\mathcal{I})(Xf, Xg) \xhookrightarrow{\underline{j}_1} (s\underline{\mathcal{I}})(Xf, Xg)];$$
$$\underline{r}_1 = r,$$
$$\underline{r}_1 = r|_{T^+s\mathcal{C}} : T^+s\mathcal{C} = s\underline{\mathcal{C}} \to T^+s\mathcal{I} = s\underline{\mathcal{I}}; \quad (10.1)$$
$$\underline{r}_k = 0 \qquad \text{for } k > 1.$$

Let us check that $\underline{}$ maps ω-globular set A_ω into itself (so that sources and targets are preserved). It suffices to notice that the correspondence $r \mapsto \underline{r}$ is additive, and if $r = [v, b]$, then $\underline{r} = [\underline{v}, \underline{b}]$. Indeed,

$$
\begin{aligned}
{}[\underline{v}, \underline{b}]_0 &= \underline{v}_0 \underline{b}_1 = v_0 \underline{j}_1 b = v_0 b_1 \underline{j}_1 = r_0 \underline{j}_1 = \underline{r}_0, \\
{}[\underline{v}, \underline{b}]_1 &= \underline{v}_1 \underline{b}_1 - (-)^v \underline{b}_1 \underline{v}_1 = vb - (-)^v bv = r = \underline{r}_1, \\
{}[\underline{v}, \underline{b}]_k &= \underline{v}_k \underline{b}_1 - (-)^v \underline{b}_k \underline{v}_1 = 0 \pm 0 = 0 = \underline{r}_k \\
&\quad \text{for } k > 1.
\end{aligned}
$$

In particular, a natural A_∞-transformation $r : f \to g : \mathcal{C} \to \mathcal{J}$ goes to the natural A_∞-transformation $\underline{r} : \underline{f} \to \underline{g} : \underline{\mathcal{C}} \to \underline{\mathcal{J}}$, and equivalent natural A_∞-transformations r, p go to equivalent $\underline{r}, \underline{p}$. We have

$$r \underline{j}^{\mathcal{J}} = \underline{j}^{\mathcal{C}} \underline{r} : f \underline{j} = \underline{j} f \to g \underline{j} = \underline{j} g : \mathcal{C} \to \underline{\mathcal{J}}.$$

We define also the A_∞-transformation conjugate to $\underline{r}$

$$\overline{r} = \mu \underline{r} \mu^{-1} : \overline{f} = \mu \underline{f} \mu^{-1} \to \overline{g} = \mu \underline{g} \mu^{-1} : \overline{\mathcal{C}} \to \overline{\mathcal{J}}$$

(not necessarily natural). Summing up, we have a commutative cylinder

$$
\begin{array}{ccccccc}
\mathcal{B} & \hookrightarrow & \mathcal{C} & \xrightarrow{\underline{j}^{\mathcal{C}}} & \underline{\mathcal{C}} & \xrightarrow{\mu^{-1}} & \overline{\mathcal{C}} \\
f' \downarrow \overset{r'}{\Longrightarrow} \downarrow g' & & f \downarrow \overset{r}{\Longrightarrow} \downarrow g & & \underline{f} \downarrow \overset{\underline{r}}{\Longrightarrow} \downarrow \underline{g} & & \overline{f} \downarrow \overset{\overline{r}}{\Longrightarrow} \downarrow \overline{g} \\
\mathcal{D} & \hookrightarrow & \mathcal{J} & \xrightarrow{\underline{j}^{\mathcal{J}}} & \underline{\mathcal{J}} & \xrightarrow{\mu^{-1}} & \overline{\mathcal{J}}
\end{array}
$$

The correspondence $\overline{}$ also maps ω-globular set A_ω into itself. Indeed, if $r = [v, b]$, then

$$\overline{r} = \mu \underline{r} \mu^{-1} = \mu [\underline{v}, \underline{b}] \mu^{-1} = [\mu \underline{v} \mu^{-1}, \mu \underline{b} \mu^{-1}] = [\overline{v}, \overline{b}].$$

10.1. Proposition. *The A_∞-transformation $\overline{r}$ has the following components*

$$
\begin{aligned}
\overline{r}_n = & \sum_{l_1 + \dots + l_t = n}^{0 \leqslant q \leqslant t} (-)^t (\mu^{(l_1)} f \otimes \dots \otimes \\
& \otimes \mu^{(l_q)} f \otimes r_0 \underline{j}_1 \otimes \mu^{(l_{q+1})} g \otimes \dots \otimes \mu^{(l_t)} g) \mu^{(t+1)} \\
& + \sum_{l_1 + \dots + l_t = n}^{1 \leqslant q \leqslant t} (-)^{t-1} (\mu^{(l_1)} f \otimes \dots \otimes \mu^{(l_{q-1})} f \otimes \\
& \otimes \mu^{(l_q)} r \otimes \mu^{(l_{q+1})} g \otimes \dots \otimes \mu^{(l_t)} g) \mu^{(t)}. \qquad (10.2)
\end{aligned}
$$

Explicitly, $\overline{r}_0 = \underline{r}_0 = r_0 \underline{j}_1$ and for $n > 0$ the restriction of $\overline{r}_n$ to $T^{k_1} s\mathcal{C} \otimes \dots \otimes T^{k_n} s\mathcal{C}$ is

$$
\begin{aligned}
\overline{r}_n = \mu^{(n)} & \sum_{(a_1, \dots, a_\alpha; k; c_1, \dots, c_\beta) \in P(k_1, \dots, k_n)} (f_{a_1} \otimes \dots \\
& \dots \otimes f_{a_\alpha} \otimes r_k \otimes g_{c_1} \otimes \dots \otimes g_{c_\beta}) \underline{j}_{\alpha+1+\beta} : \\
& T^{k_1} s\mathcal{C} \otimes \dots \otimes T^{k_n} s\mathcal{C} \to T^+ s\mathcal{J}, \qquad (10.3)
\end{aligned}
$$

$$
\begin{aligned}
P(k_1, \dots, k_n) = \sqcup_{\alpha, \beta \geqslant 0} \{ & (l_1, \dots, l_\alpha; l_{\alpha+1}; \\
& l_{\alpha+2}, \dots, l_{\alpha+1+\beta}) \in \mathbb{Z}_{>0}^\alpha \times \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{>0}^\beta \mid \\
& \forall q \in \mathbb{Z}_{>0}, q \leqslant \alpha + 1 + \beta \ \forall s \in \mathbb{Z}_{\geqslant 0}, s \leqslant n \\
l_1 + \dots + l_q = k_1 + & \dots + k_s \Leftrightarrow q = \alpha + 1 + \beta, s = n \}.
\end{aligned}
$$

These maps restrict to maps $\overline{r}_n : T^n s\mathsf{D}(\mathcal{C}|\mathcal{B}) \to s\mathsf{D}(\mathcal{J}|\mathcal{D})$, which are components of an A_∞-transformation $\mathsf{D}(r|r') = \overline{r} : \overline{f} \to \overline{g} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{J}|\mathcal{D})$. The restriction of $\overline{r}_n$ to $T^n s\mathcal{C} \xhookrightarrow{\overline{j}_1^{\otimes n}} T^n s\mathsf{D}(\mathcal{C}|\mathcal{B})$ equals $T^n s\mathcal{C} \xrightarrow{r_n} s\mathcal{J} \xhookrightarrow{\overline{j}_1} s\mathsf{D}(\mathcal{J}|\mathcal{D})$.

In particular, the correspondence $r \mapsto \mathsf{D}(r|r')$ maps natural A_∞-transformations to natural ones, and equivalent $r, p : f \to g : \mathcal{B} \to \mathcal{C}$ are mapped to equivalent

$$\mathsf{D}(r|r'), \mathsf{D}(p|p') : \overline{f} \to \overline{g} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{J}|\mathcal{D}).$$

For example,

$$
\begin{aligned}
\overline{r}_1 &= r - (f \otimes r_0 + r_0 \otimes g) \mu, \\
\overline{r}_2 &= \mu r - (f \otimes r + r \otimes g) \mu - \mu (f \otimes r_0 + r_0 \otimes g) \mu \\
&\quad + (f \otimes f \otimes r_0 + f \otimes r_0 \otimes g + r_0 \otimes g \otimes g) \mu^{(3)}.
\end{aligned}
$$

10.2. Corollary. *We have a commutative cylinder*

$$
\begin{array}{ccccc}
\mathcal{B} & \hookrightarrow & \mathcal{C} & \xrightarrow{\overline{j}^{\mathcal{C}}} & \mathsf{D}(\mathcal{C}|\mathcal{B}) \\
f' \downarrow \overset{r'}{\Longrightarrow} \downarrow g' & & f \downarrow \overset{r}{\Longrightarrow} \downarrow g & & \overline{f} \downarrow \overset{\overline{r}}{\Longrightarrow} \downarrow \overline{g} \\
\mathcal{D} & \hookrightarrow & \mathcal{J} & \xrightarrow{\overline{j}^{\mathcal{J}}} & \mathsf{D}(\mathcal{J}|\mathcal{D})
\end{array}
$$

Contractibility.

10.3. Proposition. *Let $\mathcal{B}$ be an iso-strictly unital A_∞-category. Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-functor. Then the following conditions are equivalent:*

(C1) For any $X \in \operatorname{Ob} \mathcal{A}$ and any $V \in \operatorname{Ob} \mathcal{B}$ the complex $(s\mathcal{B}(Xf, V), b_1)$ is contractible;

(C2) For any $U \in \operatorname{Ob} \mathcal{B}$ and any $Y \in \mathcal{A}$ the complex $(s\mathcal{B}(U, Xf), b_1)$ is contractible;

(C3) For any object X of $\mathcal{A}$ the complex $(s\mathcal{B}(Xf, Xf), b_1)$ is acyclic;

(C4) For any object X of $\mathcal{A}$ there is an element ${}_X v \in (s\mathcal{B})^{-2}(Xf, Xf)$ such that ${}_{Xf}\mathbf{i}_0^{\mathcal{B}} = {}_X v b_1$;

(C5) $f\mathbf{i}^{\mathcal{B}} \equiv 0 : f \to f : \mathcal{A} \to \mathcal{B}$.

10.4. Proposition. *Let $\mathcal{A}$ be an iso-strictly unital A_∞-category. Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-functor. Then the following conditions are equivalent:*

(C6) *For all objects X, Y of $\mathcal{A}$ the chain map $f_1 : (s\mathcal{A}(X,Y), b_1) \to (s\mathcal{B}(Xf, Yf), b_1)$ is homotopic to 0;*

(C7) *For any object X of $\mathcal{A}$ the chain map $f_1 : (s\mathcal{A}(X,X), b_1) \to (s\mathcal{B}(Xf, Xf), b_1)$ is homotopic to 0;*

(C8) *For any object X of $\mathcal{A}$ we have $H^\bullet(f_1) = 0 : H^\bullet(s\mathcal{A}(X,X), b_1) \to H^\bullet(s\mathcal{B}(Xf, Xf), b_1)$;*

(C9) *For any object X of $\mathcal{A}$ there is an element ${}_Xw \in (s\mathcal{B})^{-2}(Xf, Xf)$ such that ${}_X\mathbf{i}_0^{\mathcal{A}} f_1 = {}_Xw b_1$.*

10.5. Proposition. *Let $\mathcal{A}$, $\mathcal{B}$ be iso-strictly unital A_∞-categories. Let $f : \mathcal{A} \to \mathcal{B}$ be a unital A_∞-functor. Then conditions (C1)–(C9) are equivalent to the following conditions:*

(C10) *There is an isomorphism of A_∞-functors $f \simeq \mathcal{O}^f : \mathcal{A} \to \mathcal{B}$, where $\mathcal{O}^f$ is defined as follows: $X\mathcal{O}^f = Xf$, $\mathcal{O}^f_k = 0$ for all $k \geqslant 1$;*

(C11) $\mathbf{i}^{\mathcal{A}} f \equiv 0 : f \to f : \mathcal{A} \to \mathcal{B}$.

10.6. Definition. Let $\mathcal{B}$ be an iso-strictly unital A_∞-category. An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is *contractible* if it satisfies equivalent conditions (C1)–(C5) of Proposition 10.3. An iso-strictly unital A_∞-category $\mathcal{B}$ is *contractible* if $\mathrm{id}_{\mathcal{B}}$ is contractible.

If f is unital, it is contractible if and only if equivalent conditions (C1)–(C11) hold.

10.7. Example. Let $\mathcal{B}$ be a full subcategory of an iso-strictly unital A_∞-category $\mathcal{C}$. Then the A_∞-functor $\jmath' = (\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{\jmath} \mathsf{D}(\mathcal{C}|\mathcal{B}))$ is contractible according to criterion (C4): for any object X of $\mathcal{B}$

$$({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}})\bar{b}_1 = ({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}})b_2 = {}_X\mathbf{i}_0^{\mathcal{C}} = {}_X\mathbf{i}_0^{\mathcal{D}}.$$

10.8. Proposition. *An iso-strictly unital A_∞-category $\mathcal{A}$ is contractible if and only if the following equivalent conditions hold:*

(C0) *$\mathcal{A}$ is equivalent in A_∞^{isu} to an A_∞-category $\mathcal{O}$, such that $\mathcal{O}(U,V) = 0$ for all objects U, V of $\mathcal{O}$;*

(C1') *For all objects X, Y of $\mathcal{A}$ the complex $(s\mathcal{A}(X,Y), b_1)$ is contractible;*

(C2') *For any object X of $\mathcal{A}$ the complex $(s\mathcal{A}(X,X), b_1)$ is contractible;*

(C3') *For any object X of $\mathcal{A}$ the complex $(s\mathcal{A}(X,X), b_1)$ is acyclic;*

(C4') *For any object X of $\mathcal{A}$ there is an element ${}_Xv \in (s\mathcal{A})^{-2}(X,X)$ such that ${}_X\mathbf{i}_0^{\mathcal{A}} = {}_Xvb_1$;*

(C5') *$\mathbf{i}^{\mathcal{A}} \equiv 0 : \mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$;*

(C10') *There is an isomorphism of A_∞-functors $\mathrm{id}_{\mathcal{A}} \simeq \mathcal{O}^{\mathrm{id}} : \mathcal{A} \to \mathcal{A}$, where $X\mathcal{O}^{\mathrm{id}} = X$, $\mathcal{O}^{\mathrm{id}}_k = 0$ for all $k \geqslant 1$.*

10.9. Corollary. *If $\mathcal{B}$ is contractible, then for any A_∞-category $\mathcal{A}$ we have $A_\infty(\mathcal{A},\mathcal{B})(f,g) = 0$ for all A_∞-functors $f, g : \mathcal{A} \to \mathcal{B}$.*

10.10. Remark. Any A_∞-category $\mathcal{O}$, such that $\mathcal{O}(U,V) = 0$ for all objects U, V of $\mathcal{O}$ with non-empty $\mathrm{Ob}\,\mathcal{O}$ is equivalent to A_∞-category $\mathcal{O}'$ with one object $*$, such that $\mathcal{O}'(*,*) = 0$. Indeed, choose an object $Z \in \mathrm{Ob}\,\mathcal{O}$. Consider A_∞-functors $\phi : \mathcal{O} \to \mathcal{O}'$, $U \mapsto *$ and $\psi : \mathcal{O}' \to \mathcal{O}$, $* \mapsto Z$. We have $\psi\phi = \mathrm{id}_{\mathcal{O}'}$ and $\phi\psi$ is isomorphic to $\mathrm{id}_{\mathcal{O}}$ via inverse to each other 2-morphisms $0 : \phi\psi \to \mathrm{id}_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}$ and $0 : \mathrm{id}_{\mathcal{O}} \to \phi\psi : \mathcal{O} \to \mathcal{O}$.

10.11. Example. Let $\mathcal{A}$ be an iso-strictly unital A_∞-category. Then $\underline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ are contractible. Indeed, due to (C3') it suffices to prove it for a strictly unital A_∞-category $\mathcal{C}$. Since

$$(\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}})\bar{b}_1 = (\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}})\underline{b}_1 = (\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}})b_2 = \mathbf{i}_0^{\mathcal{C}},$$

we deduce contractibility by (C4').

Various units. Assume that $\mathcal{C}$ is strictly unital. Since $\underline{\mathcal{C}}$, $\overline{\mathcal{C}}$ are contractible, it is obvious that $\mathbf{i}^{\underline{\mathcal{C}}} \equiv \underline{\mathbf{i}^{\mathcal{C}}}$ and $\mathbf{i}^{\overline{\mathcal{C}}} \equiv \overline{\mathbf{i}^{\mathcal{C}}}$ (all natural A_∞-transformations in $A_\infty(\mathcal{A},\underline{\mathcal{C}})$, $A_\infty(\mathcal{A},\overline{\mathcal{C}})$ are equivalent to 0). We shall prove also that $\mathbf{i}^{\mathsf{D}(\mathcal{C}|\mathcal{B})} \equiv \mathsf{D}(\mathbf{i}^{\mathcal{C}}|\mathbf{i}^{\mathcal{B}})$ via an explicit homotopy. We begin with an explicit 3-morphism between $\mathbf{i}^{\underline{\mathcal{C}}}$ and $\underline{\mathbf{i}^{\mathcal{C}}}$. Let v denote an $(\mathrm{id}_{\underline{\mathcal{C}}}, \mathrm{id}_{\underline{\mathcal{C}}})$-coderivation $Ts\underline{\mathcal{C}} \to Ts\underline{\mathcal{C}}$ of degree -2, whose components are $v_0 = 0$,

$$v_1\big|_{T^k s\mathcal{C}} = \sum_{q+t=k}^{q,t>0} 1^{\otimes q} \otimes \mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes t} : T^k s\mathcal{C} \to T^+ s\mathcal{C},$$

$$v_2 = -(1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(4)} : s\mathcal{C} \otimes s\mathcal{C} \to s\mathcal{C},$$

$$v_2\big|_{T^{k_1}s\mathcal{C} \otimes T^{k_2}s\mathcal{C}} = -1^{\otimes k_1} \otimes \mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes k_2} : T^{k_1}s\mathcal{C} \otimes T^{k_2}s\mathcal{C} \to T^+ s\mathcal{C},$$

and $v_n = 0$ for $n > 2$. Actually, it is a 3-morphism

$$v : \mathbf{i}^{\underline{\mathcal{C}}} \to \underline{\mathbf{i}^{\mathcal{C}}} : \mathrm{id}_{\underline{\mathcal{C}}} \to \mathrm{id}_{\underline{\mathcal{C}}} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}.$$

We get another 3-morphism conjugating v with μ:

$$w = \mu v \mu^{-1} : \mathbf{i}^{\overline{\mathcal{C}}} = \mu \mathbf{i}^{\mathcal{C}} \mu^{-1} \to \overline{\mathbf{i}^{\mathcal{C}}} = \\ = \mu \underline{\mathbf{i}^{\mathcal{C}}} \mu^{-1} : \mathrm{id}_{\overline{\mathcal{C}}} \to \mathrm{id}_{\overline{\mathcal{C}}} : \overline{\mathcal{C}} \to \overline{\mathcal{C}}.$$

The 3-morphism w has the following components: $w_0 = 0$,

$$w_1 = v_1 = \sum_{q+t=k}^{q,t>0} 1^{\otimes q} \otimes \mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes t} : T^k s\mathcal{C} \to T^+ s\mathcal{C}$$

and $w_n = 0$ for $n > 1$.

Since w_1 maps $s\mathsf{D}(\mathcal{C}|\mathcal{B})$ into $s\mathsf{D}(\mathcal{C}|\mathcal{B})$, the coderivation w restricts to $w : Ts\mathsf{D}(\mathcal{C}|\mathcal{B}) \to Ts\mathsf{D}(\mathcal{C}|\mathcal{B})$, which is a 3-morphism $w : \mathbf{i}^{\mathsf{D}(\mathcal{C}|\mathcal{B})} \to \mathsf{D}(\mathbf{i}^{\mathcal{C}}|\mathbf{i}^{\mathcal{B}})$. Therefore, $\mathbf{i}^{\mathsf{D}(\mathcal{C}|\mathcal{B})}$ and $\mathsf{D}(\mathbf{i}^{\mathcal{C}}|\mathbf{i}^{\mathcal{B}})$ are equivalent.

Iso-strictly unital case. Let $\mathcal{D} \hookrightarrow \mathcal{A}$ be a full subcategory of an iso-strictly unital A_∞-category $\mathcal{A}$. Let $\phi : \mathcal{A} \to \mathcal{C}$ be an invertible A_∞-functor with strictly unital $\mathcal{C}$. Denote by $\mathcal{B}$ the full subcategory of $\mathcal{C}$ with $\operatorname{Ob}\mathcal{B} = (\operatorname{Ob}\mathcal{D})\phi$, and let $\phi' = \phi|_{\mathcal{D}} : \mathcal{D} \to \mathcal{B}$. Units of $\mathcal{A}$ and $\mathsf{D}(\mathcal{A}|\mathcal{D})$ are defined as $\mathbf{i}^{\mathcal{A}} = \phi \mathbf{i}^{\mathcal{C}} \phi^{-1}$ and $\mathbf{i}^{\mathsf{D}(\mathcal{A}|\mathcal{D})} = \overline{\phi}\mathbf{i}^{\mathsf{D}(\mathcal{C}|\mathcal{B})}\overline{\phi}^{-1}$. Commutative diagram

$$\begin{array}{ccccc}
\mathcal{D} & \hookrightarrow & \mathcal{A} & \xrightarrow{\jmath^{\mathcal{A}}} & \mathsf{D}(\mathcal{A}|\mathcal{D}) \\
\downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} \\
\mathcal{B} & \hookrightarrow & \mathcal{C} & \xrightarrow{\jmath^{\mathcal{C}}} & \mathsf{D}(\mathcal{C}|\mathcal{B})
\end{array}$$

suggests how to extend the main result of this subsection to iso-strictly unital categories:

$$\mathbf{i}^{\mathsf{D}(\mathcal{A}|\mathcal{D})} = \overline{\phi}\mathbf{i}^{\mathsf{D}(\mathcal{C}|\mathcal{B})}\overline{\phi}^{-1} \equiv \\ \equiv \mathsf{D}(\phi|\phi')\mathsf{D}(\mathbf{i}^{\mathcal{C}}|\mathbf{i}^{\mathcal{B}})\mathsf{D}(\phi^{-1}|\phi'^{-1}) = \\ = \mathsf{D}(\phi\mathbf{i}^{\mathcal{C}}\phi^{-1}|\phi'\mathbf{i}^{\mathcal{B}}\phi'^{-1}) = \mathsf{D}(\mathbf{i}^{\mathcal{A}}|\mathbf{i}^{\mathcal{D}}).$$

Therefore, $\mathbf{i}^{\mathsf{D}(\mathcal{A}|\mathcal{D})} \equiv \mathsf{D}(\mathbf{i}^{\mathcal{A}}|\mathbf{i}^{\mathcal{D}})$, and the latter A_∞-transformation can be used as the unit of $\mathsf{D}(\mathcal{A}|\mathcal{D})$.

10.12. Proposition. *(A_∞-quotient over a contractible subcategory) Let $\mathcal{F}$ be a full contractible subcategory of an iso-strictly unital A_∞-category $\mathcal{E}$. Then there exists a quasi-inverse to the canonical strict embedding $\jmath^{\mathcal{E}} : \mathcal{E} \to \mathsf{D}(\mathcal{E}|\mathcal{F})$ unital A_∞-functor $\pi^{\mathcal{E}} : \mathsf{D}(\mathcal{E}|\mathcal{F}) \to \mathcal{E}$, such that $\jmath^{\mathcal{E}}\pi^{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$. In particular, $\mathsf{D}(\mathcal{E}|\mathcal{F})$ is equivalent to $\mathcal{E}$.*

A construction of an A_∞-functor

Let $\mathcal{B}$, $\mathcal{C}$ be A_∞-categories, let $f : \mathcal{B} \to \mathcal{C}$ be an A_∞-functor and let $g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ be a map. Assume that for each object $X \in \operatorname{Ob}\mathcal{B}$ there is an element $r_X \in \mathcal{C}^0(Xf, Xg)$ such that $r_X s b_1 = 0$. For any object $Y \in \operatorname{Ob}\mathcal{B}$ this element determines a chain map

$$(r_X s \otimes 1)b_2 : s\mathcal{C}(Xg, Yg) \to s\mathcal{C}(Xf, Yg), \\ p \mapsto (-)^p (r_X s \otimes p) b_2.$$

Finally, we assume that for any chain complex of $\Bbbk$-modules of the form $N = s\mathcal{B}(X_0, X_1) \otimes_{\Bbbk} s\mathcal{B}(X_1, X_2) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} s\mathcal{B}(X_{n-1}, X_n)$, $n \geqslant 0$, the following chain map

$$u = \operatorname{Hom}(N, (r_X s \otimes 1)b_2) : \\ \operatorname{Hom}^\bullet_{\Bbbk}(N, s\mathcal{C}(Xg, Yg)) \to \operatorname{Hom}^\bullet_{\Bbbk}(N, s\mathcal{C}(Xf, Yg)) \tag{11.1}$$

is a quasi-isomorphism. For $n = 0$ we have $N = \Bbbk$, and the 0-th condition means that $(r_X s \otimes 1)b_2$ is a quasi-isomorphism.

11.1. Proposition. *In the above assumptions the map $g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ extends to an A_∞-functor $g : \mathcal{B} \to \mathcal{C}$. There exists a natural A_∞-transformation $r : f \to g : \mathcal{B} \to \mathcal{C}$ such that its 0-th component is $r_0 : \Bbbk_X \to s\mathcal{C}(Xf, Xg)$, $1 \mapsto r_X s$.*

Transformations between the constructed A_∞-functors. Let $\mathcal{B}$, $\mathcal{C}$ be A_∞-categories, let $f : \mathcal{B} \to \mathcal{C}$ be an A_∞-functor, let $g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ be a map, and assume that for each object $X \in \operatorname{Ob}\mathcal{B}$ there is a map $r_0 : \Bbbk_X \to (s\mathcal{C})^{-1}(Xf, Xg)$ such that $r_0 b_1 = 0$. Let the assumptions of Section 11 hold. Let $g, g' : \mathcal{B} \to \mathcal{C}$ be two A_∞-functors, whose underlying map is the given $g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$. Let $r : f \to g : \mathcal{B} \to \mathcal{C}$, $r' : f \to g' : \mathcal{B} \to \mathcal{C}$ be natural A_∞-transformations, whose 0-th component $r_0 = r'_0$ is the given map $r_0 : \Bbbk_X \to (s\mathcal{C})^{-1}(Xf, Xg)$.

11.2. Proposition. *In the above assumptions there exists a natural A_∞-transformation $p : g \to g' : \mathcal{B} \to \mathcal{C}$, such that $r' = (f \xrightarrow{r} g \xrightarrow{p} g')$ in the 2-category A_∞. The natural A_∞-transformation p is unique up to an equivalence.*

11.3. Corollary. *Let in the above assumptions $\mathcal{C}$ be iso-strictly unital. Then the constructed 2-morphism $p : g \to g' : \mathcal{B} \to \mathcal{C}$ is invertible in A_∞.*

11.4. Proposition. *(unitality) If A_∞-categories $\mathcal{B}$, $\mathcal{C}$ are iso-strictly unital and A_∞-functor $f : \mathcal{B} \to \mathcal{C}$ is unital, then the A_∞-functor $g : \mathcal{B} \to \mathcal{C}$ constructed in Proposition 11.1 is unital as well.*

Invertible transformations. Let $\mathcal{B}$, $\mathcal{C}$ be iso-strictly unital A_∞-categories, and let $f, g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ be maps. Assume that for each object X of $\mathcal{B}$ there are $\Bbbk$-linear maps

$$
\begin{aligned}
{}_X r_0 &: \Bbbk \to (s\mathcal{C})^{-1}(Xf, Xg),\\
{}_X p_0 &: \Bbbk \to (s\mathcal{C})^{-1}(Xg, Xf),\\
{}_X w_0 &: \Bbbk \to (s\mathcal{C})^{-2}(Xf, Xf),\\
{}_X v_0 &: \Bbbk \to (s\mathcal{C})^{-2}(Xg, Xg),
\end{aligned}
$$

such that

$$
\begin{gathered}
{}_X r_0 b_1 = 0, \quad {}_X p_0 b_1 = 0,\\
({}_X r_0 \otimes {}_X p_0) b_2 - {}_{Xf}\mathbf{i}_0^{\mathcal{C}} = {}_X w_0 b_1,\\
({}_X p_0 \otimes {}_X r_0) b_2 - {}_{Xg}\mathbf{i}_0^{\mathcal{C}} = {}_X v_0 b_1. \qquad (11.2)
\end{gathered}
$$

11.5. Lemma. *Let the above assumptions hold. Then for all objects X of $\mathcal{B}$ and Y of $\mathcal{C}$ the chain maps*

$(r_0 \otimes 1) b_2 : s\mathcal{C}(Xg, Y) \to s\mathcal{C}(Xf, Y)$ *and*

$(p_0 \otimes 1) b_2 : s\mathcal{C}(Xf, Y) \to s\mathcal{C}(Xg, Y)$,

$(1 \otimes r_0) b_2 : s\mathcal{C}(Y, Xf) \to s\mathcal{C}(Y, Xg)$ *and*

$(1 \otimes p_0) b_2 : s\mathcal{C}(Y, Xg) \to s\mathcal{C}(Y, Xf)$

are homotopy inverse to each other.

11.6. Proposition. *Let $r : f \to g : \mathcal{B} \to \mathcal{C}$ be a natural A_∞-transformation, and let p_0, v_0, w_0 be as in Section 11.2 so that equations (11.2) hold. Then p_0, w_0 extend to a natural A_∞-transformation $p : g \to f : \mathcal{B} \to \mathcal{C}$, and 3-morphisms*

$$
\begin{aligned}
w &: (r \otimes p) B_2 \to f\mathbf{i}^{\mathcal{C}} : f \to f : \mathcal{B} \to \mathcal{C}, \qquad (11.3)\\
t &: (p \otimes r) B_2 \to g\mathbf{i}^{\mathcal{C}} : g \to g : \mathcal{B} \to \mathcal{C}. \qquad (11.4)
\end{aligned}
$$

In particular, r is invertible and $p = r^{-1}$ in A_∞^{isu}.

11.7. Corollary. *Let assumptions of Section 11.2 hold and, moreover, let $f : \mathcal{B} \to \mathcal{C}$ be a unital A_∞-functor. Then the map g extends to a unital A_∞-functor $g : \mathcal{B} \to \mathcal{C}$, and given r_0 can be extended to an isomorphism $r : f \to g : \mathcal{B} \to \mathcal{C}$.*

Derived categories

Let $\mathcal{A}$ be an abelian $\Bbbk$-linear category, and let $\mathcal{C} = \mathsf{C}(\mathcal{A})$ or $\mathcal{C} = \mathsf{C}^+(\mathcal{A})$ be differential graded category of complexes (resp. bounded below complexes) of objects of $\mathcal{A}$. Denote by $\mathcal{B} = \mathsf{A}(\mathcal{A})$ its full subcategory of acyclic complexes. Let $\mathcal{D} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ be the quotient category.

Invertibility of quasi-isomorphisms. Assume that X, Y are objects of $\mathcal{C}$ and $q : X \to Y$ is a quasi-isomorphism. In particular, $q \in \mathcal{C}^0(X, Y)$, $qm_1 = 0$. Let us prove that $r = qs\bar{\jmath}_1 \in (s\mathcal{D})^{-1}(X, Y)$ is invertible in the sense of Section 11.2, that is, there are elements $p \in (s\mathcal{D})^{-1}(Y, X)$, $w \in (s\mathcal{D})^{-2}(X, X)$, $v \in (s\mathcal{D})^{-2}(Y, Y)$, such that

$$
\begin{aligned}
r\bar{b}_1 = 0, \quad p\bar{b}_1 = 0, \quad (r \otimes p)\bar{b}_2 - 1_X s &= w\bar{b}_1,\\
(p \otimes r)\bar{b}_2 - 1_Y s &= v\bar{b}_1. \qquad (12.1)
\end{aligned}
$$

Indeed, denote $C = \operatorname{Cone}(q) = (Y \oplus X[1], d^C)$, where $(y, x)d^C = (yd^Y + xq, -xd^X)$ for $y \in Y^l$, $x \in X^{l+1}$. Since q is a quasi-isomorphism, C is acyclic. There is a standard exact sequence of complexes $0 \to Y \xrightarrow{n} C \xrightarrow{k} X[1] \to 0$ with the chain maps n, k, $yn = (y, 0)$, $(0, x)k = x$. We denote by n, k the corresponding elements $n \in \mathcal{C}^0(Y, C)$, $k \in \mathcal{C}^1(C, X)$. Define p as $p = ns \otimes ks \in (s\mathcal{C})^{-1}(Y, C) \otimes (s\mathcal{C})^0(C, X) \subset (s\mathcal{D})^{-1}(Y, X)$. Denote by $h \in \mathcal{C}^{-1}(X, C)$ the following $\Bbbk$-linear embedding $X \to C$, $X^l \to C^{l-1} = Y^{l-1} \oplus X^l$, $x \mapsto (0, x)$. Define w as $w = hs \otimes ks \in (s\mathcal{C})^{-2}(X, C) \otimes (s\mathcal{C})^0(C, X) \subset (s\mathcal{D})^{-2}(X, X)$. Then equations (12.1) hold true.

K-injective complexes. A complex $A \in \operatorname{Ob}\mathcal{C}$ is K-injective if and only if for every quasi-isomorphism $t : X \to Y \in \mathcal{C}$ the chain map $\mathcal{C}(t, A) : \mathcal{C}(Y, A) \to \mathcal{C}(X, A)$ is a quasi-isomorphism [13, Proposition 1.5]. Assume that each complex $X \in \mathcal{C}$ has a right K-injective resolution $r_X : X \to Xi$, that is, r_X is a quasi-isomorphism and $Xi \in \operatorname{Ob}\mathcal{C}$ is K-injective. Moreover, if X is K-injective, we assume that $Xi = X$ and $r_X = 1_X$. By definition, $\mathcal{C}(r_X, A) : s\mathcal{C}(Y, A) \to s\mathcal{C}(X, A)$, $fs \mapsto (r_X f)s$, is a quasi-isomorphism. The assumption is satisfied, when $\mathcal{A}$ has enough injectives and $\mathcal{C} = \mathsf{C}^+(\mathcal{A})$, or when $\mathcal{A} = R$-mod, or when $\mathcal{O}$ is a sheaf of rings on a topological space, and $\mathcal{A}$ is the category of sheaves of left $\mathcal{O}$-modules, see [13].

Notice that if $\Bbbk$ is a field, then for any chain complex of $\Bbbk$-modules of the form $N = s\mathcal{C}(X_0, X_1) \otimes_\Bbbk s\mathcal{C}(X_1, X_2) \otimes_\Bbbk \cdots \otimes_\Bbbk s\mathcal{C}(X_{n-1}, X_n)$, $n \geqslant 0$, $X_i \in \operatorname{Ob}\mathcal{C}$, for any quasi-isomorphism $r_X : X \to Y$, and for any K-injective $A \in \mathcal{C}$ the following chain map

$$
\begin{aligned}
u = \operatorname{Hom}(N, \mathcal{C}(r_X, A)) &:\\
\operatorname{Hom}^\bullet_\Bbbk(N, s\mathcal{C}(Y, A)) &\to \operatorname{Hom}^\bullet_\Bbbk(N, s\mathcal{C}(X, A)),
\end{aligned}
$$

is a quasi-isomorphism (any $\Bbbk$-module complex is K-projective). Therefore, assuming that $\Bbbk$ is a field, we may apply results of Section 11 to differential graded category $\mathcal{C} = \mathsf{C}$ or C^+, and its full subcategories $\mathcal{B} = \mathsf{A}(\mathcal{A})$ (resp. $\mathcal{I} = \mathsf{I}(\mathcal{A})$, $\mathcal{J} = \mathsf{AI}(\mathcal{A})$) of acyclic (resp. K-injective, acyclic K-injective) complexes. Denote by

$e : \mathcal{I} \hookrightarrow \mathcal{C}$ the full embedding. Starting with the identity functor $f = \mathrm{id}_\mathcal{C}$ we get existence of $g = ie$ simultaneously with existence of a unital A_∞-functor $i : \mathcal{C} \to \mathcal{I}$ – "injective resolution functor", a natural A_∞-transformation $r : \mathrm{id} \to ie : \mathcal{C} \to \mathcal{C}$ (Propositions 11.1, 11.4). The said i, r are unique in the sense of Proposition 11.2 and Corollary 11.3. Moreover, while solving appropriate equations we will choose the solutions

$$i_n = g_n = \mathrm{id}_n : s\mathcal{C}(X_0, X_1) \otimes_\Bbbk \cdots \otimes_\Bbbk \otimes_\Bbbk s\mathcal{C}(X_{n-1}, X_n) \to s\mathcal{C}(X_0, X_n),$$
$$r_n = \mathbf{i}_n^\mathcal{C} : s\mathcal{C}(X_0, X_1) \otimes_\Bbbk \cdots \otimes_\Bbbk \otimes_\Bbbk s\mathcal{C}(X_{n-1}, X_n) \to s\mathcal{C}(X_0, X_n),$$

if $X_0, \dots, X_n$ are K-injective (recall that $X_0 i = X_0$, $X_n i = X_n$).

Extending e, i to A_∞-functors between quotient categories, we get a unital strict A_∞-embedding (actually, a faithful differential graded functor) $\bar{e}$: $\mathsf{D}(\mathcal{I}|\mathcal{J}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$, which is injective on objects, and a unital A_∞-functor $\bar{\imath} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{I}|\mathcal{J})$. Let us prove that these A_∞-functors are quasi-inverse to each other. First of all, $ei = \mathrm{id}_\mathcal{I}$ implies $\overline{ei} = \mathrm{id}_{\mathsf{D}(\mathcal{I}|\mathcal{J})}$. Secondly, there is a natural A_∞-transformation $\bar{r} : \mathrm{id} \to \overline{ie}$: $\mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$. Let us prove that it is invertible. The 0-th component is

$$_X\bar{r}_0 = \big[\Bbbk \xrightarrow{_X r_0} (s\mathcal{C})^{-1}(X, Xi) \xhookrightarrow{\bar{\jmath}_1} \xhookrightarrow{\bar{\jmath}_1} (s\mathsf{D}(\mathcal{C}|\mathcal{B}))^{-1}(X, Xi)\big].$$

Since r_X is a quasi-isomorphism, the above element is invertible in the sense of Section 12.1: there exist p_0, v_0, w_0 such that equations (11.2) hold. We conclude by Section 11.2 that $\bar{r}$ is invertible, hence, $\mathsf{D}(\mathcal{C}|\mathcal{B})$ and $\mathsf{D}(\mathcal{I}|\mathcal{J})$ are equivalent.

Each acyclic K-injective complex X is contractible. Indeed, $\mathsf{K}(\mathcal{A})(X, X) \simeq \mathsf{D}(\mathcal{A})(X, X) = 0$ by [13, Proposition 1.5]. Hence, $\mathcal{J}$ is a contractible subcategory of $\mathcal{I}$. Thus, $\bar{\jmath} : \mathcal{I} \to \mathsf{D}(\mathcal{I}|\mathcal{J})$ is an equivalence. We deduce that $\mathsf{D}(\mathcal{C}|\mathcal{B})$ and $\mathcal{I}$ are equivalent in A_∞^{isu}. Taking H^0 we get equivalent categories $H^0(\mathsf{D}(\mathcal{C}|\mathcal{B}))$ and $H^0(\mathcal{I})$. The latter is a full subcategory of $\mathsf{K}(\mathcal{A})$, whose objects are K-injective complexes. It is equivalent to the derived category $\mathsf{D}(\mathcal{A})$ (e.g. by [8, Proposition 1.6.5]). Hence, $H^0(\mathsf{D}(\mathcal{C}|\mathcal{B}))$ is equivalent to the derived category $\mathsf{D}(\mathcal{A})$.

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КАТЕГОРІЯ A_∞-КАТЕГОРІЙ І ПОХІДНІ КАТЕГОРІЇ

Визначаються природні A_∞-перетворення і будується A_∞-категорія A_∞-функторів. Послаблюється поняття строгих одиниць в A_∞-категорії. Описано 2-категорію A_∞-категорій, функторів і перетворень. Вивчається факторкатегорія A_∞-категорії за повною підкатегорією. Звичайну похідну категорію одержано як нульові когомології факторкатегорії диференціально-градуйованої категорії комплексів за ациклічними комплексами.