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CLASSIFICATION OF THE OBSERVABLE AND REACHABLE DISCRETE LINEAR DYNAMIC SYSTEMS WITH ONE-DIMENSION INPUT AND OUTPUT SPACES

Canonicalforms of the systems, given in the head, are described in this article.

Accordance [1], [2] a linear dynamic system is the a triple (F,G,H) of operators: $F : Q \to Q$, $G : A \to Q$, $H : Q \to B$, where A is an input vector space, B is an output vector space, Q is a state vector space. Everywhere bellow we will identify operators with their matrices in some fixed basis. If $dimA = dimB = 1$ and the matrix, which is a collection of the columns $(G|FG|\dots|F^{n-1}G)$ (the matrix $(H|HG|\dots|HF^{n-1})^t$) is a regular one then the system is called a reachable one (an observable one); here $n = dimQ$ and t means an operator of the matrix transposition. Let us construct a matrix M of the system (F,G,H) in such a manner

$$M = \left(\begin{array}{c|c} F & G \\ \hline H & 0 \end{array}\right).$$

There are invariant definitions of equivalent systems, but all of them are equivalent the next one, which will be used.

Two systems (F_1,G_1,H_1) and (F_2,G_2,H_2) are equivalent ones if there exists a regular matrix T such that $TF_1 = F_2T$, $TG_1 = G_2$, $H_1 = H_2T$.

The canonical forms of the equivalent classes for systems which are only reachable or only observable has been know (see [2], [3]).

Canonical forms of equivalent classes for the systems which have only one of these properties (reachable or observable) is long ago well known. But it is impossible to use these results for obtaining canonical forms of the systems which are reachable and observable ones, because an additional restriction (regularity condition) on matrix elements of the known canonical forms leads to the essentially more complicated forms.

The aim of this paper is to present canonical forms of reachable and observable systems, where all restrictions on matrix elements a_{ij} have the form: $a_{ij} \neq 0$, $a_{ij} = 1$, $a_{ij} = 0$ for some values i,j.

Definition 1. *Let* $M = \left(\begin{array}{c|c} F & G \\ \hline H & 0 \end{array}\right)$ *be a dynamic system with one - dimension input and output spaces. For a given elements* $p \neq 0, \alpha_1, \dots, \alpha_m$ *of a field let us define the extension of the system* M *in such a way:*

$\overline{M} = M(p;\alpha_1,\dots,\alpha_m) = \left(\begin{array}{c|c} \overline{F} & \overline{G} \\ \hline H & 0 \end{array}\right)$, *where*

$$\bar{F} = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{m-1} & \alpha_m & H \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & G & F \end{pmatrix},$$

$$\bar{G} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \bar{H} = \begin{pmatrix} 0 & 0 & \dots & 0 & p & 0 \end{pmatrix}.$$

Lemma 1. *The system* M *is reachable if and only if the system* $\bar{M}$ *is reachable.*

The direct checking gives the following equalities:

$$\bar{G} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \bar{F}\bar{G} = \begin{pmatrix} * \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \dots,$$

$$\bar{F}^{m-1}\bar{G} = \begin{pmatrix} * \\ * \\ * \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \end{pmatrix}, \quad \bar{F}^{m}\bar{G} = \begin{pmatrix} * \\ * \\ * \\ \cdot \\ \cdot \\ \cdot \\ \beta_0 \\ G \end{pmatrix},$$

$$\bar{F}^{m+1}\bar{G} = \begin{pmatrix} * \\ * \\ * \\ \cdot \\ \cdot \\ \cdot \\ \beta_1 \\ \beta_0 G + FG \end{pmatrix},$$

$$\bar{F}^{m+2}\bar{G} = \begin{pmatrix} * \\ * \\ * \\ \cdot \\ \cdot \\ \cdot \\ \beta_2 \\ \beta_1 G + \beta_0 FG + F^2G \end{pmatrix}, \ldots,$$

$$\bar{F}^{m+n-1}\bar{G} = \begin{pmatrix} * \\ * \\ * \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_{n-2}F^0G + \ldots + \beta_0 F^{n-2}G + F^{n-1}G \end{pmatrix}$$

Here by symbols "*" we marked numbers, whose value can be found, but they do not affect our reasoning. Other elements can ber expressed by the numbers $\beta_0, \beta_1, \ldots, \beta_{n-1}$, $(n = dim Q)$, though values of these numbers are also not essential.

Reachability of the system $\overline{M}$ means, that the matrix columns given above, are linearly independent ones. This condition, in view of a kind of the first m matrices of this list, is equivalent to a linear independence of the columns G, $\beta_0 G + FG$, $\beta_1 G + \beta_0 FG + F^2G, \ldots, \beta_{n-2}G + \beta_{n-3}FG + \ldots + \beta_0 F^{n-2}G + F^{n-1}G$, that is equivalent to a linear independence of the columns G, FG, $F^2G, \ldots, F^{n-1}G$. This gives the reachability of the system M.

Lemma 2. *The system M is observable if and only if the system $\bar{M}$ is observable.*

The direct checking gives the following equalities:

$$\bar{H} = p(\ 0\ \ 0\ \ \ldots\ \ 0\ \ 1\ \ 0\ )$$
$$\bar{H}\bar{F} = p(\ 0\ \ 0\ \ \ldots\ \ 1\ \ 0\ \ 0\ )$$
$$\cdots\cdots\cdots$$
$$\bar{H}\bar{F}^{m-1} = p(\ 1\ \ 0\ \ \ldots\ \ 0\ \ 0\ \ 0\ )$$
$$\bar{H}\bar{F}^{m} = p(\ \beta_0\ \ *\ \ \ldots\ \ *\ \ *\ \ H\ )$$
$$\bar{H}\bar{F}^{m+1} = p(\ \beta_1\ \ *\ \ \ldots\ \ *\ \ *\ \ \beta_0 H + HF\ )$$
$$\bar{H}\bar{F}^{m+2} =$$
$$= p(\ \beta_2\ \ *\ \ \ldots\ \ *\ \ *\ \ \beta_1 H + \beta_0 HF + HF^2\ )$$
$$\cdots\cdots\cdots$$
$$\bar{H}\bar{F}^{m+n-1} = p(\ \beta_{n-1}\ \ *\ \ \ldots\ \ *\ \ *\ \ \beta_{n-2}H +$$
$$+\beta_{n-3}HF\ \ +\ldots+\ \ \beta_0 HF^{n-2}\ \ +HF^{n-1}\ )$$

Observability of the system $\bar{M}$ means, that the matrix rows given above, are linearly independent ones. This condition, in view of a kind of the first m matrices of this list is equivalent to the linear independence condition of the rows H, $\beta_0 H + HF$, $\beta_1 H + \beta_0 HF + HF^2, \ldots, \beta_{n-2}H + \beta_{n-3}HF + \ldots + \beta_0 HF^{n-2} + HF^{n-1}$. It is equivalent to a linear independence of the rows H, HF, $HF_2, \ldots, HF_{n-1}$. This gives the observability of the system M.

Theorem 1. *An arbitrary linear dynamic system with one – dimensional input and output spaces, which is reachable and observable, is equivalent to a system which is consecutive extensions one of the systems as starting points:*

$$\mu_1 = \left(\begin{array}{c|c} * & 1 \\ \hline p & 0 \end{array}\right), \quad \mu_2 = \left(\begin{array}{cc|c} * & * & 1 \\ 1 & 0 & 0 \\ \hline 0 & p & 0 \end{array}\right), \ldots,$$

$$\mu_s = \left(\begin{array}{ccc|c} * & * & \ldots\ * & * & 1 \\ 1 & 0 & \ldots\ 0 & 0 & 0 \\ 0 & 1 & \ldots\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ 0 & 0 & \ldots\ 1 & 0 & 0 \\ \hline 0 & 0 & \ldots\ 0 & p & 0 \end{array}\right)$$

where $p \neq 0$, and by symbols "" some elements of a field are marked.*

Lemma 3. *Linear dynamic systems*

$$\left(\begin{array}{cccccccccc|c}
* & * & \dots & * & * & * & \dots & * & * & * & 1 \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & p & * & \dots & * & * & * & 0
\end{array}\right)$$

and

$$\left(\begin{array}{ccccccccccc|c}
* & * & \dots & * & * & * & * & \dots & * & * & 1 \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & * & * & \dots & * & * & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & * & \dots & * & * & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\
. & . & \dots & . & . & . & . & \dots & . & . & . \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & 0 & p & 0 & \dots & 0 & 0 & 0
\end{array}\right)$$

are equivalent.

(The undefined elements "*" belong to the rows, which numbers are 1, $n+1$ for the first matrix and 1, m, $m+1$ for the second one; the element $p \neq 0$ belongs to the m–th column, $m \geq 1$).

Really, let us change the first matrix by elementary transformations of the form – adding the m–th column multiple to the columns with the numbers $m+1, m+2, \dots, n$. In this way one can last row, gets the which coincide with the last row of the second matrix. Then under inverse transformations the $m+1, m+2, \dots, n$–th rows will be added to the m–th row. Since other rows, except first row, remain without changes we get second matrix.

Lemma 4. *Linear dynamic systems*

$$\left(\begin{array}{ccccccccccc|c}
* & * & \dots & * & * & * & * & \dots & * & * & 1 \\
1 & * & \dots & * & * & * & * & \dots & * & * & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & * & * & \dots & * & * & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\
. & . & \dots & . & . & . & . & \dots & . & . & . \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & p & 0 & 0 & \dots & 0 & 0 & 0
\end{array}\right)$$

and

$$\left(\begin{array}{ccccccccccc|c}
* & * & \dots & * & * & * & \dots & * & * & * & 1 \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & * & \dots & * & * & * & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & p & 0 & \dots & 0 & 0 & 0 & 0
\end{array}\right)$$

are equivalent.

(The uncertain elements of matrices of these systems are marked by the symbol "*" ; they stand in the rows number 1, 2 and $m+1$ of the first matrix, and also in rows number 1 and $m+1$ of the second one; the element $p \neq 0$ stands in the column number m, $2 \leq m \leq n$).

Really, if we shall change the first matrix by elementary transformations, adding a column number 1, multiplied by the appropriate numbers, to columns number $2, 3, \dots, n$, reducing its second row to a kind given in the second matrix, at completion of inverse transformations, rows number $2, 3, \dots, n$ of the first matrix will be added to its first row. As other rows, except the first and second one, will not change at such transformations, we shall receive the second matrix.

Proof of theorem 1. Taking into consideration the reachability of given system, we can conclude, (look [1], [2]), that its matrix is the first matrix given in the lemma 3. Thus, if $m = n$, the statement of the theorem is trivial, as then the matrix of the system will coincide with the one of matrices μ_s, given in the theorem. Using the lemma 3, we receive the second matrix of this lemma, which is equivalent to the first one.

Let's consider the matrix M, which is the special case of the second matrix from the lemma 3 (if $r = m$) and of the first matrix from the lemma 4 (if $m = 1$).

$$M = \left(\begin{array}{cccccccccccccc|c}
* & * & \dots & * & * & * & * & \dots & * & * & * & * & \dots & * & * & 1 \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & * & * & \dots & * & * & * & * & \dots & * & * & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 & * & * & \dots & * & * & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & \dots & . & . & . \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & p & 0 & 0 & \dots & 0 & 0 & 0
\end{array}\right)$$

– matrix, in which uncertain elements, signed by symbol "* stand in rows number 1, $r+1$ and $m+1$, where $1 \leq r \leq m$, and element $p \neq 0$ – stands in column number m.

By induction for r we shall prove, that this matrix M is equivalent to the second matrix from lemma 4.

Base of an induction gives lemma 4, as for it $r = 1$.

Let's change the matrix M by elementary transformations: adding the column with number r, multiplied by an appropriate number, to the columns number $r+1, r+2, \dots, n$. By these transformations we can get $r+1$–th row equal zero. Then under inverse transformations the rows with the numbers $r+1, r+2, \dots, n$, will be added to the r–th row. Since other rows (excepting 1–st and r–th ones) remain without changes, we get a matrix of the similar form, where parameter r has decreased on the 1.

According to the assumption of an induction such matrix is equivalent to the second matrix given in the lemma 4. It was necessary to prove.

To finish the proof of the theorem 1, it is enough to note, that the second matrix given in lemma 4, is the extension of a matrix of a smaller dimension, which according to lemmas 1 and 2 are both reachable and observable ones. Applying the given above reasons to it, we shall after all receive a matrix, for which $m = n$, that is one of the matrices μ_s, given in the theorem 1. It complete the proof.

Theorem 2. *Let systems M_1 and M_2 be a minimal linear dynamic systems with one-dimensional inputs and outputs spaces, and*

$$\bar{M}_1 = M_1(p; \alpha_1, \dots, \alpha_m) = \left(\begin{array}{c|c} \bar{F}_1 & \bar{G}_1 \\ \hline H_1 & 0 \end{array}\right),$$

$$\bar{M}_2 = M_2(q; \beta_1, \dots, \beta_k) = \left(\begin{array}{c|c} \bar{F}_2 & \bar{G}_2 \\ \hline H_2 & 0 \end{array}\right)$$

are their extension. Systems $\bar{M}_1$ and $\bar{M}_2$ are equivalent if and only if systems M_1 and M_2 are equivalent ones and $k = m$, $p = q$, $\alpha_i = \beta_i$ for all $i = 1, 2 \dots, m$.

Let a map φ gets the equivalence of M_1 and M_2, that is $\varphi F_1 \varphi^{-1} = F_2$, $\varphi G_1 = G_2$, $H_1 = H_2 \varphi$. Let's take $T = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & \varphi \end{array}\right)$, then according to the definition of the extension of linear dynamical systems we shall receive equalities $T\overline{F_1} = \overline{F_2}T$, $T\overline{G_1} = \overline{G_2}$, $\overline{H_1} = \overline{H_2}T$ so systems $\bar{M}_1$ and $\bar{M}_2$ are equivalent, as it was necessary to prove.

Lemma 5. *If systems $\bar{M}_1 = M_1(p; \alpha_1, \dots, \alpha_m)$ and $\bar{M}_2 = M_2(q; \beta_1, \dots, \beta_k)$ are equivalent, then $k = m$, $dim F_1 = dim F_2$.*

It is enough to prove that for any system, whose matrix looks like given in lemma 4, the place of an element p (in the column number m) is unique determined by this system. Really, the basis $e_1, e_2, \dots, e_n$ of state space Q gets out so that the operator F transforms subspaces, which are constructed by the base vectors, according to the circuit

$$G(A) = \langle e_1 \rangle \to \langle e_1, e_2 \rangle \to \langle e_1, e_2, e_3 \rangle \to$$
$$to \dots \to \langle e_1, e_2, \dots, e_n \rangle = Q.$$

Then the number m will be the least natural number, for which operator H acts nontrivial, that is $\dim \bar{F}_1 = \dim \bar{F}_2 H \langle e_1, e_2, \dots, e_n \rangle \neq 0$. The equality $dim F_1 = dim F_2$ follows from consequences $k = m$ and , and it was necessary to prove.

Let's *return to the proving of the theorem 2.*

Let's assume, that the systems $\bar{M}_1$ and $\bar{M}_2$ are equivalent.

Let's sign (taking into account lemma 5):

$$\varphi_1 = \begin{pmatrix} \alpha_1 & \dots & \alpha_{m-1} & \alpha_m \\ 1 & \dots & 0 & 0 \\ . & . & . & . \\ 0 & \dots & 1 & 0 \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} \beta_1 & \dots & \beta_{m-1} & \beta_m \\ 1 & \dots & 0 & 0 \\ . & . & . & . \\ 0 & \dots & 1 & 0 \end{pmatrix},$$

$$\tilde{H}_i = \begin{pmatrix} h_1^{(i)} & \dots & h_{n-1}^{(i)} & h_n^{(i)} \\ 0 & \dots & 0 & 0 \\ . & . & . & . \\ 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} H_i \\ 0 \\ \dots \\ 0 \end{pmatrix},$$

$$\tilde{G}_i = \begin{pmatrix} 0 & \dots & 0 & g_1^{(i)} \\ . & \dots & . & . \\ 0 & . & 0 & g_{n-1}^{(i)} \\ 0 & \dots & 0 & g_n^{(i)} \end{pmatrix} = \\ = \begin{pmatrix} 0 & \dots & 0 & G_i \end{pmatrix}, \quad (i = 1, 2)$$

then according to the definition of extension we'll have

$$\bar{F}_i = \left(\begin{array}{c|c} \varphi_i & \tilde{H}_i \\ \hline \tilde{G}_i & F_i \end{array}\right).$$

Let's sign also through $\mu(X)$ and $\nu(X)$ matrices, which accordingly are formed from a matrix X by withdrawing of the first row, or the last column.

The existence of a matrix $T = \left(\begin{array}{c|c} R & Q \\ \hline P & S \end{array}\right)$ follows from the equivalence of systems $\bar{M}_1$ and $\bar{M}_2$, for which $T\bar{F}_1 = \bar{F}_2 T$, $T\tilde{G}_1 = \tilde{G}_2$, $\tilde{H}_1 = \tilde{H}_2 T$.

The next lemma follows directly from the last two equalities.

Lemma 6. *The matrix P has a zero first column and the matrix Q has a zero last row. The first column and last row of a matrix R looks like*

$$\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^t, \quad \begin{pmatrix} 0 & \dots & 0 & * \end{pmatrix}.$$

Lemma 7.

$$P = 0, \quad Q = 0.$$

Let us deduce from the matrix equality $T\bar{F}_1 = \bar{F}_2 T$, the next ones $R\tilde{H}_1 + QF_1 = \varphi_2 Q + \tilde{H}_2 S$, $P\varphi_1 + S\tilde{G}_1 = \tilde{G}_2 R + F_2 P$. As follows from the lemma 6, we have $\mu(R\tilde{H}_1) = 0$, $\mu(\tilde{H}_2 S) = 0$, $\nu(S\tilde{G}_1) = 0$, $\nu(\tilde{G}_2 R) = 0$, and we shall receive equalities $\mu(QF_1) = \mu(\varphi_2 Q)$ and $\nu(P\varphi_1) = \nu(F_2\varphi_1)$.

From the first of them for rows $q_1, q_2, \dots, q_{n-1}, q_n$ of a matrix Q we get

$$\begin{pmatrix} q_2 F_1 \\ \hline q_3 F_1 \\ \hline \dots \\ \hline q_{n-1} F_1 \\ \hline q_n F_1 \end{pmatrix} = \begin{pmatrix} q_1 \\ \hline q_2 \\ \hline \dots \\ \hline q_{n-2} \\ \hline q_{n-1} \end{pmatrix},$$

whence we shall receive $q_{n-1} = q_{n-2} = \dots = q_2 = q_1 = 0$ (as $q_n = 0$ according to lemma 6), so, we can see, that $Q = 0$.

From the second equality for columns $p_1, p_2, \dots, p_{n-1}, p_n$ of a matrix P we shall receive

$$\begin{pmatrix} p_2 & p_3 & \dots & p_{n-1} & p_n \end{pmatrix} = \\ \begin{pmatrix} F_2 p_1 & F_2 p_2 & \dots & F_2 p_{n-2} & F_2 p_{n-1} \end{pmatrix}.$$

From here, taking into account, that $p_1 = 0$ according to lemma 6, we shall receive $p_2 = p_3 = \dots = p_{n-1} = p_n = 0$, hence $P = 0$.

Lemma 8. *The matrix R is the unit.*

Taking into account the previous lemma, from a condition $T\bar{F}_1 = \bar{F}_2 T$ of equivalence of systems we shall receive $R\varphi_1 = \varphi_2 R$, so, by rejecting the first row, we get the equality

$$\begin{pmatrix} r_{22} & \dots & r_{2,m-1} & r_{2,m} & 0 \\ r_{32} & \dots & r_{3,m-1} & r_{3,m} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ r_{m-1,2} & \dots & r_{m-1,m-1} & r_{m-1,m} & 0 \\ r_{m,2} & \dots & r_{m,m-1} & r_{m,m} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} r_{11} & r_{12} & \dots & r_{1,m-1} & r_{1,m} \\ r_{21} & r_{22} & \dots & r_{2,m-1} & r_{2,m} \\ . & . & . & . & . \\ r_{m-2,1} & r_{m-2,2} & \dots & r_{m-2,m-1} & r_{m-2,m} \\ r_{m-1,1} & r_{m-1,2} & \dots & r_{m-1,m-1} & r_{m-1,m} \end{pmatrix}.$$

Taking into account the lemma 6 (concerning to the matrix R), we see, that R is the unit.

To finish the proof of the theorem 2, it is necessary to use a condition of equivalence of systems, and of as lemmas 7 and 8 once more.

The theorems 1 and 2 enable to find all canonical varieties of reachable and observable linear dynamic systems with one-dimensional inputs and outputs spaces for a fixed dimension of the space Q and to find out their quantity.

Corollary 1. *If $dim Q = n$, then there exist 2^{n-1} canonical varieties of reachable and observable systems with one- dimension input and output spaces.*

Really, let a_n is a quantity of canonical varieties in dimension n. To get all canonical varieties in dimension $n + 1$ we must take advantage of the theorem 1, to find all extensions of varieties in smaller dimension and adding to them a variety μ_{n+1}. All

the systems formed in such a way are not equivalent according to the theorem 2, so they form canonical varieties. We have equality

$$a_{n+1} = a_n + a_{n-1} + \ldots + a_2 + a_1 + 1.$$

As $a_1 = 1$ (only the variety μ_1), it's easy to prove by induction that $a_n = 2^{n-1}$.

Example 1. *To write out all canonical varieties for* $n = 3$*, it is necessary at first to write out them in dimension* 2*. Let's construct extensions of a degree* 1 *of system* μ_1 *to dimension* 2*, that is system* $\bar{\mu}_1 = \left(\begin{array}{cc|c} * & p_1 & 1 \\ 1 & * & 0 \\ \hline p_2 & 0 & 0 \end{array}\right)$*, which together with system* μ_2 *will give a complete set of nonequivalent systems in dimension* 2*. Building extensions of degree* 2 *for* μ_1 *and extensions of degree* 1 *for* μ_2 *and for* $\bar{\mu}_1$*, together with* μ_3 *we shall receive a complete set of nonequivalent canonical forms in dimension* 3*:*

$$\left(\begin{array}{ccc|c} * & * & p_1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & * & 0 \\ \hline 0 & p_2 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|c} * & 0 & p_1 & 1 \\ 1 & * & * & 0 \\ 0 & 1 & 0 & 0 \\ \hline p_2 & 0 & 0 & 0 \end{array}\right),$$

$$\left(\begin{array}{ccc|c} * & p_2 & 0 & 1 \\ 1 & * & p_1 & 0 \\ 0 & 1 & * & 0 \\ \hline p_3 & 0 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|c} * & * & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & p_1 & 0 \end{array}\right).$$

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КЛАСИФІКАЦІЯ СПОСТЕРЕЖНИХ І ДОСЯЖНИХ ДИСКРЕТНИХ ЛІНІЙНИХ ДИНАМІЧНИХ СИСТЕМ З ОДНОВИМІРНИМИ ПРОСТОРАМИ ВХОДІВ І ВИХОДІВ

Описуються канонічні форми вказаних у заголовку систем.