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THE CAUCHY PROBLEM FOR ONE CLASS OF PARABOLIC PSEUDODIFFERENTIAL EQUATION WITH DEVIATION OF THE ARGUMENT

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Abstract. In this paper, we study solvability of the Cauchy problem for a parabolic pseudodifferential equation with the deviation of the argument. Parabolic pseudodifferential operator with non-smooth symbols introduced by Eidel'man and Drin' for the first time. For such equations, the initial condition is set on a certain interval. Technical and physical reasons for delays can be transport delays, delays in decision-making, delays in information transmission, etc. The most natural are delays when modeling objects in medicine, population dynamics, ecology, etc. Other physical and technical interpretations are also possible, for example, the molecular distribution of thermal energy in various media (liquids, solid bodies, etc.) is modeled by heat conduction equations. Features of the dynamics of vehicles in different environments (water, land, air) can also be taken into account by introducing a delay. The formula for the solution of the Cauchy problem is constructed for the nonlinear equation of heat conduction with a deviation of the argument, its properties are investigated.

Key words: pseudodifferential nonlinear equation, Cauchy problem, deviation argument, step method.

Mathematics Subject Classification MSC2020: 35K61.

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1. Introduction

Differential equations with deviating argument are differential equations in which unknown function and its derivatives take, generally speaking, different values of the argument. These equations are frequently encountered in the description of various processes and problems in the theories of automatic control, automatics and telemechanics, radiolocation, radionavigation, electric communications, theoretical cybernetics, missile engineering, nuclear fusion, biology, economics, and medicine. Some equations with ordinary derivatives were introduced

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by Condorcet (1771). The systematic study of these equations was originated by Myshkis, Write, and Bellman in connection with the needs of applied sciences. The extensive application of these methods resulted in a significant increase in the interest to the theory of these equations and, hence, in the appearance of numerous publications dealing with the theory of differential equations with deviating argument. The works of Ukrainian mathematicians Mitropol'skii, Samoilenko, and Perestyuk became classical in the field of differential equations with deviating argument. A brief survey of the methods used for the investigation of three-dimensional nonlocal effects caused by the delay in diffusion models of some populations placed in bounded or unbounded domains can be found in [2].

Parabolic pseudodifferential equations with nonsmooth constants symbols introduced by Eidel'man and Drin' in [3]. Cauchy problem for quasilinear systems of parabolic pseudodifferential equations is studied in [4,5] and its equations with deviation argument is studied in [7]. The theory of parabolic pseudodifferential equations, a well-developed part of the contemporary theory of pseudodifferential equations and mathematical physics, is the subject of an immense rearch activity (see [6], [8]). In this book interprets pseudodifferential operator as a hypersingular integral, i. e., as an integral with singularity whose order is higher than the dimension of the space regularized with the help of finite differences. The relationship between a pseudodifferential operator and a hypersingular integral is described in detail in [6, p. 911-914]. The behavior of oscillating integrals is being invertion by virtue of solvability Cauchy problem for the parabolic pseudodifferential equations with nonsmooth symbols are studied in [9].

2. Statement of the Cauchy Problem

In the present paper, we use the step method [1] to construct the solution of a quasilinear Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} + (Au)(x,t) = f(x,t,u(x,t-h)), \quad x \in \mathbb{R}^n, \quad t > h, \tag{2.1}$$

$$u(x,t) = u_0(x,t), \quad x \in \mathbb{R}^n, \quad 0 \le t \le h, \tag{2.2}$$

where $h > 0$ is a number, A is a pseudodifferential operator with symbol $a : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ homogeneous of order $\gamma > 0$ and infinitely differentiable with respect to σ for $\sigma \neq 0$ ($\sigma = 0$ is a point of nonsmoothness of the symbol $a(\sigma,t)$), and f and u_0 are known bounded and continuous functions of the arguments x, t, and u. The band $\Pi_{[0,h]} \equiv \{(x,t); x \in \mathbb{R}^n, 0 \le t \le h\}$ in which the initial function u_0 is defined is called the initial band and the hyperplane $\{(x,h); h > 0$ is a number; $x \in \mathbb{R}^n\}$ is called an initial hyperplane. If, in Eq. (2.1) and in the initial condition (2.2), u, f, and u_0 are vector functions and the symbol $a(\sigma,t)$ is a matrix of the corresponding order, then we get the principal initial-value problem for the system of equations with delay.

In the case of variable delay $h(t) > 0$, it is first necessary to define an initial hyperplane $\{(x, h_0)\,; x \in \mathbb{R}^n,\ h_0 > 0$ is a constant$\}$ and an initial set formed the initial hyperplane and the points of the band

$$\Pi_{[t-h(t),h_0]} \equiv \{(x,t); x \in \mathbb{R}^n, t-h(t) < h_0 \text{ for } t > h_0\}$$

For Eq. (2.1), it is necessary to find a solution $u(x,t), x \in \mathbb{R}^n, t > h_0$, which coincides with a given initial function $u_0(x,t)$ on the initial set.

For Eq. (2.1) in which the pseudodifferential operator is defined by a nonsmooth symbol independent of t, the posed problem with nonlocal condition is solved in the present paper for the first time.

We now formulate conditions for the symbol a, define the corresponding pseudodifferential operator, and interpret it as a hypersingular integral [6]. Assume that the symbol $a : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ satisfies the conditions of nonsmoothness at the origin, homogeneity $a \in C^\infty(\mathbb{R}^n \backslash \{0\})$, $a(\lambda\sigma,t) = \lambda^\gamma a(\sigma,t)$, $\lambda > 0$, $\gamma > 0$, and ellipticity

$$\operatorname{Re} a(\sigma,t) \geq a_0 > 0, \quad \sigma \in \mathbb{R}^n, \quad |\sigma| = 1, \quad t \in [0,T], \tag{2.3}$$

$$|D_\sigma^\chi a(\sigma,t)| \leq C|\sigma|^{\gamma-|\chi|}, \quad \sigma \neq 0, \quad \gamma > 0, \quad t \in [0,T]. \tag{2.4}$$

In the class of rapidly decreasing functions, a pseudodifferential operator is defined by the formula

$$(Au)(x,t) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{i(x,\sigma)} a(\sigma,t)\hat{u}(\sigma,t)\, d\sigma, \quad x \in \mathbb{R}^n, \quad t > 0,$$

$$\hat{u}(\sigma,t) = \int\limits_{\mathbb{R}^n} e^{-i(\sigma,y)} u(y,t)\, dy, \quad \sigma \in \mathbb{R}^n, \quad t > 0,$$

where the symbol a satisfies conditions (2.3) and (2.4).

In [6] interprets a pseudodifferential operator as a hypersingular integral, i.e., as an integral with singularity whose order is higher than the dimension of the space regularized with the help of finite differences.

For given complex-valued bounded functions $f \in (\mathbb{R}^n)$ and $\Omega \in C\left(\mathbb{R}^n \times S^{n-1}\right)$, a relation of the form [6]

$$\left(D_\Omega^\alpha f\right)(x) = \frac{1}{d_{n,l}(\alpha)} \int\limits_{\mathbb{R}^n} \Omega\left(x, \frac{\tilde{h}}{|\tilde{h}|}\right) \frac{\left(\Delta_{\tilde{h}}^l f\right)(x)}{|\tilde{h}|^{n+\alpha}}\, d\tilde{h}, \quad x \in \mathbb{R}^n \tag{2.5}$$

where $\alpha > 0$, l is a natural number, $d_{n,l}(\alpha)$ is a normalizing constant, S^{n-1} is a unit sphere in $\mathbb{R}^n$, and

$$\left(\Delta_{\tilde{h}}^l f\right)(x) = \sum_{k=0}^{l} (-1)^k C_l^k f(x - k\tilde{h}),$$

is called a hypersingular integral of order α with characteristic Ω. The relationship between a pseudodifferential operator and a hypersingular integral is described in detail in [6, p. 911–914].

3. Step Method

By the step method, we reduce the Cauchy problem for a pseudodifferential equation with deviating argument to the Cauchy problem for an equation with nondeviating argument (this result is announced in [7]).

Let $h < t \le 2h$, let $x \in \mathbb{R}^n$, and let $f\,(x, t, u_0(x, t-h)) \equiv f_0(x, t, h)$. Then $u(x, t-h) = u_0(x, t-h)$ and problem (2.1), (2.2) takes the form

$$\frac{\partial u(x,t)}{\partial t} + (Au)(x,t) = f_0(x,t,h), \quad x \in \mathbb{R}^n, \quad h < t \le 2h, \tag{3.1}$$

$$u(x,t)|_{t=h} = u_0(x,h), \quad x \in \mathbb{R}^n. \tag{3.2}$$

In terms of the Fourier transforms, we get the following problem:

$$\frac{dv(\sigma,t)}{dt} + a(\sigma,t)v(\sigma,t) = \hat{f}_0(\sigma,t,h), \quad \sigma \in \mathbb{R}^n, \quad h < t \le 2h, \tag{3.3}$$

$$v(\sigma,t)|_{t=h} = \hat{u}_0(\sigma,h), \quad \sigma \in \mathbb{R}^n. \tag{3.4}$$

The solution to the problem (3.3), (3.4) for $k = 1$ will be written as

$$v(\sigma,t) = \exp\{-I(\sigma,h,t)\}\left[v(\sigma,h) + \int_h^t \hat{f}_0(\sigma,z,h)\exp\{I(\sigma,h,z)\}\,dz\right],$$

where $I(\sigma,kh,t) = \int\limits_{kh}^{t} a(\sigma,\tau)\,d\tau$,

$$\hat{f}_0(\sigma,z,h) = (2\pi)^{-n}\int_{\mathbb{R}^n} f_0(\xi,z,h)\exp\{-i(\xi,\sigma)\}\,d\xi, \quad h < z < t,$$

$$v(\sigma,h) \equiv \hat{u}_0(\sigma,h) = (2\pi)^{-n}\int_{\mathbb{R}^n} u_0(\xi,h)\exp\{-i(\xi,\sigma)\}\,d\xi, \quad \sigma \in \mathbb{R}^n.$$

Considering that

$$u(x,t) = \int_{\mathbb{R}^n} \exp\{i(x,\sigma)\}v(\sigma,t)\,d\sigma,$$

we get the expression

$$u(x,t) = \int_{\mathbb{R}^n} G(x-\xi,t,h)u_0(\xi,h)\,d\xi + \int_h^t dz \int_{\mathbb{R}^n} G(x-\xi,t,h,z)f_0(\xi,z,h)\,d\xi, \ k = 1.$$

where

$$G(x-\xi,t,h) = (2\pi)^{-n}\int_{\mathbb{R}^n} \exp\{-I(\sigma,h,t) + i(x-\xi,\sigma)\}\,d\sigma,$$

$$G(x-\xi,t,h,z) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \exp\{-[I(\sigma,h,t) - I(\sigma,h,z)] + i(x-\xi,\sigma)\}\ d\sigma.$$

If the symbol $a(\sigma,t) \equiv a(\sigma)$ does not depend on t, then $I(\sigma,h,t) = a(\sigma)$, then the formula for solving the Cauchy problem (3.1), (3.2) takes the form

$$v(\sigma,t) = \exp\{-a(\sigma)(t-h)\}\hat{u}_0(\sigma,h) + \int\limits_h^t \exp\{-a(\sigma)(t-\tau)\}\hat{f}(\sigma,\tau,\tau-h)\ d\tau, \quad (3.5)$$

where $\sigma \in \mathbb{R}^n$ and $h < t \leq 2h$. It is clear that condition (3.4) is satisfied and

$$v(\sigma,2h) = \exp\{-a(\sigma)h\}\hat{u}_0(\sigma,h) + \int\limits_h^{2h} \exp\{-a(\sigma)(2h-\tau)\}\hat{f}(\sigma,\tau,\tau-h)\ d\tau, \quad \sigma \in \mathbb{R}^n. \quad (3.6)$$

Condition (3.6) follows from (3.5) and plays the role of initial condition for the Cauchy problem in the interval $2h < t \leq 3h$.

In view of the fact that

$$u(x,t) = \int\limits_{\mathbb{R}^n} \exp\{i(x,\sigma)\}v(t,\sigma)\ d\sigma, \quad x \in \mathbb{R}^n, \quad h < t \leq 2h,$$

$$\hat{f}_0(\sigma,\tau,\tau-h) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} f_0(\xi,\tau,h)\exp\{-i(\xi,\sigma)\}\ d\xi, \quad \sigma \in \mathbb{R}^n, \quad h < \tau \leq t,$$

$$\hat{u}_0(\sigma,h) = (2\pi)^{-n} \int_{\mathbb{R}^n} u_0(\xi,h)\exp\{-i(\xi,\sigma)\}\ d\xi, \quad \sigma \in \mathbb{R}^n,$$

we get the following solution of the Cauchy problem (3.1), (3.2) (in the interval $h < t \leq 2h$; it is denoted by $u_1(x,t,h)$):

$$u_1(x,t,h) = \int\limits_{\mathbb{R}^n} G(x-\xi,t-h)u_0(\xi,h)\ d\xi + \int\limits_h^t d\tau \int\limits_{\mathbb{R}^n} G(x-\xi,t-\tau)f_0(\xi,\tau,h)\ d\xi, \quad x \in \mathbb{R}^n, \quad h < t \leq 2h, \quad (3.7)$$

where

$$G(x,t) \equiv (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \exp\{i(x,\sigma) - a(\sigma)t\}\ d\sigma, \quad x \in \mathbb{R}^n, \quad t > 0. \quad (3.8)$$

By Π_k we denote the band $\{x \in \mathbb{R}^n, kh < t \leq (k+1)h, k \in \mathbb{N}\}$. Then, for $(x,t) \in \Pi_k$, the solution of the Cauchy problem (3.1), (3.2) is given by the formula

$$u_k(x,t,kh) = \int\limits_{\mathbb{R}^n} G(x-\xi,t-kh)u_{k-1}(\xi,kh)\ d\xi$$
$$+\int\limits_{kh}^{t} d\tau \int\limits_{\mathbb{R}^n} G(x-\xi,t-\tau)f_{k-1}(\xi,\tau,h)\ d\xi. \quad (3.9)$$

In the case of the symbol $a(\sigma,t)$ under the sign of integrals over spatial variables, we must write $G(x-\xi,t,kh)$ and $G(x-\xi,t,kh,z)$, which were introduced earlier. We will take this fact into account in the following, and for simplicity, the symbol $a(\sigma)$ is independent of t.

4. Substantiation of the Expression for the Solution of the Cauchy Problem

Since the symbol $a:\mathbb{R}^n\to\mathbb{R}$ is a function infinitely differentiable with respect to σ for $\sigma\neq 0$, the following estimates hold for $\gamma>0$:

$$|D_x^\chi G(x,t)| \leq Ct\left(t^{1/\gamma}+|x|\right)^{-n-\gamma-|\chi|}, \quad x\in\mathbb{R}^n, \quad t>0, \quad (4.1)$$

$$\left|\frac{\partial}{\partial t}G(x,t)\right| \leq C\left(t^{1/\gamma}+|x|\right)^{-n-\gamma}, \quad x\in\mathbb{R}^n, \quad t>0, \quad (4.2)$$

(these estimates are established in [5] for $\gamma>1$, in [6] for $\gamma\geq 1$, and in [9] for $\gamma>0$). Moreover, the following equality is true:

$$\int\limits_{\mathbb{R}^n} G(x-\xi,t-y)\ dy = 1, \quad x\in\mathbb{R}^n, \quad \xi\in\mathbb{R}^n, \quad t>0. \quad (4.3)$$

Equality (4.3) follows from (3.8) and the relation for the Fourier transform. In view of (4.2) and (4.3), we get

$$\int\limits_{\mathbb{R}^n} \frac{\partial}{\partial t}G(x-\xi,t-y)\ dy = 0.$$

The convergence of integrals (3.7)–(3.9) is guaranteed by the conditions imposed on the symbol a and the function f and the estimates for G.

We study the properties of the volume potential

$$u_k(x,t,kh) = \int\limits_{kh}^{t} d\tau \int\limits_{\mathbb{R}^n} G(x-\xi,t-\tau)f_{k-1}(\xi,\tau)\ d\xi, \quad (4.4)$$
$$x\in\mathbb{R}^n, \quad t>kh, \quad k\in\mathbb{N},$$

where $f_k(y,\tau)\equiv f\left(y,\tau,u_k(y,\tau-h)\right)$. Assume that the following conditions are satisfied:

(f_1) $|f_{k-1}(\xi,\tau)| \le C(\tau - kh)^{-\rho}$, $kh < \tau \le t$, $0 \le \rho < 1$,

(f_2) $|f_{k-1}(x,\tau) - f_{k-1}(\xi,\tau)| \le C|x-\xi|^{\lambda}(\tau - kh)^{\rho}$, $kh < \tau \le t$, $0 \le \rho < 1$, $0 < \lambda \le 1$, $k \in \mathbb{N}$.

It follows from inequalities (4.1) that integral (4.4) absolutely converges and the function u_k has continuous derivatives with respect to x of any order smaller than γ. These derivatives can be found by the direct differentiation under the integral sign.

As in [5, 6], the existence of the derivative with respect to t and the relation

$$\frac{\partial u_k(x,t,kh)}{\partial t} = \int\limits_{kh}^{t} d\tau \int\limits_{\mathbb{R}^n} \frac{\partial G(x-\xi,t-\tau)}{\partial t}\left[f_{k-1}(\xi,\tau) - f_{k-1}(x,\tau)\right] d\xi + f_{k-1}(x,t), \quad x \in \mathbb{R}^n, \quad t > kh. \tag{4.5}$$

are established by using inequality (4.2).

Consider the action of the hypersingular integral [the operator D^{α}_{Ω} of the form (2.5)] on the volume potential (4.4) in two cases: $\alpha < \gamma$ and $\alpha = \gamma$. Let α be a noninteger number and let $l \ge [\alpha] + 1$. It follows from (4.1) that, for $|\tilde{h}| \le (t-\tau)^{1/\gamma}$,

$$\left|\left(\Delta^{l}_{\bar{h}} G\right)(x-\xi,t-\tau)\right| \le C|\bar{h}|^{[\alpha]+1}(t-\tau)\sum_{v=0}^{l}\Big[(t-\tau)^{1/\gamma} + \left|x - \theta_v v\bar{h} - \xi\right|\Big]^{-(n+\gamma+[\alpha]+1)}, \tag{4.6}$$

where $x \in \mathbb{R}^n, t > \tau$, and $0 < \theta_v < 1$. By using (3.4), for $|\tilde{h}| > (t-\tau)^{1/\gamma}$, we arrive at the estimate

$$\left|\left(\Delta^{l}_{\bar{h}} G\right)(x-\xi,t-\tau)\right| \le C(t-\tau)\sum_{\nu=0}^{l}\Big[(t-\tau)^{1/\gamma} + |x - \theta_\nu v h - \xi|\Big]^{-n-\gamma}, \quad x \in \mathbb{R}^n, \quad t > \tau. \tag{4.7}$$

In view of (4.6), (4.7), and the Fubini theorem, we conclude that the hypersingular integral $D^{\alpha}_{\Omega}u$ is absolutely convergent and can be used in the integrand of the integral with respect to ξ,

$$\left(D^{\alpha}_{\Omega}u_k\right)(x,t,kh) = \int\limits_{kh}^{t} d\tau \int\limits_{\mathbb{R}^n} G_{\Omega}(x, x-\xi, t-\tau) f_{k-1}(\xi,\tau)\, d\xi, \quad x \in \mathbb{R}^n, \quad t > kh, \tag{4.8}$$

where $G_{\Omega} \equiv D^{\alpha}_{\Omega}G$, i.e.,

$$G_{\Omega}(x,z,t-y = (2\pi)^{-n}\int\limits_{\mathbb{R}^n} \tilde{\Omega}(x,\sigma)\exp\{i(z,\sigma) - a(\sigma)(t-y)\}\, d\sigma, \quad x \in \mathbb{R}^n, \quad z \in \mathbb{R}^n, \quad t > y,$$

and $\tilde{\Omega}$ is the symbol of the analyzed hypersingular integral. The convergence of integral (4.8) is guaranteed by the properties of the function G_Ω and estimates similar to (4.1) and (4.2) valid for G_Ω.

Let α be an integer. Hence (see [6, p. 911]), if α is even, then D_Ω^α can be defined by equality (2.5) and the hypersingular integral, i.e., the operator D_Ω^α, is a differential operator of order α. If α is odd, then, for $l > \alpha$, the integral in (2.5) is identically equal to zero for any function f. In this case, the hypersingular integral is defined only for the even characteristic Ω by relation (3.2) in [6, p. 911] with $l = \alpha$. We now consider the hypersingular integral of order $\gamma > 0$. Assume that $\bar{\Omega}(x, \sigma)$ is an infinitely differentiable symbol with respect to σ for $\sigma \neq 0$ and

$$\left| D_\sigma^\chi \tilde{\Omega}(x, \sigma) \right| \le C|\sigma|^{\gamma - |\chi|}, \quad \gamma > 0,$$

$$\left| D_\sigma^\chi [\tilde{\Omega}(x, \sigma) - \tilde{\Omega}(y, \sigma)] \right| \le C|x - y|^\lambda |\sigma|^{\gamma - |\chi|}, \quad \gamma > 0, \quad 0 < \lambda \le 1.$$

Moreover, we assume that $\tilde{\Omega}(x, \sigma) \neq 0$ for $\sigma \neq 0$. If the number γ is integer and the symbol $\tilde{\Omega}$ is not a polynomial in σ (this is possible only for odd γ and the even characteristic), then we additionally assume that, in the decomposition in spherical harmonics

$$[\tilde{\Omega}(x, \sigma)]^{-1} = \sum_{v=0}^{\infty} \sum_{\mu=1}^{\delta_{2v}} C_{2v,\mu}(x) Y_{2v,\mu}(\sigma), \quad |\sigma| = 1,$$

the coefficients $C_{2v,\mu}(x) = 0$ for $\gamma = n + 2v + 2k, k = 0, 1, 2, \ldots$. Since the case where $\tilde{\Omega}$ is a polynomial in σ belongs to the theory of parabolic differential equations [8], it is not considered in the present work.

Theorem 4.1. *Under the conditions listed above, the hypersingular integral* $D_\Omega^\gamma u_k$ *exists in a sense of conditional convergence*

$$\left(D_\Omega^\gamma u_k \right)(x, \cdot) = \lim_{\varepsilon \to 0} \left(D_{\Omega,\varepsilon}^\gamma u_k \right)(x, \cdot),$$

where the truncated hypersingular integral $\left(D_{\Omega,\varepsilon}^\gamma u_k \right)(x, \cdot)$ *is obtained from relation (2.5) by replacing the domain of integration with* $\left\{ \tilde{h} \in \mathbb{R}^n; |\tilde{h}| > \varepsilon, \varepsilon > 0 \right\}$ *and, furthermore,*

$$\left(D_\Omega^\gamma u_k \right)(x, t, kh) = \int_{kh}^{t} d\tau \int_{\mathbb{R}^n} G_\Omega(x, x - \xi, t - \tau) \left[f_{k-1}(\xi, \tau) - f_{k-1}(x, \tau) \right] \, d\xi, \tag{4.9}$$

$$x \in \mathbb{R}^n, \quad t > kh.$$

The proof is performed according to the following scheme (see [6, p. 922, 923]): Instead of the function u_k in (4.4), we consider the expression

$$u_k^\theta(x,t,kh) \equiv \int\limits_{kh}^{t-\theta} d\tau \int\limits_{\mathbb{R}^n} G(x-\xi,t-\tau) f_{k-1}(\xi,\tau)\, d\xi, \quad x\in\mathbb{R}^n, \quad 0<\theta<t-kh$$

Inequalities (4.6) and (4.7) guarantee the absolute convergence of the hypersingular integral $D_\Omega^\gamma u_\theta$ and the possibility of application of D_Ω^γ to G_Ω under the integral sign by the formula

$$\left(D_{\Omega'}^\gamma u_k^\theta\right)(x,t,kh) = \int\limits_{kh}^{t-\theta} d\tau \int\limits_{\mathbb{R}^n} G_\Omega(x,x-\xi,t-\tau) f_{k-1}(\xi,\tau)\, d\xi, \qquad (4.10)$$
$$x\in\mathbb{R}^n, \quad 0<\theta<t-kh.$$

In view of (4.6), (4.7), and the Fubini theorem, relation (4.3) implies that

$$\int\limits_{\mathbb{R}^n} G_\Omega(x,x-\xi,t-\tau)\, d\xi = 0, \quad x\in\mathbb{R}^n, \quad t-\tau>0 \qquad (4.11)$$

where

$$|G_\Omega(x,x-\xi,t-\tau)| \le C\left|(t-\tau)^{1/\gamma}+|x-\xi|\right|^{-n-\gamma}, \quad \gamma>0, \qquad (4.12)$$
$$x\in\mathbb{R}^n, \quad \xi\in\mathbb{R}^n, \quad t>\tau.$$

The last inequality guarantees the convergence of integral (4.11). It is proved in [6] for $\gamma\ge 1$ and in [9] for $\gamma>0$. In view of (4.11), relation (4.10) can be rewritten in the form

$$\left(D_\Omega^\gamma u_k^\theta\right)(x,t,kh) = \int\limits_{kh}^{t-\theta} d\tau \int\limits_{\mathbb{R}^n} G_\Omega(x,x-\xi,t-\tau)\left[f_{k-1}(\xi,\tau)-f_{k-1}(x,\tau)\right]\, d\xi, \qquad (4.13)$$
$$x\in\mathbb{R}^n, \quad 0<\theta<t-kh.$$

By $\Phi(x,t,kh)$ we denote the function on the right-hand side of (4.9). Note that the integrals in (4.9) are convergent by virtue of estimate (4.12) and the conditions imposed on the function f. Thus, by virtue of relation (4.13), we conclude that

$$\left(D_\Omega^\gamma u_k\right)(x,t,kh) = \lim_{\theta\to 0}\left(D_\Omega^\gamma u_k^\theta\right)(x,t,kh)$$
$$= \Phi(x,t,kh), \quad x\in\mathbb{R}^n, \quad kh<t\le(k+1)h$$

uniformly in $x\in\mathbb{R}^n$.

The theorem is proved.

Theorem 4.2. *Let $kh < t \le (k+1)h$, $k \in \mathbb{N}$, $x \in \mathbb{R}^n$. Then the solution of the Cauchy problem (3.1), (3.2) is given by relation (3.9).*

We prove the theorem by induction on $k \in \mathbb{N}$. Let $k = 1$. By virtue of estimates (4.1) and (4.2) and the conditions imposed on $u_0(x,t)$ and $f_0(x,t,h)$, the integrals in (3.7) are uniformly convergent for $t \ge h + \varepsilon$, $x \in \mathbb{R}^n$, where $\varepsilon > 0$ is an arbitrarily small number. Thus, the function $u_1(x,t,h)$ is continuous and bounded. In view of (4.5) and (4.9), we can directly show that (3.7) satisfies Eq. (3.1). We now check the initial condition (3.2). First, we consider the second term in (3.7):

$$u_1^2(x,t,h) = \int_h^t d\tau \int_{\mathbb{R}^n} G(x-\xi, t-\tau) f_0(\xi,\tau,h)\, d\xi, \quad h < t \le 2h, \quad x \in \mathbb{R}^n.$$

It is clear that

$$\lim_{t \to h+0} u_1^2(x,t,h) = 0.$$

Consider the first term in (3.7). By using (4.13), we rewrite it in the form of a difference

$$u^1(x,t,h) - u_0(x,h) = \int_{\mathbb{R}^n} G(x-\xi, t-h)\left(u_0(\xi,h) - u_0(x,h)\right)\, d\xi,$$

where $h < t < 2h$ and $x \in \mathbb{R}^n$, and perform the following change of the space variables: $z_i = (\xi_i - x_i)\,\tilde{t}^{-1/\gamma}$, $1 \le i \le n, \tilde{t} = t - h$. In view of (3.8), this yields

$$u_1^1(x,t,h) - u_0(x,h) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \exp\{i(z,\sigma) - a(\sigma)\}\, d\sigma \right\} \times \left(u_0\left(x + z\tilde{t}^{1/\gamma}, h\right) - u_0(x,h)\right)\, dz, \quad x \in \mathbb{R}^n, \quad h < t \le 2h, \tag{4.14}$$

where estimate (4.1) guarantees the uniform convergence of the integral with respect to z. By using (4.14), we get

$$\begin{aligned} \left|u_1^1(x,t,h) - u_0(x,h)\right| &\le \left| \int_{|z|\le N} G(z) \left(u_0\left(x + z\tilde{t}^{1/\gamma}, h\right) - u_0(x,h)\right)\, dz \right| \\ &\quad + \left| \int_{|z|\ge N} G(z) \left(u_0\left(x + z\tilde{t}^{1/\gamma}, h\right) - u_0(x,h)\right)\, dz \right| \\ &\equiv I_1 + I_2. \end{aligned}$$

It follows from boundedness of the initial function u_0 that there exists a number $M > 0$ such that

$$\left|u_0\left(x + z\tilde{t}^{1/\gamma}, h\right) - u_0(x,h)\right| \le M, \quad x \in \mathbb{R}^n, \quad z \in \mathbb{R}^n, \quad 0 \le \tilde{t} \le h.$$

Let $\varepsilon > 0$ be an arbitrarily small number. One can find a sufficiently large $N > 0$ such that the convergence of the integral in (4.14) with respect to z enables us to write

$$|I_2| \le M \int\limits_{|z|\ge N} |G(z)|\, dz \le \frac{\varepsilon}{2}.$$

The continuity of the function $u_0(x,h)$ implies that, for all $\tilde{t} = t - h > 0$ close to zero and all $|z| \le N$, we can write

$$\left|u_0\left(x + z\tilde{t}^{1/\gamma}, h\right) - u_0(x,h)\right| \le \frac{\varepsilon}{2c},$$

where

$$c = \int\limits_{|z|\le N} |G(z)|\, dz.$$

Then

$$I_1 \le c\left|u_0\left(x + zt^{1/\gamma}, h\right) - u_0(x,h)\right| \le c\frac{\varepsilon}{2c} = \frac{\varepsilon}{2}.$$

Thus, for all $t > h$ sufficiently close to h and $x \in \mathbb{R}^n$, we find

$$\left|u_1^1(x,t,h) - u_0(x,h)\right| \le I_1 + I_2 \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that

$$\lim_{t\to h+0} u_1(x,t,h) = \lim_{t\to h+0} u_1^1(x,t,h) = u_0(x,h)$$

because $\lim\limits_{t\to h+0} u_2(x,t,h) = 0.$

By induction, we prove that

$$\lim_{t\to h+0} u_k(x,t,h) = u_{k-1}(x,h)$$

The theorem is proved.

Remark 4.1. This result remains true if, in Eq. (1), we set

$$(Au)(x,t) \equiv (A_0 u)(x,t) + \sum_{k=1}^{m} (A_k u)(x,t),$$

where A_k are pseudodifferential operators either with symbols $a_k : \mathbb{R}^n \to \mathbb{R}, 0 \le k \le m$, homogeneous of orders $\gamma_k > 0$, $0 \le k \le m$, and such that $\gamma_0 > \gamma_1 > \ldots > \gamma_m$ are infinitely differentiable with respect to $\sigma \in \mathbb{R}^n\backslash\{0\}$ [the principal symbol a_0 satisfies condition (2.3) and the other symbols satisfy condition (2.4)], or with symbols depending on the time variable $t > 0$ and the space variables $x \in \mathbb{R}^n$ [6].

Remark 4.2. The result remains true for a system of parabolic pseudodifferential equations of the form (2.1) with condition of the form (2.2).

5. Conclusion

In this paper, we use the step method [1] to construct the solution of a quasilinear Cauchy problem for parabolic pseudodifferential equation with nonsmooth symbol depend from $t > 0$. Parabolic pseudodifferential equations with nonsmooth symbols introduced by Eidel'man and Drin' in [3]. The solvability of the Cauchy problem for autonomous quasilinear parabolic pseudodifferential equations with nonsmooth symbols independing from $t > 0$ and deviating argument is proved in [10].

Theory of Differential Equations with Deviating Argument describe in [1]. Nonlocal reaction-diffusion differential equations with delay: biological models and nonlinear dynamics describe in [2]. In [6, 8] pseudodifferential operator interprets as a hypersingular integral, i. e. as an integral with singularity was order is higher then the dimension of the space regularized with the help of finite differences. We investigation [4, 5] and the behavior of oscillating integrals from [9].

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