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On synchronization of randomly coupled oscillators

We study a network of heterogeneous randomly coupled phase oscillators described by Kuramoto model. By analyzing the continuum limit, we study the mean-field type synchronization. Using this approach, we discuss the influence of network topology on the existence and properties of synchronized state if natural frequencies are normally distributed. We show that for scale-free networks the intermediate state between coherent and desynchronized state is prevalent.

Keywords: Kuramoto model, order parameter, synchronization.

Introduction

Complex networks play an important role in many natural and technological systems. One of the most fascinating phenomena in the behavior of complex dynamical systems made up of many elements is the spontaneous emergence of order and the phenomenon of collective synchronization, where a large number of the system's constituents forms a common dynamical pattern, despite the differences in their individual dynamics. A classical model for the phase dynamics of weakly coupled oscillators is Kuramoto model, which assumes global connections. Kuramoto model formulation was motivated by the behavior of systems of chemical and biological oscillators, and it has found widespread applications such as in neuroscience. But the topology of real world networks is often very complex. Many networks have scale-free topology; the distribution of the degree obeys the power law [3], $P(k) \sim k^{-\gamma}$. Additionally, they are characterized by the existence of key nodes which drastically reduce the average distance between nodes, the so-called small-world property. In this paper we consider the case of undirected coupling networks with random coupling, and discuss the synchronization of phase oscillators using the approach, proposed in [4].

Background

We study the network with N-nodes, at each node there exists an oscillator with the phase of the oscillator θ_i. Each of the oscillators is considered to have its own intrinsic natural frequency ω_i and each is coupled equally to some other oscillators. Interactions are assumed to depend sinusoidally on the phase difference. This situation can be modeled by the equation,

$$\frac{\partial \theta_i}{\partial t} = \omega + K \sum_j a_{ij} \sin(\theta_j - \theta_i), \qquad (1)$$

where K is the coupling constant; a_{ij} is 1 if the nodes i and j are connected, and 0 otherwise; ω_i — natural frequency of oscillator, is a random number, whose distribution is given by the function $g(\cdot)$. For simplicity, we assume $g(\omega) = g(-\omega)$. (Section 3 is devoted to networks with unit normal distribution of natural frequencies.)

Let's define $P(k)$ as the distribution of nodes with degree k, and $\rho(k, \omega; t, \theta)$ the density of oscillators with phase θ at time t, for given ω and k. We assume that $\rho(k, \omega; t, \theta)$ is normalized as

$$\int_0^{2\pi} \rho(k, \omega; t, \theta) d\theta = 1. \qquad (2)$$

Under this assumption the collective oscillation corresponds to the stable solution, $\frac{d\rho}{dt} = 0$.

We use the order parameter and the continuum limit equation for the network of oscillators proposed in [4]:

Definition 1. *Order parameter r is determined by an equality*

$$re^{i\psi} = \frac{\int \int \int g(\omega) P(k) k \rho(k, \omega; t, \theta) e^{i\theta} d\omega d\theta dk}{\int dk P(k) k}.$$

By definition $0 \leq r \leq 1$, $r = 0$ corresponds to desynchronized state when there is no coherence among the oscillators, $r = 1$ agrees with complete synchronization.

Proposition 1. [4] *r satisfies the condition*

$$r \int kP(k) dk =$$

$$= Kr \int k^2 P(k) \int_{-1}^{1} g(Kkr\omega)\sqrt{1 - \omega^2} d\omega dk.$$

If $r \neq 0$, then after some transformations we have

$$Ek = K \int k^2 P(k) \int_{-1}^{1} g(Kkr\omega)\sqrt{1 - \omega^2} d\omega dk \qquad (3)$$

The l.h.s. of this equation is independent of r and let's define the r.h.s. of this equation as $f(r)$. Ichinomiya showed [4], that

$$f(r) \le \frac{Ek}{r} \quad \text{for } 0 < r \le 1, \qquad (4)$$

so $f(1)$ is not larger than Ek, therefore the sufficient condition that eq. (3) have solution at $0 < r \le 1$ is that $f(0) \ge Ek$, that is

$$\frac{Kg(0)\pi Ek^2}{2Ek} > 1. \qquad (5)$$

We use equation 3 to evaluate value of order parameter.

Proposition 2. *The following inequality holds for* $0 \le r \le 1$

$$\frac{E|\omega|}{KEk} \ge (1-r)r. \qquad (6)$$

Proof. Let's use $\sqrt{1-\omega^2} \ge 1 - |\omega|$ in (3):

$$Ek = K \int k^2 P(k) \int_{-1}^{1} g(Kkr\omega)\sqrt{1-\omega^2}d\omega dk \ge$$

$$\ge K \int k^2 P(k) \int_{-1}^{1} g(Kkr\omega)(1-|\omega|)d\omega dk \ge$$

$$\ge K \int k^2 P(k) \int_{-\infty}^{\infty} g(Kkr\omega)(1-|\omega|)d\omega dk =$$

$$= K \int k^2 P(k)\frac{1}{Kkr}dk - K \int k^2 P(k)\frac{E|\omega|}{(Kkr)^2}dk =$$

$$= \frac{Ek}{r} - \frac{E|\omega|}{Kr^2}.$$

Q.E.D.

If l.h.s. of (6) tends to 0, then possible values of r are close to $r = 0$ or $r = 1$.

Scale-free networks with normally distributed natural frequencies of oscillators

Suppose that natural frequencies have unit normal distribution:

$$g(\omega) = \frac{1}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}}.$$

For network with scale-free topology the distribution of the degree obeys the power law: $P(k) \sim k^{-\gamma}$, therefore $Ek = \zeta(\gamma - 1)$, $(\zeta(\cdot) - $ Riemann zeta function).

In random scale-free network with unit natural distribution of natural frequencies of oscillators $E|\omega| = \sqrt{\frac{2}{\pi}}$, $Ek = \zeta(\gamma - 1)$, so we have:

$$\frac{\sqrt{2}}{\sqrt{\pi}K\zeta(\gamma - 1)} \ge (1-r)r. \qquad (7)$$

If $2 < \gamma \le 3$ we have: l.h.s. of (7) is finite number; it tends to 0 is γ tends to 2; if $\gamma = 3$ l.h.s. of (7) is equal to $\frac{6\sqrt{2}}{\sqrt{\pi^5}K}$, so (7) does not matter for small K.

$$f(0) = \int k^2 P(k)dk \int_{-1}^{1} \sqrt{1-\omega^2}g(0)d\omega =$$

$$= \frac{1}{\sqrt{2\pi}}\int k^2 P(k)dk \int_{-1}^{1} \sqrt{1-\omega^2}d\omega = \frac{\sqrt{\pi}}{2\sqrt{2}}Ek^2$$

In random scale-free network Ek^2 diverges if $2 < \gamma \le 3$, so $r \ne 0$.

For network with normally distributed natural frequencies

$$\int k^2 P(k) \int_{-1}^{1} g(Kkr\omega)\sqrt{1-\omega^2}d\omega dk <$$

$$< \int k^2 P(k) \int_{-1}^{1} g(Kkr\omega)d\omega dk =$$

$$= \int dk k^2 P(k)\frac{1}{Kkr}\int_{-\infty}^{\infty} d\omega'' g(\omega'') = \frac{\int dk k P(k)}{Kr}.$$

Therefore $f(1) < Ek$ in this case.

$$f'(r) = -\int k^2 P(k) \int_{-1}^{1} \frac{\sqrt{1-\omega^2}}{\sqrt{2\pi}} \times$$

$$\times e^{-\frac{(Kkr\omega)^2}{2}}\left(K^2 k^2 r\omega^2\right)d\omega dk < 0,$$

for all $0 < r \le 1$, so $f(0) > f(r_1) > f(r_2) > f(1)$ for all $0 < r_1 < r_2 < 1$. Therefore there is only one value $r = r_0$, $0 \le r \le 1$ that satisfies $f(r_0) = Ek$.

Using that $f(r)$ is a non-increasing function, equation (3) and inequality (6), it is easy to prove the following statement:

Proposition 3. *If* $f(\frac{1}{2}) > Ek$, *then* $r > \frac{1+\sqrt{1-4\frac{E|\omega|}{KEk}}}{2}$.

If $f(\frac{1}{2}) < Ek$, *then* $r < \frac{1-\sqrt{1-4\frac{E|\omega|}{KEk}}}{2}$.

If $f(\frac{1}{2}) = Ek$, *then* $r = \frac{1}{2}$ *and* $\frac{E|\omega|}{KEk} \ge \frac{1}{4}$.

Moreover, we have shown that in random scale-free network with unit natural distribution of natural frequencies of oscillators $r_0 \ne 0$, $r_0 \ne 1$. So we have discovered a remarkable state where the population of oscillators splits into two subpopulations, where one is synchronized and the other is desynchronized. This state is analogue to *chimera* state [1], when all oscillators are identical.

Numerical simulation

The analysis above is in good agreement with the results of the numerical simulations for $N = 1000$. To check our analysis we carried out the simulations on Erdos-Renyi random network model and Barabasi–Albert scale-free network model. We considered two types of probability density for natural frequencies: normally distributed and uniformly distributed on $[-1, 1]$.

We considered scale-free network with different values of $2 < \gamma < 3$. With increasing γ, the average value of order parameter increased respectively. For any $2 < \gamma < 3$ we observed order parameter $r > 0$. We can remark that there is no threshold in the random scale-free network. These results suggest that in the infinite size scale-free network the critical coupling constant K_c tends to 0, the same as in mean-field continuum limit equation. In this paper we showed how our theory described the behavior of the order parameter r for a particular realization of the network and the frequencies. We compare the approximations described in this section with the numerical solution for different types of networks.

Conclusions

In this paper, we study the frequency synchronization of the random oscillator network. By analyzing the continuum limit equation, we find that mean-field type synchronization occurs in random network model. When K is less than K_c oscillators are uniformly distributed across all possible phases, and the population is in a statistical steady-state. When coupling K is sufficiently large, a synchronized solution is possible. In the completely synchronized state, all the oscillators share a common frequency, although their phases can be different (difference tends to 0). We evaluate order parameter and study synchronization in scale-free networks with normally distributed natural frequencies of oscillators. The results of numerical simulations and recent results [2] are in good agreement with this analysis.

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Крюкова Г. В.

Про синхронізацію випадково зв'язаних осциляторів

Досліджено систему неоднорідних випадково зв'язаних фазових осциляторів, а саме: узагальнену модель Курамото. Використовуючи підхід середнього поля, вивчено умови синхронізації, вплив структури мережі на існування та властивості синхронізованих станів, якщо власні частоти осциляторів є випадковими величинами, що відповідають нормальному розподілу. Доведено, що в топології scale-free проміжний стан між повною синхронізацією та десинхронізованим станом є домінуючим.

Ключові слова: модель Курамото, параметр впорядкованості, синхронізація.