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Primary decompositions of unital locally matrix algebras

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We construct a unital locally matrix algebra of uncountable dimension that

(1) does not admit a primary decomposition,
(2) has an infinite locally finite Steinitz number.

It gives negative answers to questions from [V. M. Kurochkin, On the theory of locally simple and locally normal algebras, *Mat. Sb., Nov. Ser.* **22(64)**(3) (1948) 443–454; O. Bezushchak and B. Oliynyk, Unital locally matrix algebras and Steinitz numbers, *J. Algebra Appl.* (2020), online ready]. We also show that for an arbitrary infinite Steinitz number s there exists a unital locally matrix algebra A having the Steinitz number s and not isomorphic to a tensor product of finite-dimensional matrix algebras.

Keywords: Locally matrix algebra; primary decomposition; Steinitz number; tensor product; Clifford algebra.

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1. Introduction

Let F be a ground field. In this paper, we consider associative unital F-algebras. Following Kurosh [6] we say that an algebra A with a unit 1_A is a locally matrix

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algebra if an arbitrary finite collection of elements $a_1, \ldots, a_s \in A$ lies in a subalgebra B, $1_A \in B \subset A$, that is isomorphic to a matrix algebra $M_n(F)$, $n \geq 1$.

In [4], Köthe proved that every countable-dimensional unital locally matrix algebra admits a decomposition into an (infinite) tensor product of finite-dimensional matrix algebras and admits a primary decomposition.

Kurosh [6] and Kurochkin [5] further studied existence and uniqueness of such decompositions of unital locally matrix algebras of arbitrary dimensions. In particular Kurochkin [5] formulated the question:

Does every locally matrix algebra have a primary decomposition?

Glimm [3] proved that a countable-dimensional unital locally matrix algebra is uniquely determined by its Steinitz number. In [1], we showed that this is no longer true for an algebra of the dimension $> \aleph_0$. As follows from [2, 3], for any Steinitz number s there exists a countable-dimensional unital locally matrix algebra A such that $\mathbf{st}(A) = s$. In [1], we proved that for any not locally finite Steinitz number s there exists an uncountable-dimensional unital locally matrix algebra A with the Steinitz number s. So, in [1] we asked:

If $\mathbf{st}(A)$ *is locally finite, does it imply that* $\dim_F A \leq \aleph_0$?

In this paper, we give negative answers to both of above questions. In Sec. 3, we prove

Theorem 1. *For an arbitrary infinite locally finite Steinitz number s there exists a unital locally matrix algebra of uncountable dimension with Steinitz number s.*

Moreover, in this case, the algebra A has no primary decomposition.

Theorem 2. *There exists a unital locally matrix algebra of uncountable dimension that has no primary decomposition.*

Kurosh in [6] constructed an example of a unital locally matrix algebra of uncountable dimension that does not admit a decomposition into an infinite tensor product of finite-dimensional matrix algebras. Another example of this kind (a Clifford algebra) is constructed in [1]. Both examples in [6, 1] have Steinitz number 2^∞. In Sec. 4 for an arbitrary odd number l we construct a unital locally matrix algebra of Steinitz number l^∞ that does not admit a decomposition into a tensor product of finite-dimensional matrix algebras. Let $\mathbb{R}$ be the set of all real numbers with natural order and let $\mathrm{Clg}(l, \mathbb{R})$ be the generalized Clifford algebra (see Sec. 4, a more general construction of this kind appeared in [7]).

Theorem 3. *If l is an odd number then* $\mathrm{Clg}(l, \mathbb{R})$ *is not isomorphic to a tensor product of finite-dimensional matrix algebras.*

Finally, we obtain

Theorem 4. *For an arbitrary infinite Steinitz number s there exists a unital locally matrix algebra A such that $\mathbf{st}(A) = s$ and A does not admit a decomposition into a tensor product of finite-dimensional matrix algebras.*

2. Steinitz Numbers

Let $\mathbb{P}$ be the set of all primes and $\mathbb{N}$ be the set of all positive integers. A *Steinitz* or *supernatural* number (see [8]) is an infinite formal product of the form

$$\prod_{p\in\mathbb{P}} p^{r_p}, \tag{1}$$

where $r_p \in \mathbb{N} \cup \{0, \infty\}$ for all $p \in \mathbb{P}$. Two Steinitz numbers

$$\prod_{p\in\mathbb{P}} p^{r_p} \quad \text{and} \quad \prod_{p\in\mathbb{P}} p^{k_p}$$

can be multiplied with

$$\prod_{p\in\mathbb{P}} p^{r_p} \cdot \prod_{p\in\mathbb{P}} p^{k_p} = \prod_{p\in\mathbb{P}} p^{r_p+k_p},$$

where we assume that $k_p \in \mathbb{N} \cup \{0, \infty\}$ and $t + \infty = \infty + t = \infty + \infty = \infty$ for all positive integers t.

Denote by $\mathbb{SN}$ the set of all Steinitz numbers. Note, that the set $\mathbb{N}$ is a subset of $\mathbb{SN}$. A Steinitz number (1) is called *locally finite* if $r_p \neq \infty$ for any $p \in \mathbb{P}$. The numbers $\mathbb{SN}\backslash\mathbb{N}$ are called *infinite* Steinitz numbers.

Let A be a locally matrix algebra with a unit 1_A over a field F and let $D(A)$ be the set of all positive integers n such that there is a subalgebra A', $1_A \in A' \subseteq A$, $A' \cong M_n(F)$. Then the least common multiple of the set $D(A)$ is called the *Steinitz number* of the algebra A and denoted as $\mathbf{st}(A)$.

3. Primary Decompositions of Unital Locally Matrix Algebras

Definition 1. A unital locally matrix algebra A over a field F is called primary if $\mathbf{st}(A) = p^s$, where p is a prime number and $s \in \mathbb{N}$ or $s = \infty$.

Recall, that if A and B are unital locally matrix algebras then the algebra $A \otimes_F B$ is a unital locally matrix and $\mathbf{st}(A \otimes_F B) = \mathbf{st}(A) \cdot \mathbf{st}(B)$ (see [1]).

Definition 2. We say that the decomposition

$$A = \bigotimes_{p\in\mathbb{P}} A_p$$

of a unital locally matrix algebra A over F is a primary decomposition if each algebra A_p is primary for all $p \in \mathbb{P}$.

Proof of Theorem 1. The crucial role in the proof will be played by Kurosh's Theorem [6, Theorem 10] which is reformulated as follows.

Let A be a countable-dimensional locally matrix algebra with a unit 1_A. Then A contains a proper subalgebra $1_A \in B \subset A$ such that $A \cong B$.

Now suppose that A is a locally matrix algebra such that $\mathbf{st}(A) = s$ and all unital locally matrix algebras of Steinitz number s are no more than countable-dimensional.

Let γ be an uncountable ordinal. For all ordinals $\alpha \leq \gamma$ we will construct an unital locally matrix algebra A_α such that

(1) $\mathbf{st}(A_\alpha) = s$,
(2) if $\alpha < \beta \leq \gamma$ then A_α is properly contained in A_β.

Let $A_1 = A$. If α is a limit ordinal then we let

$$A_\alpha = \bigcup_{\mu<\alpha} A_\mu,$$

where $\mathbf{st}(A_\mu) = s$, so $\mathbf{st}(A_\alpha) = s$.

If α is a nonlimit ordinal then $\alpha - 1$ exists and an algebra $A_{\alpha-1}$ has been constructed with $\mathbf{st}(A_{\alpha-1}) = s$. By an assumption $\dim_F A_{\alpha-1} \leq \aleph_0$. Hence, by Kurosh's Theorem there exists an unital locally matrix algebra A' such that $A_{\alpha-1}$ is properly contained in A', the unit of $A_{\alpha-1}$ is the unit of A' and $\mathbf{st}(A') = s$. Let $A_\alpha = A'$.

We have already arrived at contradiction since the algebra A_γ cannot be countable-dimensional. □

Proof of Theorem 2. Let s be an infinite locally finite Steinitz number. We have shown (Theorem 1) the existence of a unital locally matrix algebra A such that $\mathbf{st}(A) = s$ and $\dim_F A > \aleph_0$. If the algebra A admits a primary decomposition then

$$A \cong \bigotimes_{p\in\mathbb{P}} A_p, \quad \mathbf{st}(A_p) = p^{r_p} \quad \text{and} \quad r_p < \infty \quad \text{for all } p \in \mathbb{P}.$$

Hence, $A_p \cong M_{p^{r_p}}(F)$ and therefore $\dim_F A \leq \aleph_0$. This concludes the proof of Theorem 2. □

4. Decompositions into Products of Matrix Algebras

Let us recall the definition of a generalized Clifford algebra introduced in [1].

Let $l > 1$ be an integer. If $\operatorname{char} F > 0$ then we assume that l is coprime with $\operatorname{char} F$. Let $\xi \in F$ be an lth primitive root of 1. Let I be an ordered set. The generalized Clifford algebra $\operatorname{Clg}(l, I)$ is presented by generators x_i, $i \in I$, and relations:

$$x_i^l = 1, \quad x_i^{-1} x_j x_i = \xi x_j \quad \text{for } i < j,$$

$$x_i^{-1} x_j x_i = \xi^{-1} x_j \quad \text{for } i > j, \ \ i, j \in I.$$

In [1], we showed that $\operatorname{Clg}(l, I)$ is a unital locally matrix algebra of Steinitz number l^∞ and that ordered monomials

$$x_{i_1}^{k_1} \cdots x_{i_r}^{k_r}, \quad i_1 < \cdots < i_r, \ \ 1 \leq k_j \leq l-1, \ \ 1 \leq j \leq r,$$

form a basis of $\operatorname{Clg}(l, I)$.

Let $X_{\mathbb{Q}}$ be the set of generators indexed by rational numbers.

Lemma 1. *Let l be odd. Then the centralizer of $X_{\mathbb{Q}}$ in $\mathrm{Clg}(l, \mathbb{R})$ is $F \cdot 1$.*

Remark 1. The assertion of Lemma 1 is not true for $l = 2$, that is for the case of ordinary Clifford algebras. Indeed, if α, β are two distinct irrational numbers then $x_\alpha x_\beta$ lies in the centralizer of $X_{\mathbb{Q}}$.

Proof of Lemma 1. In [1], we showed that

(1) for any $i \in I$ the mapping $\varphi_i(x_k) = \xi^{\delta_{ik}} x_k$, $k \in I$, extends to an automorphism φ_i of $\mathrm{Clg}(l, I)$,
(2) any subspace V of $\mathrm{Clg}(l, I)$ that is invariant under all automorphisms φ_i, $i \in I$, is spanned by all ordered monomials lying in V.

The centralizer of $X_{\mathbb{Q}}$ is invariant under all automorphisms φ_i, $i \in \mathbb{R}$, hence, it is spanned by all ordered monomials. Let a monomial

$$v = x_{i_1}^{k_1} \cdots x_{i_r}^{k_r}, \quad i_1 < \cdots < i_r, \ \ 1 \leq k_j \leq l-1, \ \ 1 \leq j \leq r,$$

lies in the centralizer of $X_{\mathbb{Q}}$.

Let j be a rational number such that $j < i_1$. Then

$$x_j^{-1} v x_j = \xi^{(k_1 + \cdots + k_r)} v = v,$$

which implies $k_1 + \cdots + k_r = 0 \mod l$. Now let j be a rational number such that $i_1 < j < i_2$. Then

$$x_j^{-1} v x_j = \xi^{(-k_1 + k_2 + \cdots + k_r)} v = v,$$

which implies $-k_1 + k_2 + \cdots + k_r = 0 \mod l$.

Subtracting these comparisons we get $2k_1 = 0 \mod l$. Since the number l is odd we conclude that $k_1 = 0 \mod l$, a contradiction. This completes the proof of Lemma 1. $\square$

The next statement directly follows from Kurosh [6].

Lemma 2. *Let A be a locally matrix algebra of uncountable dimension. Suppose that A contains a countable subset S whose centralizer is $F \cdot 1$. Then A is not isomorphic to a tensor product of finite-dimensional matrix algebras.*

Proof. Let $A = \otimes_{i \in I} A_i$, where $\dim_F A_i < \infty$. Then $\mathrm{Card}\, I > \aleph_0$. The subset S lies in $\otimes_{j \in J} A_j$, where J is a countable subset of I. Hence, for any $i \in I \backslash J$ the subalgebra A_i lies in the centralizer of S, a contradiction. This completes the proof of Lemma 2. $\square$

Proof of Theorem 3. From Lemma 1 it follows that the centralizer of the countable subset $X_{\mathbb{Q}}$ in $\mathrm{Clg}(l, \mathbb{R})$ is $F \cdot 1$. Then by Lemma 2 the algebra $\mathrm{Clg}(l, \mathbb{R})$ does not admit a decomposition into a tensor product of finite-dimensional matrix algebras. $\square$

Proof of Theorem 4. (1) There are two examples of an uncountable-dimensional unital locally matrix algebra $A(2)$ with Steinitz number $s = 2^\infty$ that is not isomorphic to a tensor product of finite-dimensional matrix algebras. The first example was constructed by Kurosh [6]. In [1], we found such an example among Clifford algebras. For more details see [1, 6].

(2) For an odd number l the generalized Clifford algebra $A(l) = \mathrm{Clg}(l, \mathbb{R})$ with Steinitz number $s = l^\infty$ also does not admit such a decomposition (Theorem 3).

(3) Now let s be an infinite locally finite Steinitz number and let A be the algebra from Theorems 1 and 2, $\mathbf{st}(A) = s$. If A admitted a decomposition into a tensor product of finite-dimensional matrix algebras then A would admit a primary decomposition, a contradiction.

(4) Let s be an infinite Steinitz number that is not locally finite, i.e. $s = p^\infty \cdot s'$ for some prime number p. By [2] there exists a countable-dimensional (or finite-dimensional) unital locally matrix algebra A' such that $\mathbf{st}(A') = s'$. Let $A = A(p) \otimes_F A'$. Clearly, $\mathbf{st}(A) = s$. By Lemma 1 and [1, 6] (in the case $p = 2$) the algebra $A(p)$ contains a countable-dimensional subspace W whose centralizer is $F \cdot 1_{A(p)}$. The subspace $W \otimes_F A'$ of the algebra A is also countable-dimensional.

We claim that the centralizer of $W \otimes_F A'$ is $F \cdot 1_A$. Indeed, let an element $a = \sum_i a_i \otimes a'_i$ lies in the centralizer of $W \otimes_F A'$; $a_i \in A(p)$, $a'_i \in A'$. Suppose that the elements a'_i are linearly independent. For an arbitrary element $w \in W$ we have

$$\left[\sum_i a_i \otimes a'_i, w \otimes 1_{A'}\right] = \sum_i [a_i, w] \otimes a'_i = 0,$$

which implies that the elements a_i lie in the centralizer of W, that is, in $F \cdot 1_{A(p)}$. Hence, $a = 1_{A(p)} \otimes a'$, where the element a' lies in the center of A', $a' \in F \cdot 1_{A'}$. This completes the proof of the claim.

By Lemma 2 the algebra A is not isomorphic to a tensor product of finite-dimensional matrix algebras. This completes the proof of Theorem 4. □

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