Sparse matrices in computer algebra when using distributed memory: theory and applications

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J. Dongarra at his talk at International Congress ICMS-2016 [1] put attansion on the several difficult challenges. The task of managing calculations on a cluster with distributed memory for algorithms with sparse matrices is today one of the most difficult challenges.

Here we must also add problems with the type of the basic algebra: matrices can be over fields or over commutative rings. For sparse matrices, it is not true that all computations over polynomials or integers can be reduced to computations in finite fields. Such reduction may be not effective for sparse matrices.

We consider the class of block-recursive matrix algorithms. The most famous of them are standard and Strassen's block matrix multiplication, Schur and Strassen's block-matrix inversion [2].

Class of block-recursive matrix algorithms

Block-recursive algorithms were not so important as long as the calculations were performed on computers with shared memory. The generalization of Strassen's matrix inversion algorithm [2] with additional permutations of rows and columns by J. Bunch and J. Hopkroft [3] is not a block-recursive algorithm. Only in the nineties it became clear that block-recursive matrix algorithms are required to operate with sparse super large matrices on a supercomputer with distributed memory.

The block recursive algorithm for the solution of systems of linear equations and for adjoint matrix computation which is some generalisation of Schur inversion in commutative domains was discraibed in [7], [8] and [10]. See also at the book [9]. However, in all these algorithms, except matrix multiplication, a very strong restriction are imposed on the matrix. The leading minors, which are on the main diagonal, should not be zero.

This restriction was removed later. The algorithm that computes the adjoint matrix, the echelon form, and the kernel of the matrix operator for the commutative domains was proposed in [11]. The block-recursive algorithm for the Bruhat decomposition and the LDU decomposition for the matrix over the field was obtained in [12], and these algorithms were generaized for the matrices over commutative domains in [14] and in [15].

Some important areas of sparse matrix applications

Calculation of electronic circuits

The behavior of electronic circuits can be described by Kirchhoff's laws. The three basic approaches in this theory are direct current, constant frequency current and a current that varies with time. All these cases require the compilation and solution of sparse systems of equations (numerical, polynomial or differential). The solution of such differential equations by the Laplace method also leads to the solution of polynomial systems of equations [16].

Control systems

In 1967 Howard H. Rosenbrock introduced a useful state-space representation and transfer function matrix form for control systems, which is known as the Rosenbrock System Matrix [17]. Since that time, the properties of the matrix of polynomials being intensively studied in the literature of linear control systems.

Groebner basis.

Another important application is the calculation of Gröbner bases. A matrix composed of Buchberger S-polynomials is a strongly sparse matrix. Reduction of the polynomial system is performed when calculating the echelon and diagonal forms of this matrix. The algorithm F4 [18] was the first such matrix algorithm.

Solving ODE's and PDEâs.

Solving ODE's and PDE's is often based on solution of leanear systems with sparse matrices over numbers or over polynomials. One of the important class of sparse matrix is called quasiseparable. Any submatrix of quasiseparable matrix entirely below or above the main diagonal has small rank. These quasiseparable matrices arise naturally in solving PDEâs for particle interaction with the Fast Multi-pole Method (FMM). The efficiency of application of the block-recursive algorithm of the Bruhat decomposition to the quasiseparable matrices is studied in [20].

Development of the matrix recursive agorithms in integral domain

Algorithms for solution of a system of linear equations of size n in an integral domain, which served as the basis for creating recursive algorithms

(1983) Forward and backward algorithm ($\sim n^3$) [4].
(1989) One pass algorithm ($\sim \frac{2}{3}n^3$) [5].
(1995) Combined algoritm with upper left block of size r ($\sim \frac{7}{12}n^3$ for $r = \frac{n}{2}$) [6].

Recursive algorithms for solution of a system of linear equations and for adjoint matrix computation in an integral domain without permutations

(1997) Recursive algorithm for solution of a system of linear equations [7].
(2000) Adjoint matrix computation (with 6 levels) [8].
(2006) Adjoint matrix computation alternative algorithm (with 5 levels) [10].

Main recursive algorithms for sparse matrices

(2008) Computation of adjoint and inverse matrices and the operator kernel [11].
(2010) Bruhat and LEU decompositions in the feilds [12].
(2012) Bruhat and LDU decompositions in the domains [13], [14].
(2015) Bruhat and LDU decompositions in the domains (alternative algorithm) [15].

New achivements

(2013) It is proved that the LEU algorithm in the feild has the complexity $O(n^2 r^{\beta-2})$ for matrices of rank r. [19].
(2017) It is proved that the LEU algorithm in the feild has the complexity $O(n^2 s^{\beta-2})$ for quasiseparable matrix, if any it's submatrix which entirely below or above the main diagonal has small rank s [20].

Sparse matrices when using distributed memory

The block-recursive matrix algorithms for sparse matrix require a special approachs to managing parallel programs. One approach to the cluster computations management is a scheme with one dispatcher (or one master).

We consider another scheme of cluster menagement. It is a scheme with multidispatching, when each involved computing module has its own dispatch thread and several processing threads [21], [22].

We demonstrate the results of experiments with parallel programms on the base of multidispatching.

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