УДК 517.977+612.82

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CONTROL OF SYMMETRY BY LYAPUNOV EXPONENTS

In this paper we describe control systems with local and global symmetry. Recent results in control theory have demonstrated that control can lead to symmetry breaking in chaotic systems with a simple type of symmetry. In our work we analyze controllability of Lyapunov exponents using continuous control functions. We show that, by controlling Lyapunov exponents, a chaotic attractor lying in some invariant subspace can be made unstable with respect to perturbations transverse to the invariant subspace. Furthermore, a symmetry-increasing bifurcation can occur, after which the attractor possesses the system symmetry. We demonstrate control of local Lyapunov exponents for the control of symmetry in nonlinear dynamical systems. We also study the effect of noise in the system. It is shown that the small-amplitude noise can restore the symmetry in the attractor after the bifurcation and that the average time for trajectories to switch between the symmetry-broken components of the attractor scales algebraically with the noise amplitude. We demonstrate the relation between Lyapunov exponents, order parameters (Haken, 1983, 1988) and symmetry using a simple physical system and discuss the applicability of our approach to the study of state transitions in the epileptic brain.

Keywords: symmetry; optimization; control, Lyapunov exponents, brain, stimulation, epileptic seizures,

1. Introduction

Recent investigations of the epileptic human brain have shown that an effective correction of brain functions needs new control, prediction and optimization methods (Refs. 1-4). These methods are connected with reconstruction, optimization and control problems. The solution of the first problem is mainly based on the representation of electroencephalogram (EEG) time series in a state space using delay embedding methods (Refs. 5-6). The obtained quantitative information can be computed by estimating parameters which are invariants of the embedding process. The particular set of invariants we shall concentrate on in this paper is the spectrum of Lyapunov exponents (Refs. 7-10). The second problem can be reduced to the solution of a detection problem using Lyapunov exponents (Refs. 1, 3, 11-12). The third problem connects with controllability of Lyapunov exponents. The analysis of controllability of the Lyapunov exponents is an important new problem in control theory and different applications (Ref. 13). There are only few attempts of attacking this problem (Refs. 3, 14-18). A relatively new approach is to consider a control system of Lyapunov exponents as a dynamical system, where the set of control functions is part of the state space of this dynamical system (Ref. 18). For solving the above three problems, it is necessary to have effective algorithms for the control of symmetry by Lyapunov exponents.

The Lyapunov exponents of control systems are interesting because they encapsulate in an intuitive form the dynamical information contained in the EEG data. In addition, the Lyapunov spectrum can be related to other quantities derived from the experimental data. For example, the T-index is based on the largest Lyapunov exponent (Refs. 1, 11). A number of algorithms have been -proposed for estimating Lyapunov exponents from a scalar time series. Some, for example the method developed by Wolf et al. (Ref. 7) or the more recent work of Rosenstein et al. (Refs. 19-20), find the largest exponent and use this to classify a system according to whether or not it is chaotic. Algorithms which are designed to calculate the full spectrum of Lyapunov exponents have also been suggested. Most of these are derived from the Jacobian method proposed by Eckmann and Ruelle (Ref. 8) which was further developed by Eckmann et al. (Ref. 8) and Sano and Sawada (Ref. 9). These algorithms are more general than the basic 'Largest exponent' methods, since they are required to extract more information from the experimental data; and as a consequence are more inclined to difficulties in implementation.

We reformulate the problem of calculation of Lyapunov exponents as an optimization problem. Then, we present an algorithm for its solution. The algorithm is globally and quadratically convergent. This algorithm is based on earlier suggestions by the authors (Ref. 22). Here we use well-established techniques from numerical methods for dealing with the optimization problem which inevitably arise when estimating Lyapunov exponents from time series.

The paper is organized as follows. In Section 2, we propose a numerical algorithm for calculation of the Lyapunov exponents by solving the corresponding optimization problem. In Section 3, we demonstrate a number of applications to demonstrate the controllability of Lyapunov exponents using the proposed algorithm. The advantages of the algorithm are demonstrated by application to a range of data sets. In Section 4, we shown that control can lead to symmetry breaking in chaotic systems with a simple type of symmetry.

2. Calculation of the Lyapunov exponents for control systems

Let us consider control dynamical systems (Ref. 18) described by the differentiable dynamical model

$$\dot{q} = f(q, u), \quad q(t_0) = q_0, \tag{1}$$

where q is a vector in the phase space $\mathbb{R}^n$, $f(q,u)$ is a smooth vector field on a manifold M, and u is a control function. We suppose that u is a feed-back control function $u = u(q)$.

The vector field f yields a flow $\Phi = \{\Phi^t\}$ on the phase space, where Φ^t is a map

$$q \mapsto \Phi^t(q, u), \quad t \in \mathbb{R}, \quad q \in \mathbb{R}^n. \tag{2}$$

The observed trajectory, starting at q_0, is

$$\{\Phi^t(q_0, u) \,|\, t \in \mathbb{R}^+\}. \tag{3}$$

To get an information about the time evolution of arbitrarily small perturbed initial conditions, we consider the time evolution of tangent vectors in the tangent space TM. It is given by the linearization of the equation (1).

The Taylor expansion of $f(\Phi^t(q_0, u))$ for small Δq is

$$f(\Phi^t(q_0, u)) + Df(\Phi^t(q_0, u))\Delta q + \cdots . \tag{4}$$

Here $Df(\Phi^t(q_0, u))$ is the local Jacobian matrix of the vector field f at $\Phi^t(q_0, u_0)$

$$J(q_0, u_0) = Df(\Phi^t(q_0, u_0)) = \left[\frac{\partial f_i}{\partial q_j}\Big|_{\Phi^t(q_0,u_0)} \right]. \tag{5}$$

For $\Delta q \to 0$ the following first-order approximation holds [23] :

$$\dot{\delta q} = J(q_0, u_0))\delta q. \tag{6}$$

A solution of the linear variational equation (6) has the form

$$\delta q(t) = D\Phi^t(q_0, u_0)\delta q_0, \tag{7}$$

and represents the time dependence of the vector in tangent space TM. Let $A^t(q_0)$ be the $(n \times n)$ matrix of the linearized flow $D\Phi^t(q_0, u_0)$ and δq_0 an initial perturbation. We consider matrix A^t as a linear map from the tangent space TM at (q_0) to the tangent space at $\Phi^t(q_0)$.

The spectrum of Lyapunov exponents is the set of logarithms of the eigenvalues of the self-adjoint matrix

$$\Lambda_{q_0} := \lim_{t\to\infty} [(A^t(q_0))^* A^t(q_0)]^{1/2t}, \tag{8}$$

where $(A^t(q_0))^*$ is the transpose of $A^t(q_0)$. The existence of the limit in equation (8) is proved by Oseledec's theorem (Ref. 25).

Let $E := (e^1, \ldots, e^n)$ be an $(n \times n)$ matrix, where the column vectors are a basis of the tangent space. If the limit exists

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln ||A^t(q_0)e^i|| \tag{9}$$

then the λ_i's are called *Lyapunov exponents*. They are ordered by their magnitudes $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$.

If the limit is independent of q_0, the system is called *ergodic* (Ref. 25).

For calculation of the Lyapunov exponents using (9) with an arbitrary set of basis vectors it is necessary to use a renormalization procedure after some time Δt. One can write $A^t(q_0)$ as the product of $(n \times n)$ matrices $A^{\Delta t}(q_j)$, each of which represents the linearization of the flow $\Phi^{\Delta t}$, and maps $q_j \equiv \Phi^{j\Delta t}(q_0)$ to q_{j+1}:

$$A^{k\Delta t}(q_0) = A^{\Delta t}(q_{k-1}) \circ \cdots \circ A^{\Delta t}(q_j) \circ \cdots \\ \cdots \circ A^{\Delta t}(q_1) \circ A^{\Delta t}(q_0), \tag{10}$$

with $k\Delta t = t$. After every time step of the evolution time Δt any renormalization method can be applied.

Optimization Algorithm. For the calculation of Lyapunov exponents we need to reconstruct an attractor in phase space from a single time series of an observable using the method of time shifted samples. Let an observable be a function p, that maps any point $\Phi^t(x_0)$ in the state space to a (measurable) real value $p(\Phi^t(x_0))$. It has been shown for compact manifolds of dimension m, that the set

$$\{p(\Phi^t(x_0)), p(\Phi^{t+\tau_d}(x_0)), \ldots, p(\Phi^{t+2m\tau_d}(x_0))| \\ \tau_d \in \mathbb{R}^+ \backslash \{0\} \to \infty\} \tag{11}$$

is diffeomorphic to the positive limit set of $\Phi^t(x_0)$ under generic conditions.

The matrix of the linearized flow $A^{\Delta t} = D\Phi^{\Delta t}(x_j)$ can be approximated from a single trajectory by using the recurrent structure of strange attractors. This is done by averaging over the time evolution of difference vectors between x_j and points of the same trajectory on the attractor, that are within a small distance r.

The set of N difference vectors in a ball centered at $q_j = \Phi^{j\Delta t}(q_0)$, with $j\Delta t \equiv t$ is

$$L(r) = \{|\Phi^t(q_0) - \Phi^{t+t_i}(q_0)| \times \\ \times \|\Phi^t(q_0) - \Phi^{t+t_i}(q_0)\| \leq r, \\ t_i \geq -t, i = 1, \ldots, N\} \tag{12}$$

and will be denoted by $\{y^i | i = 1, \ldots, N\}$. After the evolution time Δt it is mapped to the set $\{\Phi^{t+\Delta t}(q_0) - \Phi^{t+t_i+\Delta t}(q_0)\} \equiv \{z^i | i = 1, \ldots, N\}$.

We will consider the approximate matrix $\hat{A}$ as a vector x. Now the $(n \times n)$ elements of the matrix $\hat{A}$ are determined, such that

$$\min_x F = \|g(x)\|^2 = \frac{1}{N} \sum_{i=1}^{N} |\|g_i(x)\|^2 \tag{13}$$

$$\text{Subject to} \quad x \in \mathbb{R}^{n^2}, \tag{14}$$

where $g_i : \mathbb{R}^{n^2} \mapsto \mathbb{R}$, $g_i = ||z^i - \hat{A}y^i||^2$. Then, the matrix A can be estimated by the Gauss-Newton method

$$x^{\gamma+1} = x^\gamma - \alpha^\gamma(\nabla g(x^\gamma)\nabla g(x^\gamma)g(x^\gamma)), \tag{15}$$

where ∇g is gradient matrix of g. For solution of the optimization problem we can use different numerical algorithms (Refs. 24–27).

If the ball of radius r and the evolution time Δt are short enough to represent a mapping in the tangent space (7), A should be a good approximation of the matrix of the linearized flow $D\Phi^{\Delta t}(x_j)$. Note that the evolution times Δt in the renormalization and the approximation process do not necessarily have to be the same, but are chosen equal for convenience. This approximation method seems to be the most flexible one in analyzing a data set, because several parameters [i. e., the evolution time Δt (see below)] can be controlled separately.

Each invertible $(n \times n)$ matrix can be split uniquely into a product of an upper triangular matrix R with positive diagonal elements and an orthogonal matrix Q, such that

$$\hat{A}(q_j)E_j = Q_j R_j = E_{j+1} R_j, \tag{16}$$

with $E_j := (e_j^1, \ldots, e_j^n)$. The matrix Q_j serves as the new basis E_{j+1} and the logarithms of the diagonal elements of R_j are (local) expanding coefficients, whose time-averaged values are the Lyapunov exponents. Using

$$\hat{A}^{k\Delta t}(q_0)E_0 = \prod_{j=0}^{k-1} \hat{A}^{\Delta t}(q_j)E_0 = Q_{k-1} \prod_{j=0}^{k-1} R_j \tag{17}$$

in (8) we obtain

$$\lambda_i = \lim_{k \to \infty} \frac{1}{k\Delta t} \sum_{j=0}^{k-1} \ln r_{ii}^j, \tag{18}$$

where r_{ii}^j are the diagonal elements of the matrix R_j.

3. Application of the Numerical Algorithm to the Control of Nonlinear Systems

Using the optimization algorithm, data sets of different control systems have been analyzed.

Discrete Control Systems. We have applied the method to control systems. The first system is the map:

$$x_{k+1} = a{x_k}^2 + cx_k + bu_k. \tag{19}$$

In the first simulation, we choose $p = 0.3$ and let the control gain sequence u_k be picked from the interval (Ref. 15)

$$[-g'(x_k - e^{kp}(T_{k-1})^{-1} - d, \\ - (g)'(x_k - e^{kp}(T_{k-1})^{-1}d]. \tag{20}$$

The control gain sequence u_k^i is picked from this interval to satisfy condition (20) that makes the Lyapunov exponent the larger then $p = p'$. Mathematical simulations show the possibility to control of the positive Lyapunov exponents.

The second control system is the map:

$$x_{k+1} = ax_k^2 + cx_k + bu_k, \qquad u_{k+1} = u_0 x_k, \quad (21)$$

where u_0 is a fixed parameter, u_k is control function.

Control of Lyapunov Exponents in Lattice System. Consider a nonlinear lattice system (Refs. 31–32) of the form

$$x_{k+1}^i = r(x_k^{i-1} - x_k^i) - r(x_k^i - x_k^{i+1}), \qquad (22)$$

where $k = 1, 2, \ldots, N$ are the discrete time steps, $i = 1, 2, \ldots, L$ is the discrete lattice sites with periodic boundary conditions, r is some nonlinear function, and $x \in \mathbb{R}^2$.

Let us rewrite the system (22) in the following form

$$x_{k+1}^i = g_k^i(x_k^{i-1}, x_k^i, x_k^{i+1}), \qquad (23)$$

where g_k is assumed to be continuously differentiable.

The goal towards control of the dynamical system (23) is to design an input sequence $\{u_k^i\}$ such that the output (state vectors) of the controlled system

$$x_{k+1}^i = g_k^i(x^{i-1}, x_k^i, x^{i+1}) + u_k^i, \qquad (24)$$

where x_0^i is given, behaves chaotically, in the sense that all the Lyapunov exponents of this controlled system achieve some value while the controlled system orbits remain to be bounded.

We assume that the linear state-feedback controls have the standard structure $u_{k+1}^i = \gamma_k^i(x_k^i)$, where $\{\gamma_k^i\}$ are $1 \times n$ nonlinear functions to be determined, without tuning any of the system parameters. Using this u_k^i, the controlled system (24) becomes

$$\begin{aligned} x_{k+1}^i &= g_k^i(x^{i-1}, x_k^i, x^{i+1}) + u_k^i, \\ u_{k+1}^i &= \gamma_k^i(x_k^i), \quad x_0^i \text{ given.} \end{aligned} \qquad (25)$$

Let

$$J_j^i(z) := (\tilde{g}_j^i)' \qquad (26)$$

be the Jacobian of $\tilde{g}_j^i(\cdot) = (g_j^i, \gamma_j^i)$ evaluated at z, $j = 0, 1, 2, \ldots$, and let

$$\begin{aligned} T_j^i = T^i{}_j(x_0^i, \ldots, x^i{}_j) :=& \\ J^i{}_j(x_j^i) J_{j-1}^i(x_{j-1}^i) \cdots & J_1^i(x_1^i) J_0^i(x_0^i). \end{aligned} \qquad (27)$$

Let $\xi_l^j = \xi(T^i{}_j^T T_j^i)$ be the l^{th} eigenvalue of the $j^t h$ product matrix $[T^i{}_j^T T_j^i]$, where $l = 1, \ldots, n$ and $j = 0, 1, 2, \ldots$

The $l^t h$ Lyapunov exponent of the orbit $\{x_k^i\}_{k=0}^{\infty}$ of the controlled system (24), starting from the given x_0^i, is defined by (Ref. 3, 14–18) [25]

$$\begin{aligned} \lambda_l^i(x_0^i) &= \lim_{k\to\infty} \frac{1}{2k} \ln |\xi_l^i T^i{}_k^T T_k^i| \\ &= \lim_{k\to\infty} \frac{1}{2k} \ln |\mu_i(J^i{}_0^T(x_0^i) \cdots J^i{}_k^T(x_k^i) J^i{}_k(x^i{}_k))|, \\ & \qquad k = 1, \ldots, n. \end{aligned} \qquad (28)$$

In the controlled system (24) we can design the control $\{\tilde{\gamma}_k^i(x_k^i)\}_{k=0}^{\infty}$ such that all the Lyapunov exponents of the orbit $\{x_k^i\}_{k=0}^{\infty}$ are positive in a suitable region

$$0 < p^i \le \lambda_m^i(x^i) < \infty, \quad m = 1, \ldots, n, \qquad (29)$$

where p^i is some predesigned constant.

At the initial step, $k = 0$, we determine the control-gain $\tilde{\gamma}_k^i(x)$ such that $[T_0^i T_0^{i\,T}]^{-1} > 0$. Then, for each $k = 1, 2, \ldots$, we determine the control $\tilde{\gamma}_k^i$ such that

$$(i) \quad [T_{k-1}^i T_{k-1}^i{}^T]^{-1} > 0, \qquad (30)$$

and

$$(ii) \quad [J_k^i{}^T J_k] - e^{2kp^i}[T_{k-1}^i T_{k-1}^i{}^T]^{-1} \ge 0. \qquad (31)$$

where the constant $p^i > 0$ is the one given in (29).

The designed controller can be obtained by the algorithm as follows.

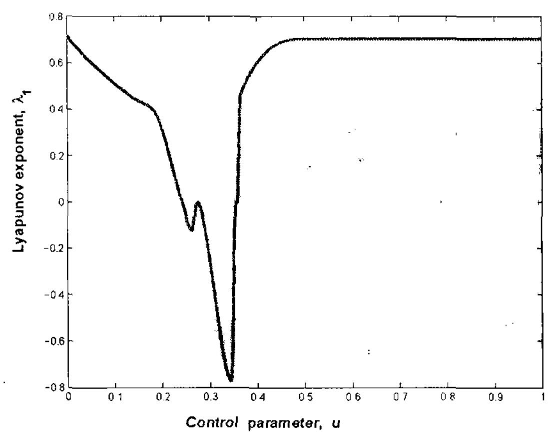

Fig. 1. First Lyapunov exponent λ_1 for the lattice system.

Start with the feedback controlled system (24), where x_0^i is initially given. Let $T_0^i = J_0^i(x_0^i)$. Design a feedback control such that the matrix $[T_0^i T_0^{i\,T}]$ is finite and diagonally dominant.

For $k = 0, 1, 2, \ldots$, $B_k^i = \sigma_k^i I_n > 0$, start with the controlled system (24):

Step 1. Compute the Jacobian $J_k^i(x_k^i)$, and then let $T_k^i = J_k^i T_{k-1}^i$;

Step 2. Design a positive feedback controller by choosing the positive number σ_k^i such that the matrix $[T_{k-1}^i T_{k-1}^i{}^T]^{-1} - e^{2kp^i} I_n$ is finite and diagonally dominant, where the constant $p^i > 0$ is the one given in (27).

The control sequence $\{B_k^i\} = \{\sigma_k^i I_n\}$ is chosen such that at step k, σ_k^i satisfies

$$J_k^T J_k^i - e^{2kp^i}[T_{k-1}^i T_{k-1}^i{}^T]^{-1} =$$
$$[(f_k^i)'(x_k i)]^T[(f_k^i)'(x_k^i)] + \sigma_k^i([(f_k^i)'(x_k^i)]^T +$$
$$[(f_k^{i\prime}(x_k^i)] + (\sigma_k^i)^2 I_n - e^{2kp^i}[T_{k-1}T_{k+1}^T]^{-1}. \quad (32)$$

This can be achieved if we let the matrix in (33) to be diagonally dominant by appropriately choosing a real number σ_k^i. Figure 1 shows the first Lyapunov exponent of the lattice system.

4. Control of Symmetry by Lyapunov Exponents in Dynamical System

When a dynamical system with control functions possesses certain symmetry, there can be an invariant subspace with a chaotic attractor in the phase space. As the Lyapunov exponent changes through a critical value, the chaotic attractor can lose stability with respect to perturbations transverse to the invariant subspace. Furthermore,a symmetry-increasing bifurcation can occur, after which the attractor acquires the system symmetry. It may therefore be possible to use Lyapunov exponents for control of symmetry in nonlinear dynamical systems. In this case the loss of the transverse stability can lead to a symmetry-breaking bifurcation characterized by lack of the system symmetry in the asymptotic attractor. An accompanying physical phenomenon is an extreme type of temporally intermittent bursting behavior. The mechanism for this type of symmetry-breaking bifurcation is elucidated.

We simulated the following control system

$$\begin{aligned} x_{n+1} &= r x_n (1 - x_n) \\ y_{n+1} &= \tfrac{1}{2\pi} u x_n \sin[2\pi(y_n - b)] + b, \end{aligned} \quad (33)$$

where r and b are parameters; u is control function. The values chosen were $r = 3.8$; $b = 0.05$ and $u \in [1, 4]$. Figure 2 shows the largest Lyapunov exponent λ_y and the $|y_{max}| - 0.5$ for the system (33). Figure 3 shows the transverse Lyapunov exponent $\lambda_\perp$ for the system (33).

Preliminary simulations suggests that it is possible to change the local Lyapunov exponent with noise as the control input. Changing Lyapunov exponents may restore the symmetry in the attractor after the bifurcation and the average time for trajectories to switch between the symmetry-broken components of the attractor depends on the range of Lyapunov exponents. The realization of control using our approach is based on the relation between Lyapunov exponents, order parameters (Haken, 1983, 1988) and symmetry. This idea may be extended to the case of complex biological systems such as the epileptic brain which demonstrates intermittent state transitions as it moves into and out of seizure states. We hypothesize that such state transitions involve corresponding changes in the symmetry of the system and hence our approach may be applied to control of such dynamical systems.

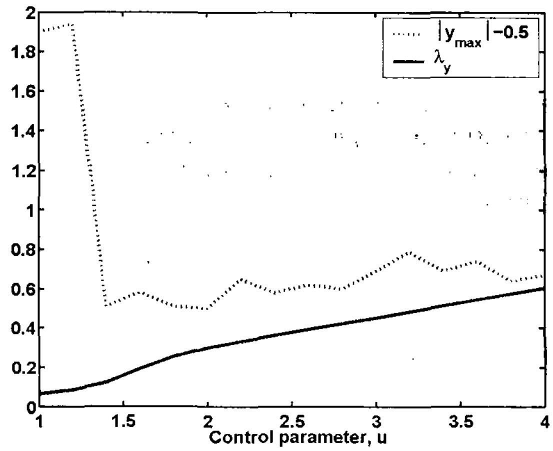

Fig. 2. The largest Lyapunov exponent λ_y and the $|y_{max}| - 0.5$ for the system (33).

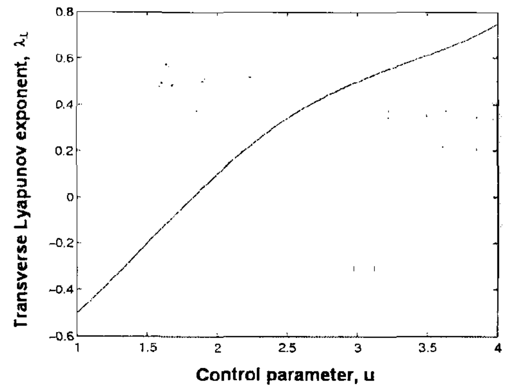

Fig. 3. The transverse Lyapunov exponent $\lambda_\perp$ for the system (33).

It is possible to control a scenario of symmetry-breaking bifurcation in chaotic dynamical systems by transverse Lyapunov exponent. We assume that symmetry breaking bifurcation occurs if the transverse Lyapunov exponent crosses zero from the negative side. When such a symmetry-breaking bifurcation occurs, the largest Lyapunov exponent exhibits an on-off intermittency. As the control parameter varies further, symmetry-increasing bifurcation occurs when trajectories start switching intermittently among the coexisting symmetric chaotic components.

5. Discussions and Conclusions

The Lyapunov exponents are conceptually the most basic indicators of deterministic chaos of dynamical systems with control. For the analysis of such dynamics, several numerical algorithms for estimating the spectrum of Lyapunov exponents have been proposed. In this paper we have focused on control of symmetry and Lyapunov exponents using optimization techniques.

By using the new method we have obtained good estimates of the Lyapunov spectrum from the observed time series in a very systematic way. We also investigate the possibility of modifying the symmetry of a dynamical system by changing its Lyapunov exponents. It is hoped that the new method will have wide applicability to systems whose dynamic equations are not available.

Preliminary results suggest that it may be possible to use our algorithm for controlling the Lyapunov exponents and symmetry in a nonlinear dynamical system. This research has been motivated by the practical necessity to change Lyapunov exponents in biological systems, where the maintenance of chaos provides the key to the avoidance of undesirable paralogical behaviour. We have therefore suggested algorithms which can maintain a desired level of chaoticity by achieving a prescribed value of the largest Lyapunov exponent.

Acknowledgements This work was supported by CRDF Grant No. UM2-2547-KV-03.

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КЕРУВАННЯ СИМЕТРІЄЮ ЗА ДОПОМОГОЮ ПОКАЗНИКІВ ЛЯПУНОВА

Дослідження останніх років у галузі систем керування показують, що зовнішні збурення можуть призводити до порушення симетрії в системах з хаотичною динамікою з певним типом симетрії. В роботі проаналізовано можливість керування показниками Ліяпунова за допомогою неперервного зовнішнього впливу. Показано, що хаотичний атрактор може стати нестабільним по відношенню до трансверсальних до інваріантного підпростору збурень. При цьому можуть виникати біфуркації, після яких утворюється нова симетрія атрактора. Ми також: показуємо існування співвідношення між: показниками Ляпунова, параметрами порядку (Хакен, 1983, 1988) та симетрією на прикладі простої фізичної системи. Обговорюється можливість використання нашого підходу до вивчення перехідних режимів в епілептичному головному мозку.