SOLVING CLASSICAL (SYSTEMS OF) LINEAR ALGEBRAIC EQUATIONS USING NON-CLASSICAL METHODS OF COMPUTER MATHEMATICS

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Since the beginning of time people have made some optimal, more or less, decision, based only on pure human experience. Over time it was no longer possible to realize this action without special mathematical methods that carry out global search for necessary optimum. Nowadays, in crazy time of global computerization, these methods allow us to take, in fact, innovative look at some complicated problems of classical mathematics. The given paper presents basic results of analysis of well known classical Gauss method (or Gaussian elimination method) [1, 2] and new optimization method of gradient descent [3] for solving general systems of linear algebraic equations, obtained after testing our own program, written in «Visual Basic for Applications».

The essence of proposed numerical optimization method of gradient descent is that solving an arbitrary system of the form $A\vec{x} = \vec{b}$ is reduced to searching of minimizer $\vec{x}^*$, i.e. such vector, that $\min\limits_{x\in\mathbb{R}^n} f(\vec{x}) = f(\vec{x}^*)$, for the next function

$$f(\vec{x}) = \big(A\vec{x} - \vec{b}, A\vec{x} - \vec{b}\big) = ||A\vec{x} - \vec{b}||^2,$$

$$A \in \mathbb{R}^{m\times n},\ \vec{b} \in \mathbb{R}^m,\ m \le n.$$

Formally, this method consists in iterative generation of some sequence of such points $\{\vec{x}_k\}_{k\ge 0}$, i.e. some descent trajectory, that is sometimes called a *relaxation trajectory*, that converges to our real solution $\vec{x}^*$ while $k \to \infty$, that $f(\vec{x}_{k+1}) \le f(\vec{x}_k)$, $k \ge 0$, according to the following iterated scheme.

(1) An arbitrary point is selected as an initial approximation $\vec{x}_0$.

(2) The approximation $\vec{x}_{k+1}$, $k \ge 0$, is determined by the formula

$$\vec{x}_{k+1} = \vec{x}_k - \lambda_k \vec{g}_k,\ \lambda_k > 0,\ k \ge 0,$$

where $\vec{g}_k = grad\big(f(\vec{x}_k)\big)$, $k \ge 0$.

At each step we have movement along the vector of anti-gradient, in the direction of the fastest decrease of f, and, as a result, we get our necessary solution. Namely, if it turns out that the modulus of our anti-gradient is zero (less than predetermined accuracy), then we are at the minimum point we are looking for. If the criterion for the end of the iteration is not true (the modulus of our anti-gradient is more than predetermined accuracy), then we

return to the first step, otherwise we return the exact value of $\vec{x}_{k+1}$, $k \geq 0$. In this formula λ_k, $k \geq 0$, determines the distance between $\vec{x}_k$ and $\vec{x}_{k+1}$, $k \geq 0$. This distance is usually called *the step* of the proposed gradient scheme.

The main problem in the process of choosing this step λ_k, $k \geq 0$, is to ensure that the inequality $f(\vec{x}_{k+1}) \leq f_(\vec{x}_k)$, $k \geq 0$, is true. There are different ways to choose this step multiplier λ_k, $k \geq 0$. Depending on this, different variants of numerical optimization method of gradient descent can be obtained. We have considered the next four variational methods: *the method with adaptive step selection*, *the method with adaptive step correction*, *the modified descent method with adaptive step selection*, and *the fastest descent method*. All these methods contribute to the accelerated approximation of our converging sequence of points $\{\vec{x}_k\}_{k\geq 0}$ to necessary real solution $\vec{x}^*$ we are looking for.

The main idea of the proposed method is to optimize, that is, to move to minimum in the direction of the fastest descent, and this direction is given by the vector of anti-gradient $grad\big(f(\vec{x}_k)\big)$, $k \geq 0$. In details, we choose some starting point in an arbitrary way, calculate the gradient of the considered function in this point, and take some small step in the opposite, anti-gradient, direction. As a result, we arrive to some point where the value of our function is less than the value in the previous point. At this new point we repeat the mentioned procedure: we again calculate the gradient of the function and take a step in the opposite direction. Continuing this process, we move towards the decreasing function. The relaxation sequence is built, according to the following mathematical conclusions: due to special choice of the direction of our movement, reducing the value of our function at each step, when moving from the first point $\vec{x}_k$, $k \geq 0$, to the second point $\vec{x}_{k+1}$, $k \geq 0$, we approach the minimum. Since at each step we move along the vector of anti-gradient, in the direction of the fastest decrease of our function f, thus, as a result, we will finally arrive to our necessary explicit solution we are looking for. After testing various systems of linear algebraic equations with the help of our program, it was found that the Gaussian method algorithm gives results in a fraction of seconds, while proposed optimization method produces the most accurate result in a short time not for every system. This fact experimentally proves that effectiveness of this descent method depends on value of its chosen step.

References

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