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5.	Discovering properties of line digraphs of polytrees.	01.03.2022	
6.	Typography work.	01.05.2022	
7.	Preliminary analysis and correction of mistakes.	03.06.2022	

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1 Introduction

1.1 Work outline

Line graphs are among the most extensively studied topics in graph theory. Harary was one of the first to study it [1], but many followed. In one of the recent articles Bagga and Beineke offer a good introduction for those unfamiliar with the concept [2]. Line graphs have numerous applications in various fields, including other branches of graph theory - for example, Liu, Sun, and Meng use them to define and describe a certain family of dense digraphs [3], while Geller and Harary show that arrow diagrams are nothing else but line digraphs [4].

The line graph operation arises naturally by mapping the set of lines of one graph to the set of points of the other, preserving adjacency. Defining line graphs formally, though, turns out to be more complicated; for that, we view them as a special kind of intersection graphs (refer to **Definition 2.1** and **Definition 2.2** for better understanding). the goal of this work is to describe the main properties of the operation. We give the most widely used characterization of line graphs, along with the characterization of line graphs of trees. We go on to Whitney isomorphism theorem, which shows that the root graph of a line graph is unique (with 1 exception), characterization of graphs isomorphic to their line graphs, and several results about bipartite graphs. The first section ends with a discussion about connection between the concept of line graphs ans that of eulerian (and hamiltonian) graphs.

In the second section we talk about line digraphs. The concept is similar to the concept of line graphs, but is defined differently (more simply) and have different properties. We characterize line digraphs in general, then do the same for digraphs isomorphic to their line digraphs. The better part of the section, though, is dedicated to polytrees, which are orientations of undirected trees. We highlight polytrees whose line digraphs are weak, and then design an algorithm that breaks a polytree into subgraphs with weak line digraphs. We finish the work by characterizing line digraphs of different classes of polytrees.

1.2 Basic notations

Let's start with basic notations we shall use throughout the work. Although we shall mostly refer to graph vertices as *points* and to edges as *lines*, we will still use notations $V(G)$ for set of points and $E(G)$ for set of lines (instead of $P(G)$ and $L(G)$). The author believes these notations to be more familiar to the reader, and also wishes to avoid confusion with the notation for line graph.

Also denote $G[A]$ the induced subgraph on set of points $A \subset V(G)$. If u is a point of G, then $E_G(u)$ denotes the set of lines adjacent to u in G.

2 Line graphs

2.1 Basic properties

Naturally we shall begin our discussion of line graphs with their definition. For that, we will first need to introduce the concept of intersection graphs.

Definition 2.1. Let X be a set, and $F \subset 2^X$. Then $\Omega(X, F)$ is called an *intersection graph* if $V(\Omega) = F$, $E(\Omega) = \{AB | A, B \in F; A \cap B \neq \emptyset\}$.

With that in mind, let's define line graphs themselves.

Definition 2.2. Let G be a graph. Then the graph $L(G) = \Omega(V(G), E(G))$ is called its *line graph*. Conversely, G is called the *root graph* of $L(G)$.

Throughout the work we shall denote $V'(G)$ and $E'(G)$ the number of points and lines of G respectively.

Given this definition, one can immediately see that $|V'(G)| = |E(G)|$ (as $V'(G) = E(G)$ in the first place). The task of finding $|E'(G)|$ is less trivial; the next theorem gives answer to it.

Theorem 2.3. *Let G be a graph. Then $|E'(G)| = \frac{1}{2} \sum_{u \in V(G)} d^2(u) - |E(G)|$.*

The proof of this theorem relies on a similar formula for line digraphs, and thus will be given in the next section.

Theorem 2.4. *(Line graph characterization). Let H be a graph. Then $H \simeq L(G)$ for some graph $G \iff \exists F \subset 2^{V(H)}$:*

1) $\forall x \in V(H) : |F_x \stackrel{def}{=} \{A \in F : x \in A\}| = 2$;

2) $\forall A \in F : H[A]$ *is a complete graph;*

3) $\forall e \in E(H)\ \exists! A \in F : e \in E(H[A])$.

Proof. $\implies$ Let $H = L(G)$. Then $F = \{E_G(u) : u \in V(G)\}$ satisfies the 3 conditions stated above, thus proving the theorem.

$\impliedby$ Set $G = \Omega(V(H), F)$, where F is a set satisfying conditions 1-3, existent by assumption. To show $H \simeq L(G)$, we shall use the definition of isomorphism, i.e. we build a bijection $f : V(H) \to V'(G) = E(G)$ which should preserve the lines. Let $f(x) = F_x$ (obviously, F_x can be viewed as a function of x, which is a point of graph H. F_x outputs a set of 2 elements of F which both contain x - that is, a set of 2 connected points of G, or a line of G).

First, prove that F_x is bijective:

a) f is injective: let $x, y \in V(H), x \neq y$ and $F_x = F_y = \{A, B | A, B \in F; x, y \in A, B\}$. By condition 2 $H[A]$ is a complete graph $\implies xy \in E(H[A])$. Similarly $xy \in E(H[B])$. But that contradicts 3) with $e = xy$. So f is indeed injective.

b) f is surjective: let $A, B \in F, AB \in E(G)$. $G = \Omega(V(H), F) \implies \exists x \in V(H) : x \in A, B \implies \{A, B\} \subset F_x$. But 1) $\implies |F_x| = 2$. Thus $\{A, B\} = F_x$. As A and B are arbitrary elements of F, this proves surjectiveness of f.

All that is left is to show the preservation of lines, namely $xy \in E(H) \iff F_xF_y \in E'(G)$:

a) Necessity: $xy \in E(H)$. By 3) $\exists A \in F : xy \in H[A]$. The necessary implication is $x, y \in A$, whence we conclude $A \in F_x, F_y$. In other words, lines F_x and F_y share a point A, which means F_x and F_y are connected in $L(G)$. Thus the necessity is proven.

b) Sufficiency: $F_xF_y \in E'(G) \implies \exists A \in F : x, y \in A$. As $H[A]$ is complete, this implies $xy \in E(H)$.

$\square$

Line graph characterization may strike with its complexity. Thankfully, there are certain types of graphs whose line graphs are characterized a little more elegantly. Prominent among them are trees.

A useful concept here is that of a block graph.

Definition 2.5. An induced subgraph $B(G)$ of a graph G is called a *block* if it is a maximal biconnected subgraph.

Denote $\beta(G)$ the set of blocks of graph G.

Definition 2.6. Call the intersection graph $B(G) \overset{def}{=} \Omega(V(G), \beta(G))$ the *block graph* of G.

With that in mind, we can characterize the line graphs of trees.

Theorem 2.7. *(Tree line graph characterization). For a graph H there exists a tree T with $L(T) \simeq H \iff H$ is a connected block graph such that its every cut point belongs to exactly 2 blocks.*

Proof. $\implies$ We have $H = L(T)$. As T is connected, therefore (obviously) $L(T)$ is connected as well. Also, $L(T) = B(T)$, since tree lines are also blocks. Now lets consider block B in $B(T)$, which is complete by the characterization of block graphs. Every its point is a line of T. If $|V(B)| = 2$, then B represents 2 lines that share a point. Elsewise it consists of several pairwise connected points; lets choose 3 of them. The corresponding lines of T are pairwise adjacent; thus they either form a cycle of length 3, or all 3 lines share a single point. As T is a tree, the latter holds true. So every block in $B(T)$ corresponds to a point in T. If $B(T)$ contains a cut point that belongs to at least 3 blocks, it means there are at least 3 pairwise connected points in $T \implies T$ contains a cycle of length 3. So H must satisfy every condition from theorem statement.

$\impliedby$ Set $F = \{V(B) : B \in \beta(H)\} \cup \{\{x\} : x$ is NOT a cut point in $H\}$ and show that F satisfies the 3 conditions from **Theorem 2.4**. We shall refer to a set A as a set of *Type* 1 if $A = V(B)$ and as a set of *Type* 2 if $A = \{x\}$.

a) $\forall x \in V(H) : |F_x| = 2$. Indeed, if x is a cut point, then by the statement of the theorem there are 2 sets of *Type* 1 that contain x, while none of the sets of *Type* 2 contain x; otherwise x is in exactly 1 set of *Type* 1 and in exactly 1 set of *Type* 2.

b) $\forall A \in F : H[A]$ is complete: if A is a set of *Type* 2, then the statement is obvious; elsewise it follows from the characterization of block graphs.

c) $\forall e \in E(H)\, \exists! A \in F : e \in E(H[A])$. Taking into account that $\forall A$ of *Type* 2 $: E(H[A]) = \emptyset$, the previous statement is equivalent to 'every line lies in exactly one block', which follows from the maximality of blocks.

□

2.2 Main results

Let's begin the main section with discussion of graph isomorphisms. Firstly, it is obvious that if $G_1 \simeq G_2$, then $L(G_1) \simeq L(G_2)$. The next theorem, often referred to as Whitney isomorphism theorem, shows that the inverse statement is also true but for one case.

Theorem 2.8. *Let G and G^* be connected graphs such that $L(G) \simeq L(G^*)$. Then either $G \simeq G^*$, or they are K_3 and $K_{1,3}$.*

Proof. Suppose for simplicity that $L(G) = L(G^*) = H$. First notice that $|E(G)| = |E(G^*)|$. Thus there is a one-to-one correspondence between the lines of G and G^*. Note that if at least one of the graphs G and G^* has no more than 4 points, then both graphs have no more than 6 lines. It can be shown that among such graphs only one pair satisfies the condition of the theorem: K_3 and $K_{1,3}$.

Let $|V(G)| \geq 5$ and $|V(G^*)| \geq 5$. We shall show that there exists an isomorphism $\phi : V(G) \to V(G^*)$. First we prove the next: let $x_1 = uv_1$, $x_2 = uv_2$ and $x_3 = uv_3$ be lines of G that compose a $K_{1,3}$ subgraph, then the lines x_1^*, x_2^*, and x_3^* of G^* which correspond to the same points in H also compose a $K_{1,3}$ subgraph. Since G is connected and contains at least 1 other point, there is a line y adjacent to exactly 1 or exactly 3 of the given lines. The corresponding point y' of H will be adjacent to one or three of x_1', x_2' and x_3', therefore in G^* y^* is connected to one or three of x_1^*, x_2^*, and x_3^*. Yet if x_1^*, x_2^* and x_3^* form a triangle in G^* then y^* can only be adjacent to exactly 2 of these 3 lines. We conclude that the points form a claw (i.e., $K_{1,3}$) in G^*.

Remember that $E_G(u)$ is the set of lines at u in G; let also $E'_G(u)$ be the set of associated points in H. We shall prove that there exists a point u^* of G^* such that $E'_G(u) = E'_{G^*}(u^*)$. Consider the case $deg(u) \geq 2$. Then let y_1 and y_2 be a pair of adjacent lines. It is easily shown that the associated points y_1^* and y_2^* are adjacent in G^*; name the shared point u^*. Let x be another line at u in G. Thus in G these 3 lines form a claw $\implies$ y_1^*, y_2^*, and x^* form a claw in G^*. So $E'_G(u) \subset E'_{G^*}(u^*)$. Similarly it can be proven that $E'_{G^*}(u^*) \subset E'_G(u)$. Therefore $E'_G(u) = E'_{G^*}(u^*)$. Let $deg(u) = 1$ and v be its only adjacent point. Then $deg(v) \geq 2$. Thus $E'_G(v) = E'_{G^*}(v^*)$ for some point v^* in G^*. Let x be the line between u and v in G and x^* the line associated with the same point x' of H (v^* is one of 2 points adjacent to x^* in G^*; call the other one u^*). What is left is to show that $deg(u^*) = 1$. By construction $deg(u^*) \geq 1$. Let z be another line incident to u^*. Then $deg(x^*) = |E_{G^*}(v^*)|$ in G^*, whereas in G $deg(x) = |E_G(v)| - 1 = |E'_G(v)| - 1 = |E'_{G^*}(v^*)| - 1 = |E_{G^*}(v^*)| - 1$. But these must be equal, as the lines correspond to the same point x' in H. Again we have $E'_G(u) = E'_{G^*}(u^*)$.

Let $\phi : V(G) \to V(G^*)$ be a mapping such that $\phi(u) = u^*$ if $E'_G(u) = E'_{G^*}(u^*)$. The mapping is obviously one-to-one. The fact that it is onto follows from the next sequence of equalities: $\sum_{u\in V(G)} E_G(u) = 2|E(G)| = 2|E(G^*)| = \sum_{u^*\in V(G^*)} E_{G^*}(u^*)$. Had the mapping not been onto, the last expression would be strictly greater than the first. Now let $e = uv$ be a line of G, and let $\phi(uv) = \{u^*, v^*\}$. From $E'_G(u) = E'_{G^*}(u^*)$ and $E'_G(v) = E'_{G^*}(v^*)$ follows that u^* and v^* are both incident to a single line. So ϕ is a bijection that preserves lines $\implies$ ϕ is a graph isomorphism, which completes the proof. □

Theorem 2.9. *If G is a connected graph, then $G \simeq L(G) \iff G$ is a cycle.*

Proof. If G is a cycle, the result is obvious.

Let G be a connected graph such that $G \simeq L(G)$. Then from equality $|V'(G)| = |E(G)|$ we have $|V(G)| = |E(G)|$. Thus G is unicyclic (it contains a single cycle Z). Now assume that G is not a cycle. Then G contains a point a that belongs to Z, but is adjacent to line α which does not belong to Z. Of course a is adjacent to 2 other lines β and γ, both of which belong to Z. Theorem statement implies that conditions of line graph characterization apply to G. Let $F \subset 2^{V(G)}$ be the set for which these conditions hold true. Therefore $\exists A, B \subset F : a \in A, B$ and A, B are the only such sets. For each of α, β, γ there is exactly one set of F which contains it. Say $\alpha \in A$. Since $G[A]$ is complete, we can say the next: if $\beta \in A$, then α and β belong to the same complete subgraph of $G \implies$ there is a cycle that contains both lines. But G contains only one cycle Z, and α is not in it. Thus β (and likewise γ) doesn't belong to A, which only leaves the choice $\beta, \gamma \in B$. So β and γ belong to a complete subgraph of G. If the subgraph is built on at least 4 points, it contains more than one cycle; as G is unicyclic, this implies that the length of its only cycle is 3.

Note that every line that does not belong to Z is a block; otherwise the lines from a single block would belong to a common cycle. Therefore every block in G is either a cycle of length 3 (Z) or a line, that is every block is complete. This means G is a block graph. Also if any cut point belongs to at least 3 blocks, then there are at least 3 lines (each from different block) adjacent to it, all 3 contained in 2 sets out of F. As shown previously, this implies that 2 of the lines belong to a single cycle despite being in different blocks. Here we conclude that every cut point belongs to exactly 2 blocks. Then by tree line graph characterization we have: G is a line graph of a tree. But G is also the line graph of itself, and G is not a tree. By Whitney's theorem this implies that G is either K_3 or $K_{1,3}$. Neither of these satisfies our initial assumption (that G is a unicyclic graph, but G is not a cycle). Thus the assumption was false, so G is indeed a cycle. □

Now a brief note about bipartite graphs.

Proposition 2.10. *Let H be a connected graph such that $H \simeq L(G)$ for some graph G. Then H is bipartite $\iff$ H is either a path or a cycle of even length.*

Proof. $\implies$ Consider the partition of H into complete subgraphs. If H has a point of degree at least 3, then at least 2 of its incident lines belong to a common complete subgraph S. Therefore S has at least 3 lines, and so it includes a cycle of length 3. Thus H cannot be bipartite. Conclude that every point of H is of degree at most 2. This implies that H is either a path or a cycle. The only case which doesn't satisfy the condition of the theorem is a cycle of odd length, and that is exactly when H is not bipartite.

$\impliedby$ Both a path and a cycle of even length are bipartite. Also path of length n is the line graph of path of length $n+1$, while a cycle is the line graph of itself (see the previous theorem for reference). □

Corollary 2.11. *Let G be a connected graph. Then $L(G)$ is bipartite $\iff$ G is either a path or a cycle of even length.*

Proof. By **Proposition 2.10** $L(G)$ is bipartite $\iff$ it is a path or a cycle of even length. Thus G is also a path or a cycle of even length respectively. □

Let's see what can we say about $L(G)$ and $L(L(G))$ given that G is eulerian or hamiltonian.

Theorem 2.12. *Let G be a connected graph. Then $L(G)$ is eulerian if and only if the degrees of all points in G are of equal parity.*

Proof. $\impliedby$ Let u and v be 2 connected points of G. Then $deg(uv) = deg(u)+deg(v)-2$. Since $deg(u)$ and $deg(v)$ are either both even or both odd, the previous equation implies $2 \mid deg(uv)$. Every point of $L(G)$ is associated with a line of G, so degrees of all the points of $L(G)$ are even. Thus $L(G)$ is eulerian.

$\Longrightarrow$ Let u and v be such points of G that $2 \mid deg(u)$ but $2 \nmid deg(v)$. We know that no to adjacent points of G have degrees of different parity: otherwise the corresponding line of $L(G)$ would be of odd degree, and $L(G)$ itself not eulerian. Therefore u and v are disconnected. However, from connectedness of G we deduce that there exists a path $u - x_1 - x_2 - ... - x_n - v$ that connects u and v. As x_1 is connected to u, $2 \mid deg(x_1)$. From this we get $2 \mid deg(x_2)$, as x_2 is connected to x_1. By induction we can show $2 \mid deg(x_n)$, but that means we have 2 adjacent points x_n and v with degrees of different parity. This implies that our initial assumption was false, proving the statement of the theorem. □

Corollary 2.13. *Let G be a connected graph. Then $L(L(G))$ is eulerian if and only if the degrees of all the lines in G are of the same parity.*

Proof. The degrees of all the lines in G are of the same parity if and only if the same holds for the points of $L(G)$, which by **Theorem 2.12** is in turn equivalent to $L(L(G))$ being eulerian. □

Proposition 2.14. *G is hamiltonian $\Longrightarrow$ $L(G)$ is hamiltonian.*

Proof. Let x_1, x_2, ..., x_n be the points of G, and $(x_1, x_2, ..., x_n, x_1)$ the hamiltonian cycle. We shall construct a hamiltonian cycle Z in $L(G)$. Let e_1, e_2, ..., e_k be the lines at x_1, with $e_k = x_1x_2$. Begin Z with a trail $(e_1, e_2, ..., e_k)$. It can be done, since the lines are pairwise adjacent. Now let f_1, f_2, ..., f_l be lines at x_2 not equal to e_k, with $f_l = x_2x_3$. Continue the cycle with a similar trail, so that Z begins with $(e_1, e_2, ..., e_k, f_1, f_2, ..., f_l)$. Note that, as e_k is incident to x_2, it is adjacent to f_1. Continue in similar manner, by choosing the set of lines at x_i which do not yet belong to Z and adding them to the cycle. The last line to add at each point should be x_ix_{i+1} (or x_nx_1, if $i = n$). That last line x_nx_1 is adjacent to e_1, thus Z indeed is a cycle. Since every line is adjacent to some point x_i (actually, to 2 such points), Z must contain every line and thus every point of $L(G)$. So Z is a hamiltonian cycle $\Longrightarrow$ $L(G)$ is hamiltonian. □

3 Line digraphs

3.1 Definitions

We will stick to the notations used in the previous section, with the exception of notation for set of lines: instead of denoting it $E(D)$ for a digraph D, we shall use notation $A(D)$, as E is a common way to call a directed draph.

Mostly we shall use the same terminology as in [5]. Still we shall outline it explicitly to avoid possible (and probable) confusion.

Points a and b are called *adjacent* if there is a line α between them. If the line goes from a to b, we shall say that a is *adjacent to* b and b is *adjacent from* a. Both a and b are *adjacent with* each other. We will also say that a is *incident to* α, b is *incident from* α, and both a and b are *incident at* α. Likewise α is *incident from* a, *incident to* b, and *incident at* both. α may be called an *outline* from a or an *inline* to b. The number of inlines to a we shall call the *indegree* of a and denote $id(a)$, while the respective number of outlines we call *outdegree* of a and denote $od(a)$. The terminology of lines is similar to that of points: if α and β are subsequent lines, then α is *adjacent to* β, β is *adjacent from* α, and both α and β are *adjacent with* each other.

Definition 3.1. A *polypath* is a sequence of points $a_1, a_2, ..., a_n$ such that $\forall i \in [1, n-1]$ points a_i and a_{i+1} are connected, i.e., any orientation of a path. We shall denote undirected paths as $a_1 - a_2 - ... - a_n$. A *polycycle* is a polypath where exactly 2 points are equal: a_1 and a_n, or, in other words, any orientation of a cycle. Equivalently, unless it is a directed cycle, a polycycle can be viewed as a sequence of $2k$ directed trails $W_1, W_2, ..., W_{2k}$ for which the next 2 conditions hold:

- W_{2i} and $W_{(2i+1)\%2k}$ share the source point $\forall i \in [1, k]$;
- W_{2i-1} and W_{2i} share the sink $\forall i \in [1, k]$.

Definition 3.2. Let T be a directed graph. Then T is called a *directed tree* (or *polytree*) if it contains no polycycles. Equivalently, a polytree is any orientation of an undirected tree.

Note that unlike with graphs, not every acyclic digraph is a polytree.

Definition 3.3. Digraph D is called *bipartite* if $\exists A, B \subset V(D)$, $A \sqcup B = V(D)$, such that every line of D is incident from a point of A to a point of B.

A special case is the *two-port digraph*, which is derived from a complete bipartite graph. Formally, if A and B are 2 disjoint sets of points, then digraph $K(A, B)$ is called a two-port digraph if $V(D) = A \sqcup B$ and $A(D) = A \times B$ (here $\times$ denotes the Cartesian product of 2 sets).

Definition 3.4. A *polytrail* is any sequence of adjacent lines. Note that a polytrail may not be necessarily traversible (2 subsequent lines may share a starting point or an endpoint).

In other words, a polytrail is a sequence of directed trails where each 2 consecutive trails share a point.

Definition 3.5. Call a *directed star* (or *polystar*) an orientation of an undirected star (a graph in which all lines are incident to a single point).

3.2 Main results

Just as with ordinary graphs, the number of points in $L(D)$ is equal to the number of lines in D. For the number of lines in $L(D)$, refer to the following theorem:

Theorem 3.6. *If D is an arbitrary digraph, then $|E'(D)| = \sum_{u \in V(D)} id(u) \times od(u)$.*

Proof. For every point p of D view all the pairs of subsequent lines α and β such that α is incident to p, whereas β is incident from p. Clearly for every point p there are $id(p) \times od(p)$ such pair of lines. Since every line of $L(D)$ is associated with a pair of subsequent lines of D, the previous statement proves the theorem. $\square$

This result allows us to prove a theorem from the first section.

Proof of ***Theorem 2.3***. Derive a digraph D from G by orienting every line of G in both directions. Then $|A'(D)| = \sum_{u \in V(D)} id(u) \times od(u) = \sum_{u \in V(G)} d^2(u)$. But every point of $L(G)$ corrsponds to 2 points of $L(D)$ with 2 lines between them. If to subtract $2|V'(G)| = 2|E(G)|$ from $|A'(D)|$, we obtain a number exactly twice greater than $E'(G)$, since every other line of $L(G)$ corresponds to 2 lines of $L(D)$. Then $|E'(G)| = \frac{1}{2}(\sum_{u \in V(G)} d^2(u) - 2|E(G)|) = \frac{1}{2} \sum_{u \in V(G)} d^2(u) - |E(G)|$. $\square$

Theorem 3.7. *(Line Digraph Characterization). Let D be a digraph. Then $\exists D' : D \simeq L(D') \iff \exists\, 2$ improper partitions A_i and B_i of $V(D)$ such that $A(D) = \bigcup_i A(K(A_i, B_i))$ and $\forall i, j\ |A_i \cap B_j| \leq 1$.*

Proof. $\implies$ Let $p_1, p_2, ..., p_n$ be the set of points in D' with nonzero indegree and nonzero outdegree. Consider inlines to p_i; they are associated with some points of D; let A_i be the set of those points. For lines incident to points with outdegree 0, let A_{n+1} be the set of corresponding points, and A_{n+2} be an empty set. Then A_i is an improper partition of $V(D)$. Construct B_i

in a similar way from outlines from p_i, only let B_{n+1} be an empty set, and B_{n+1} the set of points of D corresponding to lines incident from points of zero indegree in D'. Then every induced subgraph $D[A_i \cup B_i]$ is a two-port subgraph $K(A_i, B_i)$, and by construction every line of D belongs to one of these subgraphs. Also, the set of inlines at p_i (which corresponds to A_i), and the set of outlines at p_j (which corresponds to B_j) may have at most 1 line in common - otherwise we would have 2 parallel lines (p_j, p_i). Thus $|A_i \cap B_j| \leq 1$, which completes the proof.

$\Longleftarrow$ Construct D' the next way: associate every $K(A_i, B_i)$ with a point p_i of D', and let every point of A_i correspond to an inline to p_i, and every point of B_i to an outline from p_i. Since $|A_i \cap B_j| \leq 1$, we will get no double lines and the construction is valid. We have obviously gotten a bijection between points of D and lines of D'. By construction every pair of subsequent lines in D' corresponds to a line of D: indeed, the point joining the lines has both nonzero indegree and nonzero outdegree, and thus is associated with $K(A_i, B_i)$ for some i. The existing of such a line in D then follows from the definition of a two-port digraph. The reverse is also true: every line of D belongs to some $K(A_i, B_i)$, and thus corresponds with 2 subsequent lines in D', both incident at p_i. Conclude that $D \simeq L(D')$. □

Remark 3.8. As can be seen from the proof, the last condition ($\forall i, j\, |A_i \cap B_j| \leq 1$) serves only one purpose: to make sure that the root digraph has no parallel lines. Thus, if to expand the concept of digraphs to multidigraphs (that is, digraphs with allowed parallel lines), we obtain a simpler characterization, stated in the following theorem.

Theorem 3.9. *Digraph D is a line digraph of a multidigraph $\iff$ $\exists\, 2$ improper partitions A_i and B_i of $V(D)$ such that $A(D) = \bigcup_i A(K(A_i, B_i))$.*

Thus we obtained a way to characterize line digraphs, but not the only one. Another way can be found in [4]. Here we will restate it together with its proof.

Theorem 3.10. *Digraph D is a line digraph of a multigraph if and only if $\forall p, q \in V(D)$ if there exists a point adjacent to both p and q, then $\forall r \in V(D)$ $rp \in A(D) \iff rq \in A(D)$.*

Proof. $\Longrightarrow$ Let F be the root digraph of D, and let p and q be 2 points of D. If there is a point adjacent to both p and q, then in F p and q share the starting point. Thus every line adjacent to p or q in F is also adjacent to the other; or equivalently, every point adjacent to p or q in D is also adjacent to the other.

$\Longleftarrow$ For every point p_i of D define sets A_i and B_i in one of the next ways:

- If $id(p_i) \times od(p_i) > 0$, then $A_i = \{q \in V(D) : qp \in A(D)\}$ and $B_i = \{r \in V(D) : \exists q \in A_i \; qr \in A(D)\}$. In other words, A_i is the set of all points adjacent to p_i, and B_i the set of all points r such that r is adjacent from some point of A_i. Note that $A_i \neq \emptyset$ and $B_i \neq \emptyset$.
- If $id(p_i) = 0$, then $A_i = \emptyset$ and $B_i = \{r \in V(D) : id(r) = 0\}$.
- If $od(p_i) = 0$, then $A_i = \{q \in V(D) : od(q) = 0\}$ and $B_i = \emptyset$.

Now lets prove that A_i and A_j are either equal or disjoint. Suppose $A_i \cap A_j \neq \emptyset$. Then $id(p_i) \neq 0$ and $id(p_j) \neq 0$. If $od(p_i) = 0$, then A_i is the set of all points with outdegree 0. Then if p_j has outdegree 0, then $A_i = A_j$; otherwise A_j is the set of points adjacent ot p_j, none of which has outdegree 0. So in such case $A_i \cap A_j = \emptyset$, a contradiction. We are left with the case $id(p_i) \times od(p_i) > 0$ and $id(p_j) \times od(p_j) > 0$. Let $q \in A_i \cap A_j$. Then $qp_i \in A(D)$ and $qp_j \in A(D)$, which by condition of the theorem implies $\forall r \in V(D) : rp_i \in A(D) \iff rp_j \in A(D)$. Thus $A_i = A_j$.

Lets now show that B_i and B_j are either equal or disjoint. Proof for the cases other than $id(p_i) \times od(p_i) > 0$ and $id(p_j) \times od(p_j) > 0$ are similar to that for A_i and A_j. Let $B_i \cap B_j \neq \emptyset$. Choose a point $p_k \in B_i \cap B_j$. Then $\exists q \in A_i : qp_k \in A(D)$ and $qp_i \in A(D) \implies A_i \cap A_k \ni q \implies A_i = A_k$. Similarly $A_j = A_k$, thus $A_i = A_j$. This means that $\forall r \in V(D) : \exists q \in A_i \; qr \in A(D) \iff \exists q \in A_j \; qr \in A(D)$, which implies $B_i = B_j$. Note that in the process we also showed that if $id(p_i) \times od(p_i) > 0$ and $id(p_j) \times od(p_j) > 0$, then $A_i = A_j \iff B_i = B_j$. It easily follows that the last statement holds true for all the points of D.

Define equivalence classes $[p_i]$ the following way: $p_j \in [p_i] \iff A_i = A_j$ and $B_i = B_j$. Consider A_i and B_i for all such classes. Then we have that every two A_i and A_j are disjoint; similarly with B_i and B_j. Also by construction of A_i and B_i $D[A_i \cup B_i] = K(A_i, B_i)$. Clearly $V(D) = \bigcup A_i = \bigcup B_i$ and $A(D) = \bigcup A(K(A_i, B_i))$. Therefore A_i and B_i are improper partitions which satisfy the conditions of **Theorem 3.9**, which implies D is a line digraph of a multigraph. $\square$

Proposition 3.11. *Graph G is bipartite if and only if there exists a digraph D, derived from G by orienting every its line, such that $L(D)$ is an edgeless digraph.*

Proof. $\Longrightarrow$ Let A and B be the parts of G. Construct D in such a way that every line is a line from A to B. Then D contains no dipath of length greater than 1, thus $L(D)$ is edgeless.

$\Longleftarrow$ Since $L(D)$ has no lines, every point of D is either a source or a sink. Let A be the set of sources of D, and B the set of sinks. Then every line's tail belongs to A, and every line's head to B. Therefore by neglecting the orientation of lines we get a bipartite graph G with parts equal to A and B. $\square$

Similarly to undirected graphs, with digraphs we also pose the question of characterizing digraphs D such that $L(D) \simeq D$. For ordinary graphs we were able to accomplish that in simple terms; here, as it often is with digraphs, everything is a little more complicated. We will need several results related to connectedness of iterated line digraphs. The results can be seen at [5]. Here we shall only provide proof to several of them, to give an understanding of the main idea.

We shall denote a graph $L(L(L(...(D)...)))$ (the operation being repeated n times) as $L^n(D)$.

Theorem 3.12. *Let D be an acyclic digraph. Then $\exists n \in \mathbb{N}$ such that $L^n(D)$ is a null-digraph.*

Proof. Let W' be the longest directed walk in $L(D)$, and let n be its length. There exists a walk W in D such that $L(W) = W'$. Every point of W' corresponds to a line of W, so W has n lines. Since D is acyclic, all points and all lines in W are distinct, and thus W has $n+1$ points. Then $|W| > |W'|$. We see that the longest walk in $L(D)$ is shorter than the longest walk in D for any acyclic digraph D. But in such case $L(D)$ is also acyclic; therefore the longest walk in $L(L(D))$ will be shorter still. Then the longest walk in $L^{n+1}(D)$ will be of length 0, which means $L^{n+1}(D)$ contains no lines. Then $L^{n+2}(D)$ is a null-digraph. $\square$

Theorem 3.13. *Let D be a digraph containing 2 cycles and a path between them. Then $|A(L^n(D))| \to \infty$ as $n \to \infty$ (and thus $|V(L^n(D))| \to \infty$ as $n \to \infty$).*

Proof. Let $D*$ be the subgraph containing directed cycles C_1 and C_2 and a path of length k between them. Then $L(C_1) = C_1$, $L(C_2) = C_2$, and the path between them will have length $k+1$. So $L(D*)$ will have exactly 1 line more than $D*$. Similarly $L^2(D*)$ will have 2 more lines than $D*$, and generally $L^n(D*)$ will have n more lines than $D*$. Thus $|A(L^n(D*))| \to \infty$ as $n \to \infty$, and so it is with D. $\square$

Theorem 3.14. *Let D be a digraph containing multiple cycles, with no 2 cycles having a dipath between them. Then $\exists n \in \mathbb{N}$ such that $L^n(D)$ is disconnected.*

Theorem 3.15. *Let D be a digraph containing a single cycle of length n. Then $L(D)$ also contains a single cycle of length n. Moreover, let S be the set of points of D which either belong to the cycle or are pathwise connected to it. Then $\exists n \in \mathbb{N}$ such that $|L^n(D)| = |S|$.*

The next concept will be used in our characterization.

Definition 3.16. A digraph D is called *functional* (*contrafunctional*) if every its point has outdegree (indegree) 1.

Theorem 3.17. *Let D be a weak digraph. Then $D \simeq L(D) \iff D$ is either functional or contrafunctional.*

Proof. $\Longleftarrow$ Let D be a functional digraph with $V = \{a_1, a_2, ..., a_n\}$ its set of points and $E = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ its set of lines ($|V(D)| = |A(D)|$ follows from its functionality). Then the map $\phi : V(D) \to A(D)$ such that $\phi(a_i) = \alpha_i \; \forall i \in [1, n]$ is an isomorphism between D and $L(D)$. Indeed, if point a_i is adjacent to point a_j in D, then line α_i is adjacent to line α_j in D (and so are the corresponding points of $L(D)$). The proof for contrafunctional digraphs is similar.

$\Longrightarrow$ We have $|V(D)| = |A(D)|$. Now assume D has no source. Then $id(u) \geq 1 \forall u \in V(D)$, so $|E'(D)| = \sum_{u \in V(D)} id(u) \times od(u) \geq \sum_{u \in V(D)} od(u) = |A(D)| = |E'(D)|$. Thus the $\geq$ sign turns to $=$, which can only happen if $id(u) = 1 \; \forall u \in V(D)$. This means that D is contrafunctional. Similarly if D has no sink, then it is functional. We are left with only 1 case to consider: D has a source a and a sink b. From the previous 4 theorems we have that D has a single cycle C, and every its point is pathwise connected to C.

Since there is a dipath P_a from a to C of length n_a and a dipath P_b from C to b of length n_b, a and b are connected. Denote by P the path between them. Let P_c be the path between the endpoint of P_a and the starting point of P_b, and n_c its length. If α is an outline at a, and β an inline at b, then in $L(D)$ α is a source and β a sink. Let similarly P'_a be the dipath from α to the cycle, and n'_a its length. Define P'_b, P'_c, n'_b, and n'_c in the natural way. Then $n'_a = n_a$, $n'_b = n_b$, $n'_c = n_c - 1$. Continuing this way, after n_c iterations we get a path between a source and a sink which passes through exactly 1 point and 0 lines of C. Keeping in mind $D \simeq L(D)$, we may say $n_c = 0$ from the very beginning. Then let $P' = L(P)$. P' has $n'_a - 1$ lines in common with P'_a, $n'_b - 1$ lines in common with P'_b, and 1 line in common with neither. If $n_{min} = min\{n'_a, n'_b\}$ and $n_{max} = max\{n'_a, n'_b\}$, then after taking $L(D)$ n_{min} times P' will have $n_{max} - n_{min}$ lines in common with P'_a or P'_b and no lines in common with the other of the 2. Thus after $n_{max} - n_{min} + 1$ more iterations the resulting digraph will definitely be disconnected, which completes the proof. □

Proposition 3.18. *For a digraph D there exists a digraph P which is a polycycle but not a directed cycle such that $L(P) = D \iff D$ has an even number of weak components C_1, C_2, ..., C_{2k} with C_i being a dipath $\forall i \in [1, 2k]$.*

Proof. $\implies$ By **Definition 3.1** P consists of $2k$ directed paths W_1, W_2 ..., W_{2k}, where W_i and $W_{(i+1)\,mod\,2k}$ share the starting point or the endpoint. In any case, $L(W_1)$, $L(W_2)$, ..., $L(W_{2k})$ will be disconnected, and $L(W_i)$ is a dipath of length $|W_i| - 1$ $\forall i \in [1, 2k]$. Thus $C_i := L(W_i)$.

$\impliedby$ $\forall i \in [1, 2k]$ let W_i be a dipath of length $|C_i|+1$. Let also W_1 and W_2 share the endpoint, W_2 and W_3 share the starting point, and so on alternately. Then, since $2k$ is even, W_{2k} and W_1 will share the starting point, and we will have built a weak digraph P consisting of W_1, W_2, ..., W_{2k}. By construction $L(P) = D$, which completes the proof. □

Let's consider the problem of characterizing $L(T)$, where T is a directed tree.

Theorem 3.19. *Let D be a weak line digraph. Then $\exists T : T$ is a directed tree, $L(T) = D \iff$ every polycycle in D is a bipartite subgraph.*

Proof. $\implies$ Let Z' be a polycycle in D. Clearly Z' can't be a directed cycle. Thus Z' consists of $2k$ trails $W'_1, W'_2, ..., W'_{2k}$, trails W'_i and W'_{i+1} being connected at a single point, which is either their common source or their common sink, depending on the parity of i (here and further we write $i+1$ instead of $(i+1) \mod 2k$ for simplicity). Then $L^{-1}(D)$, which is a subgraph of T, consists of $2k$ trails $W_1, W_2, ..., W_{2k}$, with $|W_i| = |W'_i| + 1$. W_i and W_{i+1} share a line - the very line that generates the common point of W'_i and W'_{i+1}. All the lines in $L^{-1}(D)$ save for the shared ones form a polycycle Z whose trails are of length $|W_i| - 2 = |W'_i| - 1$ (the first and last lines of each trail W_i are shared with other trails and thus not included). Therefore there is a single case when $|W'_i| = 1\,\forall i \in [1, 2k]$ and Z becomes a single point. Since T is a directed tree, it contains no polycycles, so Z is indeed a point and $|W'_i| = 1\,\forall i$. But that means that Z' is a bipartite subgraph, A being the set of sources of Z and B the set of its sinks.

$\Longleftarrow$ As D is a line digraph, there exist improper partitions A_i and B_i such that $A(D) = \cup K_i$, where $K_i \overset{def}{=} K(A_i, B_i)$. Let $E = L^{-1}(D)$. Each K_i corresponds to a point of E. Every other point of E is a source or a sink; construct E in such a way that all those points are of degree 1, therefore not being a part of a polycycle. We shall prove by contradiction that in such a case E has no polycycles. Let Z be a polycycle in E, and $a_1 - a_2 - ... - a_n - a_1$ its points, α_i - the line between a_i and a_{i+1}. As previously shown, each point a_i is associated with a two-port subgraph K_i of D. Every line α_i corresponds to a point α'_i of D, which is common for K_i and K_{i+1}. Importantly, no 2 such points coincide; elsewise a point α'_j would belong to at least 3 two-port subgraphs, and therefore by pigeonhole principle to at least 2 sets A_i or at least 2 sets B_i, which is impossible (A_i and B_i are improper partitions). So α'_{i-1} and α'_i are distinct points that belong to K_i. If α'_{i-1} and α'_i are from different ports, then there is a line between them. Otherwise if they both are from port A_i, there exists a polypath $\alpha'_{i-1} - \beta - \alpha'_i$, where β is any point from port B_i (similarly if they are both from B_i). In every case there is a polypath between α'_{i-1} and α'_i. So points $\alpha'_1, \alpha'_2, ..., \alpha'_n$ together with some intermediate points form a polycycle Z' of D. Z' is not a bipartite subgraph, as every weak bipartite subgraph of D is either K_i or a subgraph of K_i. Thus we have a contradiction. □

Let T be a directed tree. If $L(T)$ is weak, then it is characterized in theorem above. Now we characterize such directed trees.

Proposition 3.20. *$L(T)$ is weak $\Longleftrightarrow$ every source and every sink of T is a leaf.*

Proof. $\Longrightarrow$ Let s be a source of degree n (the proof is similar for a sink). As T is a directed tree, there is no point to which we could get from a by 2 different polypaths; thus each of the neighbors of a belongs to a different induced subgraph, all the subgraphs sharing exactly 1 point - a. If we take the line graph, these subgraphs will become different weak components.

$\Longleftarrow$ Divide the lines of T into 2 subsets $A \sqcup B = A(T)$. As T is weak, there is a point a such that $\exists \alpha \in A, \exists \beta \in B$: a is incident at both α and β. Since a is no source nor sink, we may presume that a is incident from α to β (If, for instance, α and β are both inlines at a, then, as a a is no sink, $\exists \gamma \in B : \gamma$ is an outline at a [if $\gamma \in A$, swap A and B]. In that case $\beta := \gamma$). Let A' be the set of points of $L(T)$ generated by the lines from A, and B' the set of points generated by lines from B. Then $A' \sqcup B' = P'(T)$. It is easy to see that there is a line between A' and B' - it is the line between points which correspond to α and β. Since every such pair of sets A' and B' can be obtained in similar way, this means that for every such partition of points of $L(T)$ into 2 sets there is a line between points from different sets. This implies that $L(T)$ is weak. $\square$

Here we provide an algorithm that given a directed tree T constructs its induced subgraphs that become weak components after taking the line graph of T.

Algorithm 3.21.

1. *$S_{sources} \leftarrow$ the set of outlines from sources of T.*
2. *$s \in S_{sources} \leftarrow$ previously unvisited outline from a source; $f \leftarrow$ an inline to sink reachable from s; $S \leftarrow$ set of lines on the directed trail from s to f.*
3. *$S_{prev} \leftarrow S$; $S \leftarrow \{\alpha \in A(T) | \exists \beta \in S : \exists \alpha - \beta\}$, where $\alpha - \beta$ denotes a directed path containing both α and β.*
4. *If $S_{prev} \neq S$, go to 3; else store the subgraph containing the lines from S, mark s and every other outline from a source that belongs to the subgraph as visited, and continue.*
5. *If there are any unvisited outlines from sources, go to 2.*

Remark 3.22. Every stored subgraph is generated by repeating step 3 (adding lines according to the rule of the step) until $S_{prev} = S$. Since the total number of lines is finite, the process cannot continue indefinitely, and so the algorithm terminates.

Lemma 3.23. *Let T_1 be a subgraph generated by the previous algorithm. Then T_1 is weak, and it contains no source nor sink of degree greater than 1.*

Proof. As shown in remark above, T_1 is generated by accumulating lines through consecutive iterations according to the rule of step 3. The rule implies that $\forall k > 1$ a line generated on iteration k is connected to some line generated on step $k-1$, and all the lines generated on iteration 1 are connected. The connectedness of T_1 follows easily by induction. We prove the second part of the statement by contradiction. Let α_n and β_m be 2 lines incident from a single source (the proof for sinks is sufficiently similar to be omitted). Let also α_n and β_m have been generated on iterations n and m respectively. $\forall i \in [1, n-1]$ let α_i be a line generated on iteration i, connected with α_{i+1} (if there are several, choose any). Let α_0 be the initial line (the one whose value was assigned on step 2 of algorithm and from which T_1 was obtained). $\forall j \in [0, m-1]$ β_j shall be similarly defined (note that β_0 is the very same initial line). We have just obtained a polycycle $\alpha_n - \alpha_{n-1} - ... - \alpha_0 = \beta_0 - ... - \beta_{m-1} - \beta_m - \alpha_n$. If one of the sources α_n or β_m is the initial line, then the polycycle looks like $\alpha_0 = \beta_0 - ... - \beta_{m-1} - \beta_m - \alpha_0$ (here $\alpha_n = \alpha_0$ is the initial line). Thus we get a contradiction, as T is a directed tree. $\square$

The next theorem justifies the algorithm above.

Theorem 3.24. *Let $T_1, T_2, ..., T_n$ be subgraphs obtained by applying* ***Algorithm 3.21*** *to directed tree T. Then the next statements hold:*

1. *T_k is a weak subgraph $\forall k \in [1, n]$.*
2. *$A(T_1) \sqcup A(T_2) \sqcup ... \sqcup A(T_n) = A(T)$.*
3. *$L(T_k)$ is a weak graph $\forall k \in [1, n]$.*
4. *$L(T_k + \alpha)$ is not weak, $\forall k \in [1, n], \forall \alpha \in A(T) \backslash A(T_k)$.*

Proof. *1* follows immediately from the lemma.

Let's now prove that all sets $A(T_i)$ are pairwise disconnected. Let $A(T_1) \cap A(T_2) = S$. T_1 is weak $\implies \exists a \in P(T_1) : a$ is incident at lines $\alpha \in S$ and $\beta \in T_1 \backslash S$. By lemma a is no source nor sink, so assume a is incident from α to β. This implies that there is a dipath containing both α and β, namely, $\alpha - \beta$ is a dipath. Thus, since α is in T_2 ($S \subset A(T_2)$), $\beta \in T_2$ as well - it will be generated on the next iteration after α. To prove *2* we still need to show that $\forall \alpha \in A(T) \exists k \in \mathbb{N} : \alpha \in A(T_k)$. If α is an outline from a source than the algorithm doesn't terminate until it is marked as visited, that is, it belongs to some T_k. Otherwise we assert that α is connected by a dipath to some outline from a source: as α is not an outline from a source, there is a line β_1 which is adjacent to α. Now we have that either β_1 is an outline from a source, or there is a line β_2 adjacent to β_1. The process terminates when we reach a line β_m that is an outline from a source (it cannot continue indefinitely, since the total number of lines in T is finite). Thus α will be generated on the next iteration after β_m (or if β_m was chosen as the initial line, then possibly on the same one). In any case, as shown earlier, $\beta_m \in A(T_k)$ for some $k \in \mathbb{N}$, and so $\alpha \in A(T_k)$.

3 follows from lemma together with **Proposition 3.20**.

If $\alpha \notin A(T_k)$, then α isn't connected with any line from T_k (if it were connected with a line β, it would be generated on the next iteration after β and therefore belong to T_k), and so the point of $L(T)$ associated with it will lie in different weak component than $L(T_k)$. This completes the proof. $\square$

Now we shall consider several special cases of polytrees, to see how their line digraphs look like.

Proposition 3.25. *$D \simeq L(T)$ for an in-tree (out-tree) $T \iff$ every weak component of D is an in-tree (out-tree).*

Proof. We shall offer proof for in-trees, considering the proof for out-trees sufficiently similar to be omitted.

$\Longrightarrow$ Let E be a weak component of D. Let also s be the only sink of T. Since s is reachable from every point of T, there is an inline α to s associated with a point α' of E. The inline α is single, as otherwise there would be a point in T from which we could reach s by 2 different polypaths. Thus α' is the only sink of E, and so it can be reached from every point. Then E is an in-tree.

$\Longleftarrow$ Let again E be a weak component of D. As it is an in-tree, there is no point of E with outdegree greater than 1 - otherwise there would be 2 different dipaths from it to s, s being the sink of E. To show that E is a line digraph of a digraph, we shall construct improper partitions A_i and B_i. For every point p of E with indegree of at least 1 let $B_p = \{p\}$ and A_p be the set of points connected to p. As no 2 points are connected from a single point, the sets A_i will be disjoint; obviously $|B_i \cap A_j| \leq 1$. Thus we indeed have an improper partition satisfying the condition of line digraph characterization. Let then F be a digraph such that $L(F) = E$. If α is the line of F associated with s, then it is reachable from every other line of F. Thus point b of F, which is incident from α, is reachable from every other point of F. Conclude that F is an in-tree. Now let F_1, F_2, ..., F_n be in-trees for which $L(F_i) \simeq E_i \forall i \in [1, n]$, E_i being the weak components of D. Let all F_i have a common sink b, and let it be the only common point of different F_i (therefore there will be no common lines). This way we obtain a digraph T, which is still an in-tree. Also, $L(T) \simeq D$, which completes the proof. $\square$

Proposition 3.26. *$D \simeq L(T)$ for a polypath $T \iff$ every weak component of D is a dipath.*

Proof. $\Longrightarrow$ By definition a polypath is a sequence of directed paths where each 2 consecutive paths share a common point (alternately starting and ending points). After taking the line digraph every directed path T_n of length k will become a separate weak component, which by itself is a dipath of length $k-1$. That completes the proof.

$\Longleftarrow$ Let D_1, D_2, ..., D_n be the weak components of D, each of them being a dipath of length k_1, k_2, ..., k_n respectively. Let T_i, $i \in [1, n]$ be digraphs such that $D_i = L(T_i) \forall i \in [1, n]$. Then every T_i is a directed path of length $k_i + 1$. Let digraph T be a sequence of T_1, T_2, ..., T_n with T_1 and T_2 sharing an endpoint, T_2 and T_3 sharing the starting point, and so on alternately. Then T is a polypath, and $L(T) = D$. $\square$

Proposition 3.27. *$D \simeq L(S)$ for a polystar $S \iff D$ is either a two-port digraph or an edgeless digraph.*

Proof. $\Longrightarrow$ If the internal point p of S is either the common source or the common sink of all its lines, then D is edgeless. Otherwise let α_1, α_2, ..., α_n be lines ending at p, and β_1, β_2, ..., β_m be lines starting at p. Let a_1, ..., a_n, b_1, ..., b_m be corresponding points of $D = L(S)$. Then D is a two-port digraph, its ports being $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$. So D is either a two-port or an edgeless digraph.

$\Longleftarrow$ If D is a two-port digraph, then S is a point together with lines incident to and from it (which is, a polystar). If D is edgeless with n points, then $D = L(S)$ for a polystar S with n lines, all of them adjacent to (or all adjacent from) the internal point. In any case, D is a line digraph of a polystar. $\square$

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