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ANALYSIS OF THE SHAPE OF WAVE PACKETS IN THE “HALF SPACE–LAYER–LAYER WITH RIGID LID " THREE-LAYER HYDRODYNAMIC SYSTEM

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We study the process of propagation of weakly nonlinear wave packets on the contact surfaces of a “half space–layer–layer with rigid lid” hydrodynamic system by the method of multiscale expansions. The solutions of the weakly nonlinear problem are obtained in the second approximation. The condition of solvability of this problem is established. For each frequency of the wave packet, we construct the domains of sign constancy for the coefficient for the second harmonic on the bottom and top contact surfaces. The regularities of wave formation are determined depending on the geometric and physical parameters of the hydrodynamic system. We also analyze the plots of the shapes of deviations of the bottom and top contact surfaces typical of the constructed domains of sign-constancy of the coefficient. We discover the domains where the waves become $\bigcup$- and $\bigcap$-shaped and reveal a significant influence of wavelength on the shapes of deviations of the contact surfaces of the analyzed hydrodynamic system.

Keywords: wave packet, hydrodynamic system, shape of deviation of the contact surface.

Introduction

The investigation of the wave processes running in fluids, their general properties and characteristics attracts significant attention of numerous researchers. At present, we observe the extensive development and application of the packages of symbolic calculus. These packages enable us to study various classes of problems that were not analyzed earlier due to the awkwardness of transformations and significant difficulties encountered in getting the analytic results. As an example, we can mention the class of problems in which the effect of surface tension, which significantly affects the gravity-capillary waves, is taken into account. We now present a brief survey of the general state-of-the-art of the problem of propagation of waves in layered hydrodynamic systems.

In [12], the authors present a survey of the properties of internal solitary waves and the transient processes of generation and evolution of these waves from the viewpoint of weakly nonlinear theory. The authors analyzed the processes of instability of waves that are important for oceanography and cannot be described by using other models. The cited study also revealed the existence of strongly nonlinear waves whose properties can be explained only with the help of nonlinear models.

The results of investigation of internal interface waves in three-layer stratified incompressible fluids performed with the help of the singular method of perturbations are also of great interest [10]. On the basis of the theory of small-amplitude waves, the asymptotic solutions of the third order were obtained for the velocity potentials and the Stokes wave solutions of the third order were constructed. It was discovered that the wave ve-

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locity depends not only on the wave number and the thickness of each layer but also on the wave amplitude.

In [11], the authors considered a model of potential flow of axisymmetric waves propagating along a ferrofluid jet. This model is of significant interest because the mechanism of stabilization allows the motion of axisymmetric magnetohydrodynamic solitons and, moreover, the presented numerical scheme enables one to find steady periodic and generalized solutions for solitary waves. It was also established that the space of solutions of this model is similar to the space of solutions of the classical problem for the two-dimensional gravity-capillary waves.

A quite comprehensive analysis of the wave motion can be found in [14], where the method of multiscale expansions (up to the third order) was used to deduce the evolutionary Schrödinger equation for the wave motion on the contact surface of two fluid half spaces. A similar problem of propagation of wave packets in a "layer–half space" system was studied by I. T. Selezov and O. V. Avramenko [4] who analyzed the problem of stability of wave packets by the method of multiscale expansions up to the third order [5, 7]. In addition, various aspects of the problem of evolution of nonlinear wave packets up to the fourth approximation were analyzed in [2], whereas the evolutionary equation for the wave numbers close to the critical value was deduced in [6].

In [13], the problem of propagation of internal wave packets was investigated for "layer with rigid lid–layer–layer with rigid bottom" three-layer hydrodynamic systems. In particular, a weakly nonlinear model of interacting waves propagating along the contact surfaces was developed, the first three linear approximations were obtained, and the conditions of propagation of waves along the contact surfaces were established.

The problem of propagation of wave packets in "layer with lid–layer with rigid bottom" hydrodynamic systems in the presence of surface tension was studied by the method of multiscale expansions. The evolutionary equation in the form of a nonlinear Schrödinger equation was constructed in [8] for the envelope on the contact surface of two fluid layers. On the basis of this equation, the analysis of dependence of the shape of wave packet on the physical parameters of the system was carried out in [3].

In [1, 9], the authors studied the problem of propagation of waves in a "half space–layer–layer with rigid lid" three-layer hydrodynamic system. Three linear approximations were constructed for a weakly nonlinear problem and a dispersion equation was obtained. The roots of this equation were found and the analysis of their dependence on the physical and geometric parameters of the system was carried out. The dependences of the amplitudes of waves running along the contact surfaces on the thickness of the top layer and on the wave number were analyzed, and the structure of wave motions was described.

In the present work, we continue the investigation of the weakly nonlinear problem of propagation of wave packets in three-layer hydrodynamic systems. We establish the condition of solvability, obtain the solutions in the second approximation and analyze the dependence of the shapes of wave packets moving on the bottom and top contact surfaces on the physical and geometric parameters of the system.

1. Statement of the Problem and the Procedure of Its Solution

We consider the problem of propagation of two-dimensional wave packets with finite amplitudes on the surface of the fluid half space $\Omega_1 = \{(x,z)\colon |x| < \infty,\ -\infty \le z < 0\}$ with density ρ_1, the fluid layer $\Omega_2 = \{(x,z)\colon |x| < \infty,\ 0 \le z \le h_2\}$ with density ρ_2, and the top fluid layer $\Omega_3 = \{(x,z)\colon |x| < \infty,\ h_2 \le z \le h_2 + h_3\}$ with density ρ_3.

The layers Ω_1 and Ω_2 are separated by the contact surface $z = \eta_1(x,t)$, the layers Ω_2 and Ω_3 are separated by the contact surface $z = h_2 + \eta_2(x,t)$, and the layer Ω_3 is bounded by a rigid lid $z = h_2 + h_3$. In our calculations, we take into account the forces of surface tension on the contact surfaces. The gravity force is directed to the negative direction of the z-axis, and the fluids are incompressible (Fig. 1).

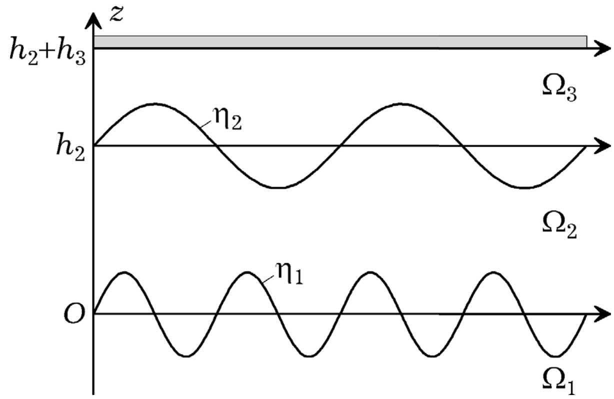


Fig. 1

The mathematical statement of the problem takes the following form:

–*equation of motion*

$$\frac{\partial^2 \varphi_j}{\partial x^2} + \frac{\partial^2 \varphi_j}{\partial z^2} = 0 \quad \text{in} \quad \Omega_j, \quad j = 1, 2, 3; \tag{1}$$

– *kinematic conditions on the contact surfaces*

$$\frac{\partial \eta_1}{\partial t} - \frac{\partial \varphi_j}{\partial z} = -\frac{\partial \eta_1}{\partial x}\frac{\partial \varphi_j}{\partial x} \quad \text{for} \quad z = \eta_1(x, t), \quad j = 1, 2, \tag{2}$$

$$\frac{\partial \eta_2}{\partial t} - \frac{\partial \varphi_j}{\partial z} = -\frac{\partial \eta_2}{\partial x}\frac{\partial \varphi_j}{\partial x} \quad \text{for} \quad z = h_2 + \eta_2(x, t), \quad j = 2, 3; \tag{3}$$

– *dynamic conditions on the contact surfaces*

$$\rho_1 \frac{\partial \varphi_1}{\partial t} - \rho_2 \frac{\partial \varphi_2}{\partial t} + g(\rho_1 - \rho_2)\eta_1 + 0.5\rho_1(\nabla\varphi_1)^2$$

$$- 0.5\rho_2(\nabla\varphi_2)^2 - T_1\left[1 + \left(\frac{\partial \eta_1}{\partial x}\right)^2\right]^{-3/2} \frac{\partial^2 \eta_1}{\partial x^2} = 0 \quad \textit{for} \quad z = \eta_1(x, t), \tag{4}$$

$$\rho_2 \frac{\partial \varphi_2}{\partial t} - \rho_3 \frac{\partial \varphi_3}{\partial t} + g(\rho_2 - \rho_3)(h_2 + \eta_2) + 0.5\rho_2(\nabla\varphi_2)^2$$

$$- 0.5\rho_3(\nabla\varphi_3)^2 \; T_2\left[1 + \left(\frac{\partial \eta_2}{\partial x}\right)^2\right]^{-3/2} \frac{\partial^2 \eta_2}{\partial x^2} = 0 \quad \textit{for} \quad z = h_2 + \eta_2(x, t); \tag{5}$$

– *condition of impermeability on the rigid lid*

$$\frac{\partial \varphi_3}{\partial z} = 0 \quad for \quad z = h_2 + h_3; \tag{6}$$

– *condition of vanishing at infinity*

$$|\nabla \varphi_1| \to 0 \quad for \quad z \to -\infty. \tag{7}$$

Here, φ_j, $j = 1,2,3$, are the velocity potentials in Ω_j; T_1 and T_2 are the coefficients of surface tension on the contact surfaces, respectively; and g is the gravitational acceleration.

The dimensionless quantities are introduced by using the characteristic length H equal to the thickness of the middle layer h_2, characteristic wavelength L, maximum deviation a of the contact surface between the layers Ω_2 and Ω_3, characteristic time $\frac{L}{\sqrt{gH}}$, and density of the bottom layer ρ_1. In this case, the dimensionless coefficient of surface tension takes the form $T_{1,2} = L^2 \rho_1 g T^*_{1,2}$ (in what follows, the asterisk is omitted).

To determine the approximate solution of problem (1)–(7) for small amplitudes, we use the method of multiscale expansions up to the third order [14]:

$$\eta_i(x,t) = \sum_{n=1}^{3} \alpha^{n-1} \eta_{in}(x_0, x_1, x_2, t_0, t_1, t_2) + O(\alpha^3), \quad i = 1,2, \tag{8}$$

$$\varphi_j(x,z,t) = \sum_{n=1}^{3} \alpha^{n-1} \varphi_{jn}(x_0, x_1, x_2, z, t_0, t_1, t_2) + O(\alpha^3), \quad j = 1,2,3, \tag{9}$$

where $x_k = \alpha^k x$ and $t_k = \alpha^k t$, $k = 1,2,3$.

As a result of the substitution of relations (8) and (9) in Eqs. (1)–(7), we get three linear problems for the unknown functions η_{11}, η_{21}, φ_{11}, φ_{21}, φ_{31}, η_{12}, η_{22}, φ_{12}, φ_{22}, φ_{32}, η_{13}, η_{23}, φ_{13}, φ_{23}, and φ_{33} [1].

In what follows, we present the statement of the problem of propagation of waves in a three-layer “half space–layer–layer with rigid lid” hydrodynamic system in the second approximation and find its solutions.

2. Solutions and the Condition of Solvability of the Problem in the Second Approximation

In the first approximation, the solutions of the problem are as follows [1]:

$$\varphi_{11} = -\frac{i\omega}{k}\left(A e^{i\theta + kz} - \bar{A} e^{-i\theta + kz}\right),$$

$$\varphi_{21} = -\frac{i\omega}{k}\left\{(\rho_1 \omega^2 - k\rho_1 + k\rho_2 - T_1 k^3)\cosh k(h_2 - z)\right.$$

$$+\left[\rho_2\omega^2\cosh kh_2+(\rho_1\omega^2-k\rho_1+k\rho_2-T_1k^3)\right.$$

$$\left.\times\sinh kh_2\right]\sinh kz\Big\}\frac{Ae^{i\theta}-\bar{A}e^{-i\theta}}{\rho_2\omega^2\cosh kh_2},$$

$$\varphi_{31} = \frac{i\omega}{k}\Big\{\Big[\rho_2\omega^2\cosh kh_2+(\rho_1\omega^2-k\rho_1$$

$$+k\rho_2-T_1k^3)\sinh kh_2\Big]\cosh k(h_2+h_3-z)\Big\}\frac{Ae^{i\theta}-\bar{A}e^{-i\theta}}{\rho_2\omega^2\sinh kh_3},$$

$$\eta_{11} = Ae^{i\theta}+\bar{A}e^{-i\theta},$$

$$\eta_{21} = \left[\rho_2\omega^2\cosh kh_2+(\rho_1\omega^2-k\rho_1+k\rho_2-T_1k^3)\sinh kh_2\right]\frac{Ae^{i\theta}+\bar{A}e^{-i\theta}}{\rho_2\omega^2},$$

where k is the wave number, ω is the frequency of a wave packet, $\theta = kx+\omega t$, and $A(\rho_1, \rho_2, \rho_3, k, h_2, h_3, T_1, T_2, \omega)$ is the envelope of a wave packet on the bottom contact surface.

The problem in the first approximation was studied in [1], where the authors established the condition of propagation of waves with two pairs of frequencies in the wave packet $\pm\omega_1$ and $\pm\omega_2$, $|\omega_1| < |\omega_2|$.

By using the presented solutions of the problem in the first approximation, the condition of impermeability of the rigid lid (6), and the condition of vanishing at infinity (7), we represent the problem in the second approximation [9] in the following form:

$$\varphi_{12,x_0x_0}+\varphi_{12,zz} = -2\omega A_{,x_1}e^{i\theta+kz}+\text{c.c.} \quad \text{in} \quad \Omega_1, \tag{10}$$

$$\varphi_{22,x_0x_0}+\varphi_{22,zz} = -2\omega\big\{(\rho_1\omega^2-k\rho_1+k\rho_2-T_1k^3)\cosh k(h_2-z)$$

$$+\left[\rho_2\omega^2\cosh kh_2+(\rho_1\omega^2-k\rho_1+k\rho_2-T_1k^3)\right.$$

$$\left.\times\sinh kh_2\right]\sinh kz\big\}\frac{A_{,x_1}e^{i\theta}}{\rho_2\omega^2\cosh kh_2}+\text{c.c.} \quad \text{in} \quad \Omega_2, \tag{11}$$

$$\varphi_{32,x_0x_0}+\varphi_{32,zz} = 2\omega\big\{\left[\rho_2\omega^2\cosh kh_2+(\rho_1\omega^2-k\rho_1+k\rho_2-T_1k^3)\sinh kh_2\right]$$

$$\times\cosh k(h_2+h_3-z)\big\}\frac{A_{,x_1}e^{i\theta}}{\rho_2\omega^2\sinh kh_3}+\text{c.c.} \quad \text{in} \quad \Omega_3, \tag{12}$$

$$\eta_{12,t_0} + \varphi_{12,z} = -A_{,t_1} e^{i\theta} - 2i\,k\,\omega\,A^2 e^{2i\theta} + \text{c.c.} \quad \text{at} \quad z=0, \tag{13}$$

$$\eta_{12,t_0} + \varphi_{22,z} = -A_{,t_1} e^{i\theta} - 2i\,k\omega\left(\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1 k^3\right)\frac{A^2 e^{2i\theta}}{\rho_2\omega^2} + \text{c.c.} \quad \text{at} \quad z=0, \tag{14}$$

$$\begin{aligned}\eta_{22,t_0} + \varphi_{22,z} = &-\left[\rho_2\omega^2 \cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\right]\frac{A_{,t_1}e^{i\theta}}{\rho_2\omega^2}\\ &- 2i\,k\omega\Big\{\left[\rho_2\omega^2 \cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\right]\\ &\times(\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\\ &+\left[\rho_2\omega^2 \cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\right]^2\\ &\times \sinh kh_2\Big\}\frac{A^2e^{2i\theta}}{\rho_2^2\omega^4\cosh kh_2} + \text{c.c.} \quad \text{at} \quad z=h_2,\end{aligned} \tag{15}$$

$$\begin{aligned}\eta_{22,t_0} + \varphi_{32,z} = &-\left[\rho_2\omega^2 \cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\right]\frac{A_{,t_1}e^{i\theta}}{\rho_2\omega^2}\\ &+ 2ik\omega\Big\{\left[\rho_2\omega^2 \cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\right]^2\\ &\times \cosh kh_3\Big\}\frac{A^2e^{2i\theta}}{\rho_2^2\omega^4\sinh kh_3} + \text{c.c.} \quad \text{at} \quad z=h_2,\end{aligned} \tag{16}$$

$$\begin{aligned}&\rho_1\varphi_{12,t_0} - \rho_2\varphi_{22,t_0} + (\rho_1-\rho_2)\,\eta_{12} - T_1\eta_{12,x_0x_0}\\ &= 0.5\left[\rho_2\left(\frac{(\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)^2}{\rho_2^2\omega^2} - \rho_2\omega^2\right)\right]A\bar{A}\\ &\quad+\left[\left[i\rho_1\omega^2 - i(\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\right]\frac{A_{,t_1}}{\omega k} - 2iT_1kA_{,x_1}\right]e^{i\theta}\\ &\quad+\Bigg[(\rho_1-\rho_2)\,\omega^2 + 0.5\rho_2\Bigg((\rho_1\omega^2 - k\rho_1\end{aligned}$$

$$+ k\rho_2 - T_1 k^3)^2 \frac{1}{\rho_2^2\omega^2} - \omega^2\Big)\Big]\Big] A^2 e^{2i\theta} + \text{c.c.} \quad \text{at} \quad z = 0, \tag{17}$$

$$\rho_2\varphi_{22,t_0} - \rho_3\varphi_{32,t_0} + (\rho_2-\rho_3)\eta_{22} - T_2\eta_{22,x_0x_0}$$

$$= 0.5\Bigg[(\rho_2-\rho_3)\Big(\big[\rho_2\omega^2\cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\big]^2 \frac{1}{\rho_2^2\omega^2}\Big)$$

$$+ \rho_3\Bigg(\Big[\Big(\rho_2\omega^2\cosh kh_2 + (\rho_1\omega^2 - k\rho_1$$

$$+ k\rho_2 - T_1k^3)\sinh kh_2\Big)\cosh kh_3\Big]\frac{1}{\rho_2\omega\sinh kh_3}\Bigg)^2$$

$$- \rho_2\Bigg(\Big[(\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3) + \Big(\rho_2\omega^2\cosh kh_2$$

$$+ (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\Big)\sinh kh_2\Big]$$

$$\times \frac{1}{\rho_2\omega\cosh kh_2}\Bigg)^2\Bigg] A\bar{A} + \Bigg[\Bigg(\Big(i\rho_2\Big[(\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)$$

$$+ \Big(\rho_2\omega^2\cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\Big)$$

$$\times \sinh kh_2\Big]\frac{1}{\rho_2\omega k\cosh kh_2} + i\rho_3\Big[\Big(\rho_2\omega^2\cosh kh_2 + (\rho_1\omega^2$$

$$- k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\Big)\cosh kh_3\Big]\frac{1}{\rho_2\omega k\sinh kh_3}\Bigg) A_{,t_1}$$

$$+ \Big(\Big[-(i\rho_2h_2 + i\rho_3h_3)\Big(\rho_2\omega^2\cosh kh_2 + \Big(\rho_1\omega^2 - k\rho_1 + k\rho_2$$

$$- T_1k^3\Big)\sinh kh_2\Big)\Big]\frac{1}{\rho_2 k} + \Big[2iT_2k\Big(\rho_2\omega^2\cosh kh_2 + (\rho_1\omega^2$$

$$- k\rho_1 + k\rho_2 - T_1k^3\Big)\sinh kh_2\Big)\Big]\frac{1}{\rho_2\omega^2}\Big) A_{,x_1}\Bigg] e^{i\theta}$$

$$+\Bigg[(1.5\rho_2 - 1.5\rho_3)\Big[\rho_2\omega^2 \cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)$$

$$\times \sinh kh_2\Big]^2 \frac{1}{\rho_2^2\omega^2} - 0.5\rho_2\Big[\rho_2\omega^2 \sinh kh_2 + (\rho_1\omega^2 - k\rho_1$$

$$+ k\rho_2 - T_1k^3)\cosh kh_2\Big]^2 \frac{1}{\rho_2^2\omega^2} + 0.5\rho_3\Big[\Big(\rho_2\omega^2 \cosh kh_2$$

$$+ (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\Big)\cosh kh_3\Big]^2$$

$$\times \frac{1}{(\rho_2\omega \sinh kh_3)^2}\Bigg]A^2e^{2i\theta} + \text{c.c.} \quad \text{at} \quad z = h_2. \tag{18}$$

Here and in what follows, by c.c., we denote the quantities complex conjugate to the preceding terms and

$$A_{,x_1} = \frac{\partial A}{\partial x_1} \quad \text{and} \quad A_{,t_1} = \frac{\partial A}{\partial t_1}$$

are the partial derivatives of the envelope of wave packet on the bottom contact surface $z = \eta_1(x,t)$.

We seek the solution of system (10)–(18) in the form

$$\varphi_{12} = \Big(B_{10}^{(2)} + B_{11}^{(2)}\cdot z\Big)e^{i\theta+kz} + B_{20}^{(2)}e^{2i\theta+2kz} + \text{c.c.},$$

$$\varphi_{22} = \Big(C_{10}^{(2)} + C_{11}^{(2)}\cdot z\Big)e^{i\theta+k(h_2-z)} + C_{20}^{(2)}e^{2i\theta+2k(h_2-z)}$$

$$+ \Big(D_{10}^{(2)} + D_{11}^{(2)}\cdot z\Big)e^{i\theta-k(h_2-z)} + D_{20}^{(2)}e^{2i\theta-2k(h_2-z)} + \text{c.c.},$$

$$\varphi_{32} = E_{10}^{(2)} \cosh k(h_2+h_3-z)e^{i\theta}$$

$$+ E_{11}^{(2)}(h_2+h_3-z)\sinh k(h_2+h_3-z)\,e^{i\theta}$$

$$+ E_{20}^{(2)} \cosh 2k(h_2+h_3-z)\,e^{2i\theta} + \text{c.c.}, \tag{19}$$

$$\eta_{12} = F_0^{(2)} + F_1^{(2)}e^{i\theta} + F_2^{(2)}e^{2i\theta} + \text{c.c.},$$

$$\eta_{22} = G_0^{(2)} + G_1^{(2)}e^{i\theta} + G_2^{(2)}e^{2i\theta} + \text{c.c.},$$

where $B_{ij}^{(2)}$, $C_{ij}^{(2)}$, $D_{ij}^{(2)}$, $E_{ij}^{(2)}$, $F_i^{(2)}$, and $G_i^{(2)}$ are unknown coefficients.

Substituting relations (19) for the unknown functions and the solutions of the problem in the first approximation [9] in Eqs. (10)–(12), we can easily determine the coefficients $B_{11}^{(2)}$, $C_{11}^{(2)}$, $D_{11}^{(2)}$, and $E_{11}^{(2)}$.

Further, substituting (19) in conditions (13)–(18) and equating the coefficients of the same functions, we arrive at two independent systems of equations for the remaining unknown coefficients. Thus, equating the expressions at the function $e^{i\theta}$, we obtain a system of equations for the coefficients $B_{10}^{(2)}$, $C_{10}^{(2)}$, $D_{10}^{(2)}$, $E_{10}^{(2)}$, $F_1^{(2)}$, and $G_1^{(2)}$:

$$
\begin{aligned}
&-kB_{10}^{(2)} - i\omega F_1^{(2)} = b_1,\\
&ke^{kh_2}C_{10}^{(2)} - ke^{-kh_2}D_{10}^{(2)} - i\omega F_1^{(2)} = b_2,\\
&kC_{10}^{(2)} - kD_{10}^{(2)} - i\omega G_1^{(2)} = b_3,\\
&k\sinh kh_3 E_{10}^{(2)} - i\omega G_1^{(2)} = b_4,\\
&-i\rho_1\omega B_{10}^{(2)} + i\rho_2\omega\, e^{kh_2}C_{10}^{(2)} + i\rho_2\omega\, e^{-kh_2}D_{10}^{(2)} + (\rho_1-\rho_2+T_1k^2)\,F_1^{(2)} = b_5,\\
&-i\rho_2\omega C_{10}^{(2)} - i\rho_2\omega D_{10}^{(2)} + i\rho_3\omega\cosh kh_3 E_{10}^{(2)} + (\rho_2-\rho_3+T_2k^2)\,G_{10}^{(2)} = b_6
\end{aligned}
\tag{20}
$$

with the free terms

$$b_1 = -A_{,t_1} - \frac{\omega}{k}A_{,x_1},$$

$$b_2 = -A_{,t_1} - \frac{\omega}{k}A_{,x_1},$$

$$
\begin{aligned}
b_3 = {} & -\left[\rho_2\omega^2\cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\right]\frac{A_{,t_1}}{\rho_2\omega^2}\\
& -\left[\rho_2\omega^2\,(kh_2\sinh kh_2 + \cosh kh_2) + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\right.\\
& \left.\times\,(kh_2\cosh kh_2 + \sinh kh_2)\right]\frac{A_{,x_1}}{k\rho_2\omega},
\end{aligned}
$$

$$b_4 = -\left[(\rho_2\omega^2\cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2)\right]\frac{A_{,t_1}}{\rho_2\omega^2}$$

$$- \left[\rho_2 \omega^2 \cosh kh_2 + (\rho_1 \omega^2 - k\rho_1 + k\rho_2 - T_1 k^3) \sinh kh_2 \right]$$

$$\times \left(1 + \frac{kh_3 \cosh kh_3}{\sinh kh_3} \right) \frac{A_{,x_1}}{k\rho_2 \omega},$$

$$b_5 = -i \left[\rho_1 \omega^2 - (\rho_1 \omega^2 - k\rho_1 + k\rho_2 - T_1 k^3) \right] \frac{A_{,t_1}}{k\omega} + 2iT_1 k A_{,x_1},$$

$$b_6 = \left(\frac{i\rho_2 (\rho_1 \omega^2 - k\rho_1 + k\rho_2 - T_1 k^3)}{\rho_2 \omega^2 k \cosh kh_2} + \left(i\rho_2 \frac{\sinh kh_2}{\cosh kh_2} + i\rho_3 \frac{\cosh kh_3}{\sinh kh_3} \right) \right.$$

$$\left. \times \left[\rho_2 \omega^2 \cosh kh_2 + (\rho_1 \omega^2 - k\rho_1 + k\rho_2 - T_1 k^3) \sinh kh_2 \right] \frac{1}{\rho_2 \omega^2 k} \right) A_{,t_1}$$

$$+ \left(2iT_2 k \left[\rho_2 \omega^2 \cosh kh_2 + (\rho_1 \omega^2 - k\rho_1 \right. \right.$$

$$\left. + k\rho_2 - T_1 k^3) \sinh kh_2 \right] \frac{1}{\rho_2 \omega^2} - i(\rho_2 h_2 + \rho_3 h_3)$$

$$\left. \times \left[\rho_2 \omega^2 \cosh kh_2 + (\rho_1 \omega^2 - k\rho_1 + k\rho_2 - T_1 k^3) \sinh kh_2 \right] \frac{1}{k\rho_2} \right) A_{,x_1}.$$

System (20) is inconsistent. For $F_1^{(2)} = 0$, the condition of its solvability takes the following form:

$$\begin{vmatrix} -k & 0 & 0 & 0 & b_1 & 0 \\ 0 & ke^{kh_2} & -ke^{-kh_2} & 0 & b_2 & 0 \\ 0 & k & -k & 0 & b_3 & -i\omega \\ 0 & 0 & 0 & k \sinh kh_3 & b_4 & -i\omega \\ -i\rho_1 \omega & i\rho_2 \omega\, e^{kh_2} & i\rho_2 \omega\, e^{-kh_2} & 0 & b_5 & 0 \\ 0 & -i\rho_2 \omega & -i\rho_2 \omega & i\rho_3 \omega \cosh kh_3 & b_6 & \rho_2 - \rho_3 + T_2 k^2 \end{vmatrix} = 0.$$

Separating the terms with the derivatives $A_{,x_1}$ and $A_{,t_1}$ of the envelope of the wave packet, we rewrite the condition of solvability in the following form:

$$V_1 A_{,t_1} + V_2 A_{,x_1} = 0, \tag{21}$$

where V_i, $i = 1, 2$, are coefficients depending on ρ_1, ρ_2, ρ_3, k, h_2, h_3, T_1, T_2, and ω.

If condition (21) is satisfied, then the system for the coefficients $B_{10}^{(2)}$, $C_{10}^{(2)}$, $D_{10}^{(2)}$, $E_{10}^{(2)}$, and $G_1^{(2)}$ takes the form

$$-kB_{10}^{(2)} = b_1,$$

$$ke^{kh_2}C_{10}^{(2)} - ke^{-kh_2}D_{10}^{(2)} = b_2,$$

$$kC_{10}^{(2)} - kD_{10}^{(2)} - i\omega G_1^{(2)} = b_3,$$

$$k\sinh kh_3 E_{10}^{(2)} - i\omega G_1^{(2)} = b_4,$$

$$-i\rho_2\omega C_{10}^{(2)} - i\rho_2\omega D_{10}^{(2)} + i\rho_3\omega\cosh kh_3 E_{10}^{(2)} + (\rho_2 - \rho_3 + T_2k^2)G_{10}^{(2)} = b_6$$

and is solvable. Its solution was obtained in the package of symbolic calculations and we do not present it here.

The system for the coefficients $B_{20}^{(2)}$, $C_{20}^{(2)}$, $D_{20}^{(2)}$, $E_{20}^{(2)}$, $F_2^{(2)}$, and $G_2^{(2)}$ obtained by equating the coefficients of $e^{2i\theta}$ is consistent. The unknown coefficients are also found in the package of symbolic calculations.

We consider the last two conditions (17) and (18). Equating the coefficients of e^0, we get the values of the coefficients F_0 and G_0:

$$F_0 = \left(\frac{0.5}{\rho_1 - \rho_2}\left(-\rho_2\left((\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)^2\frac{1}{\rho_2^2\omega^2}\right) + \rho_2\omega^2\right)\right)A\bar{A},$$

$$\begin{aligned} G_0 = {} & \left(\frac{0.5}{\rho_2 - \rho_3}\left(-(\rho_2 - \rho_3)\left(\left(\rho_2\omega^2\cosh kh_2 + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\right.\right.\right.\right. \\ & \left.\times\sinh kh_2\right)^2\frac{1}{\rho_2^2\omega^2}\Bigg) - \rho_3\Bigg(\Big(\rho_2\omega^2\cosh kh_2 + (\rho_1\omega^2 \\ & - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\Big)\cosh kh_3\frac{1}{\rho_2\omega\sinh kh_3}\Bigg)^2 \\ & + \rho_2\Bigg(\Big\{(\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3) + \Big[\rho_2\omega^2\cosh kh_2 \\ & + (\rho_1\omega^2 - k\rho_1 + k\rho_2 - T_1k^3)\sinh kh_2\Big]\sinh kh_2\Big\}\frac{1}{\rho_2\omega\cosh kh_2}\Bigg)^2\Bigg)\Bigg)A\bar{A}. \end{aligned}$$

Thus, the solutions in the second approximation take the form:

$$\eta_{12} = F_0 A\bar{A} + \Lambda_1 A^2 e^{2i\theta} + \text{c.c.},$$

$$\eta_{22} = G_0 A\bar{A} + S_1 A_{,t_1} e^{i\theta} + S_2 A_{,x_1} e^{i\theta} + \Lambda_2 A^2 e^{2i\theta} + \text{c.c.},$$

$$\varphi_{12} = \frac{1}{k}\left(A_{,t_1} + \frac{\omega}{k}(1-kz)\, A_{,x_1} \right) e^{i\theta+kz} + \frac{i\omega}{k} S_3 A^2 e^{2(i\theta+kz)} + \text{c.c.}, \tag{22}$$

$$\varphi_{22} = \frac{1}{k}\left(S_{41} A_{,t_1} + \frac{\omega}{k}(S_{42} + S_{43} z) A_{,x_1} \right) e^{i\theta} e^{k(h_2-z)}$$

$$+ \frac{1}{k}\left(S_{44} A_{,t_1} + \frac{\omega}{k}(S_{45} + S_{46} z) A_{,x_1} \right) e^{i\theta} e^{-k(h_2-z)}$$

$$+ \frac{i\omega}{k}\left(S_{47} e^{2k(h_2-z)} + S_{48} e^{-2k(h_2-z)} \right) A^2 e^{2i\theta} + \text{c.c.},$$

$$\varphi_{32} = \frac{1}{k}\Big(S_{51} \cosh k(h_2+h_3-z) A_{,t_1} + \frac{\omega}{k}(S_{52} \cosh k(h_2+h_3-z)$$

$$+ S_{53}(h_2+h_3-z) \sinh k(h_2+h_3-z))\, A_{,x_1} \Big) e^{i\theta}$$

$$+ \frac{i\omega}{k} S_{54} \cosh 2k(h_2+h_3-z) A^2 e^{2i\theta} + \text{c.c.},$$

where S_1, S_2, S_3, S_{41}, S_{42}, S_{43}, S_{44}, S_{45}, S_{46}, S_{47}, S_{48}, S_{51}, S_{52}, S_{53}, S_{54}, Λ_1, and Λ_2 are coefficients depending on ρ_1, ρ_2, ρ_3, k, h_2, h_3, T_1, T_2, and ω. In this case, in view of the conditions of solvability (21), we get $S_1 = S_2 = 0$.

3. Analysis of the Shape of Wave Packet on the Contact Surfaces

To determine the shape of deviations of the contact surfaces, it is important to know the signs of the coefficients Λ_1 and Λ_2 of the other harmonics on the surfaces. By using the expressions for the first [1] and second (22) approximations to the deviations of contact surfaces, we construct the domains of sign constancy for Λ_1 and Λ_2 and analyze the shapes of waves on the contact surfaces. The analysis is performed for a fixed value of density of the bottom layer $\rho_1 = 1$.

We consider the following two cases:

Case 1. Assume that the densities of the bottom half space ρ_1 and the middle layer ρ_2 are fixed: $\rho_1 = 1$ and $\rho_2 = 0.9$, respectively, and that the density of the top layer ρ_3 varies from 0 to ρ_2. The other parameters of the system are as follows: $0 \le k \le 3.5$, $h_2 = 1$, and $T_1 = T_2 = 0$.

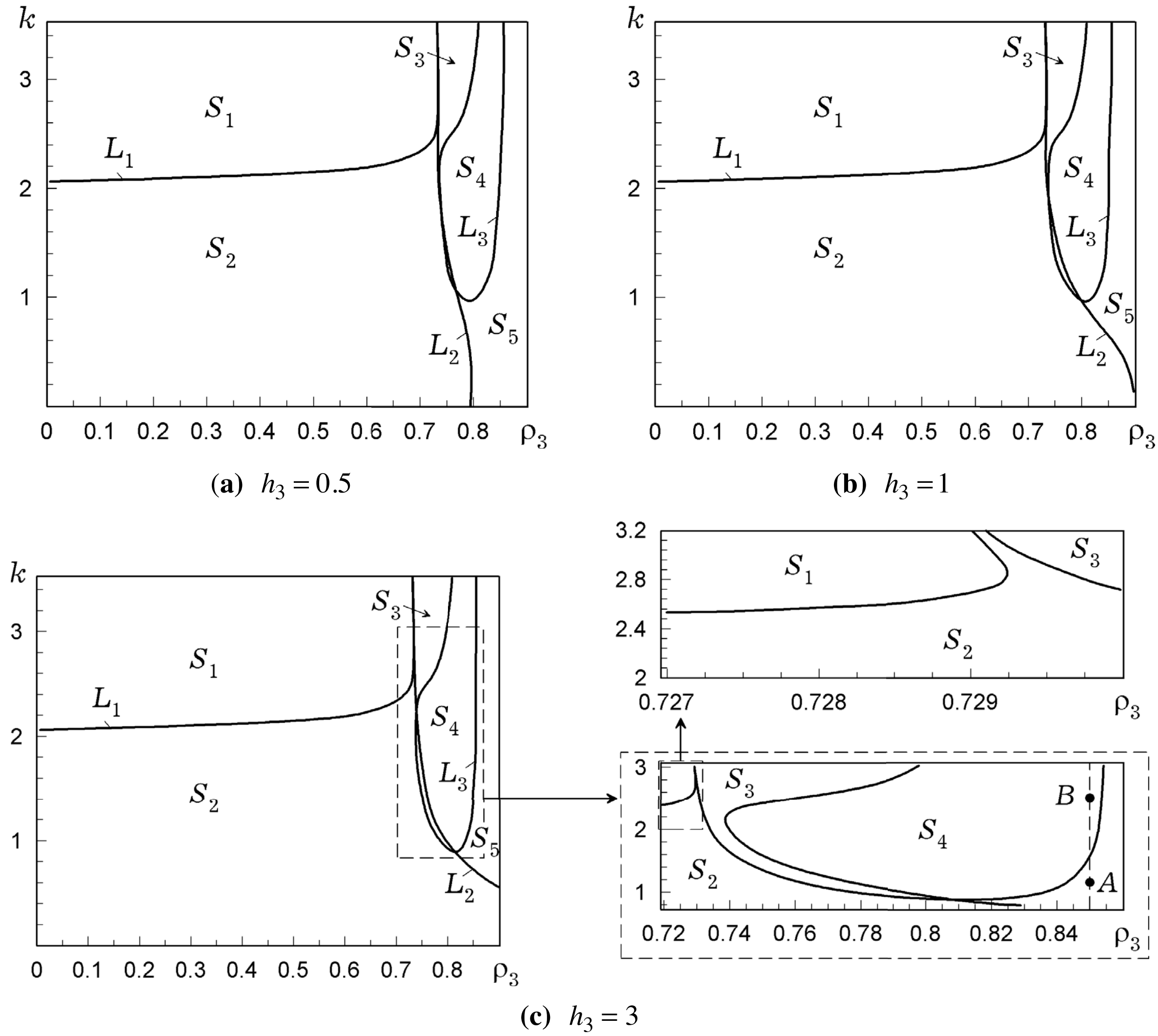
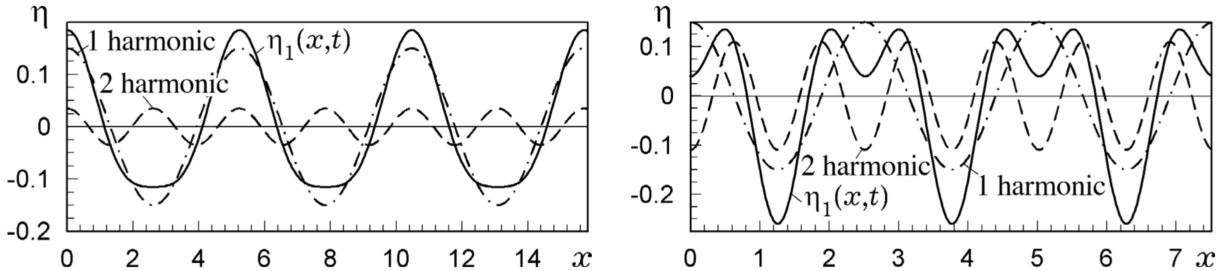


Fig. 2. Domains of sign constancy for Λ_1 in the plane (ρ_3, k).

1.1. Shape of Deviation of the Bottom Contact Surface. The shape of deviation of the bottom contact surface $\eta_1(x,t)$ depends on the sign of the coefficient Λ_1 of the second harmonic. The analysis of the sign of Λ_1 shows that the curves L_1 and L_2 along which Λ_1 is equal to zero exist in the plane (ρ_3, k). Moreover, there exists a curve L_3 in the vicinity of which Λ_1 takes arbitrarily large values. In passing through the curves L_1, L_2, and L_3, the sign of the coefficient Λ_1 changes into the opposite.

As follows from Fig. 2, the curves L_1, L_2, and L_3 split the plane (ρ_3, k) into five domains. In the domains S_1, S_3, and S_5, where the coefficient Λ_1 of the second harmonic takes positive values, the waves have sharp crests and blunt troughs. In the remaining two domains S_2 and S_4, where $\Lambda_1 < 0$, we observe waves with blunt crests and sharp troughs.

It was discovered that the area of the domain S_2 in which long waves are $\cap$-shaped increases with the thickness of the top layer h_3 (Figs. 2a, b). The accumulated results also demonstrate that the domains S_1 and S_3 are separated by a narrow part of the domain S_2; moreover, a narrow part of the domain S_3 is located between the domains S_2 and S_4 (Fig. 2c).



(a) $k = 1.2$ **(b)** $k = 2.5$

Fig. 3. Plots of the shape of deviations of the inner bottom contact surface $\eta_1(x,t)$.

In Fig. 2c, we mark the points A and B for which the plots of the shape of deviations of the bottom contact surface $\eta_1(x,t)$ are presented in Fig. 3. These points are chosen to analyze the changes in the shape of waves in passing the curve L_3. In the vicinity of the curve L_3, the value of Λ_1 changes its sign from positive to negative and takes arbitrarily large values. In this case, we observe a significant influence of the second harmonic on the shape of the contact surface. The plots of the shape of deviations of the bottom contact surface $\eta_1(x,t)$ (Fig. 3) are shown for the frequency of the wave packet ω_1 and the following parameters of the system: $\rho_1 = 1$, $\rho_2 = 0.9$, $\rho_3 = 0.85$, $h_2 = 1$, $h_3 = 3$, $T_1 = T_2 = 0$, $a = 0.15$, and $\alpha = 0.1$ at the point A (Fig. 3a), where $k = 1.2$, and at the point B (Fig. 3b), where $k = 2.5$ (the positions of these points are shown in Fig. 2c).

As follows from Fig. 3a, the amplitude of the first harmonic is larger than the amplitude of the second harmonic and the period of the second harmonic is twice smaller. Therefore, the overlapping of the minimum of the first harmonic and the maximum of the second harmonic results in the effect of blunting of the troughs on the contact surface. In the case of overlapping of the maximum of the first harmonic and the next maximum of the second harmonic, we observe the effect of sharpening of the wave crest. Hence, the deviations of the bottom contact surface in the domain S_5 become $\bigcup$-shaped.

The plot of deviations of the contact surface in Fig. 3b shows the shape of waves in the domain S_4, where Λ_1 takes negative values. In this case, we can see the picture in which the effect of overlapping of minima of the first and second harmonics results in sharpening of the wave troughs. The overlapping of the maximum of the first harmonic and the next minimum of the second harmonic results in blunting of the wave crests. Hence, the domain S_4 contains $\bigcap$-shaped waves.

It was also discovered that, within the limits of a single domain, there exists a significant influence of the wave number k on the amplitudes of harmonics for the same remaining parameters of the system.

1.2. Shape of Deviations of the Top Contact Surface. In Fig. 4, we present the constructed domains of sign constancy for the coefficient Λ_2, the frequency of the wave packet ω_2, and the following values of the parameters: $\rho_1 = 1$, $\rho_2 = 0.9$, $0 \le \rho_3 \le \rho_2$, $0 \le k \le 4.5$, $h_2 = 1$, and $T_1 = T_2 = 0$.

As follows from Fig. 4, the curves L_4 and L_5 along which the coefficient Λ_2 is equal to zero decompose the plane (ρ_3, k) into three domains. In the domains S_6 and S_8, the coefficient Λ_2 takes positive values. In the domain S_7, the coefficient $\Lambda_2 < 0$. In this case, the area of the domain S_7 decreases, as the thickness of the top layer h_3 increases. We discovered no curves in the vicinities of which Λ_2 may take arbitrarily large values.

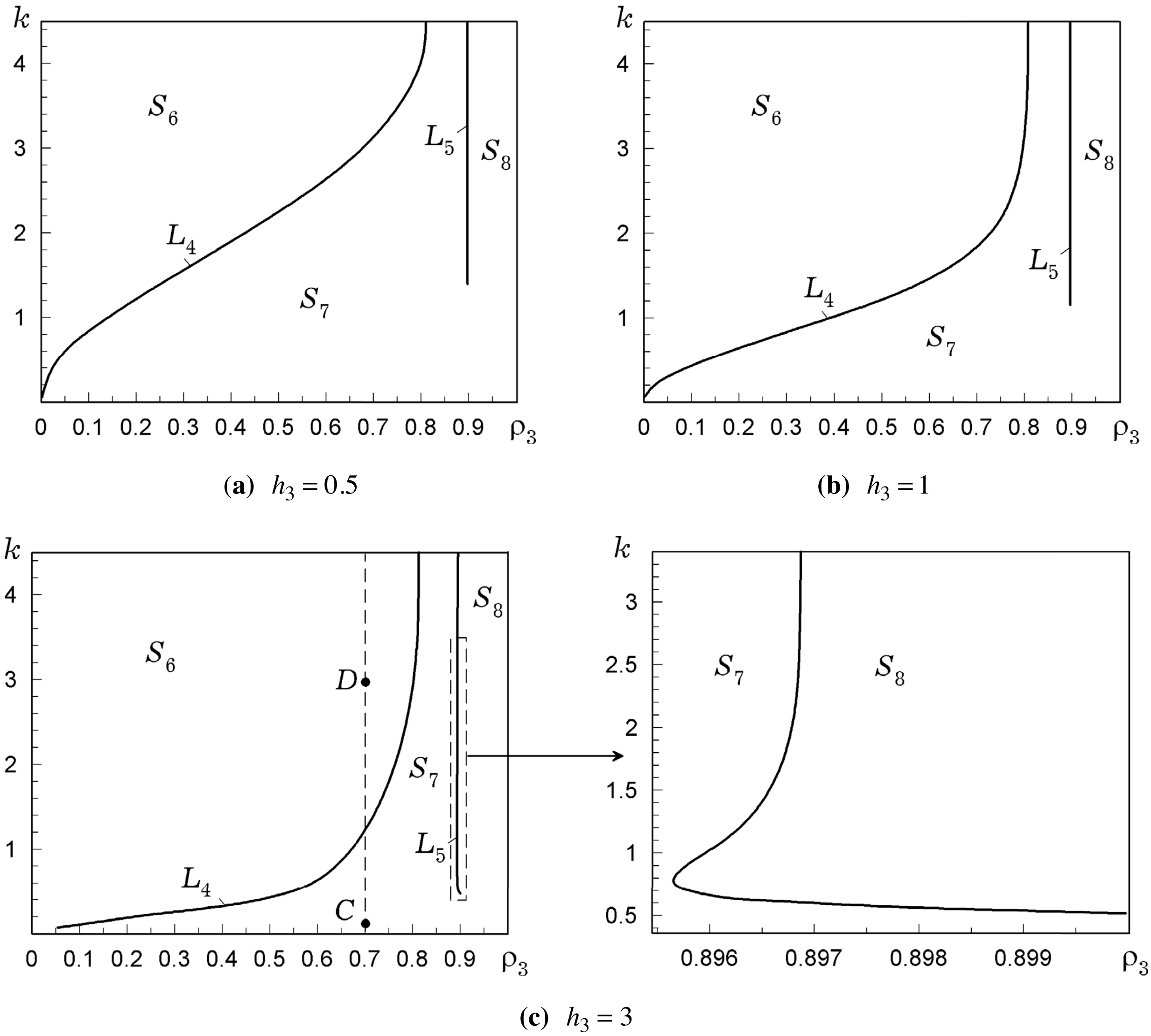


Fig. 4. Domains of sign constancy for Λ_2 in the plane (ρ_3, k).

We construct the plots of deviations of the top contact surface for the following parameters of the system: $\rho_1 = 1$, $\rho_2 = 0.9$, $\rho_3 = 0.7$, $h_2 = 1$, $h_3 = 3$, $T_1 = T_2 = 0$, $a = 0.15$, and $\alpha = 0.1$ at the points C and D (see Fig. 4c) for which $k = 0.1$ and $k = 3$, respectively.

As follows from Fig. 5a, in the case where the coefficient Λ_2 takes negative values, we observe the overlapping of minima of the first and second harmonics creating the effect of sharpening of the wave troughs. Moreover, as a result of overlapping of the maximum of the first harmonic with the next minimum of the second harmonic, we observe the effect of blunting of the wave crests.

In Fig. 5b, we present the case where Λ_2 is positive. In this case, the overlapping of the maxima of the first and second harmonics leads to sharpening of the wave crests. Moreover, the overlapping of the minimum of the first harmonic with the next maximum of the second harmonic results in the effect of blunting of the wave troughs. Hence, the wave has the $\bigcup$-like shape in the domains S_6 and S_8 and the $\bigcap$-like shape in the domain S_7.

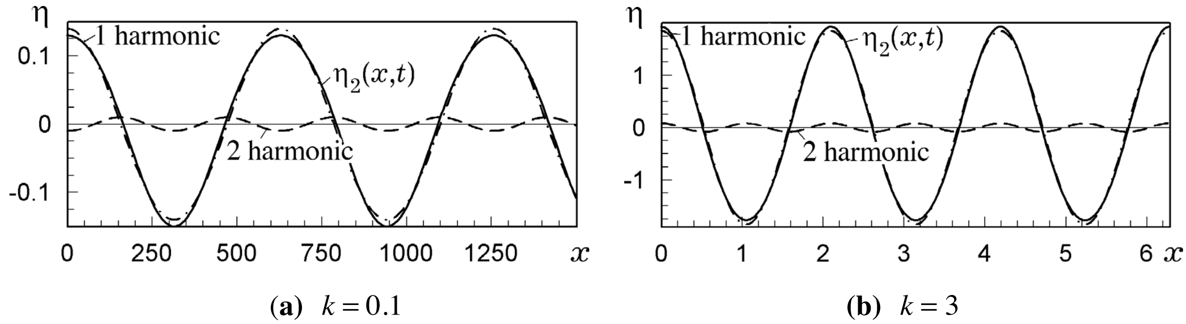


Fig. 5. Plots of the shape of deviations of the top inner contact surface $\eta_2(x,t)$.

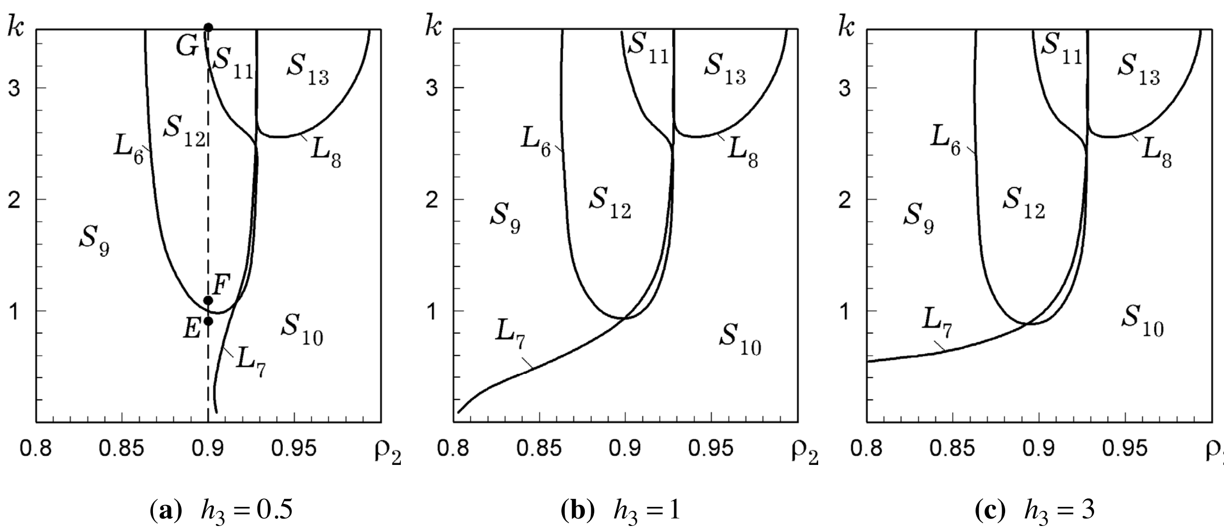
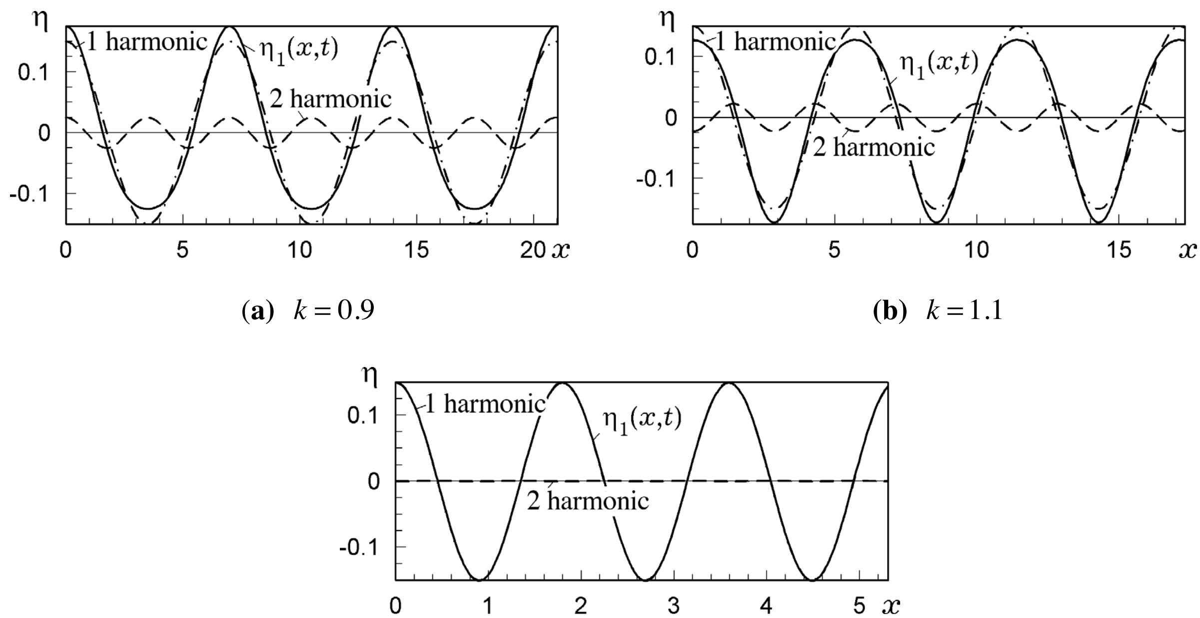


Fig. 6. Domains of sign constancy for Λ_1 in the plane (ρ_2, k).

Case 2. Assume that the density of the bottom half space $\rho_1 = 1$ and that the density of the top layer $\rho_3 = 0.8$. Suppose that the density of the middle layer ρ_2 varies from ρ_3 to ρ_1. The other parameters are as follows: $0 \le k \le 3.5$, $h_2 = 1$, and $T_1 = T_2 = 0$.

2.1. Shape of Deviations of the Bottom Contact Surface. In Fig. 6, we present the domains of sign constancy for the coefficient Λ_1 of the second harmonic in the bottom contact surface. The plots are constructed for different values of h_3 and the following parameters of the system: $\rho_1 = 1$, $\rho_3 \le \rho_2 \le \rho_1$, $\rho_3 = 0.8$, $h_2 = 1$, and $T_1 = T_2 = 0$.

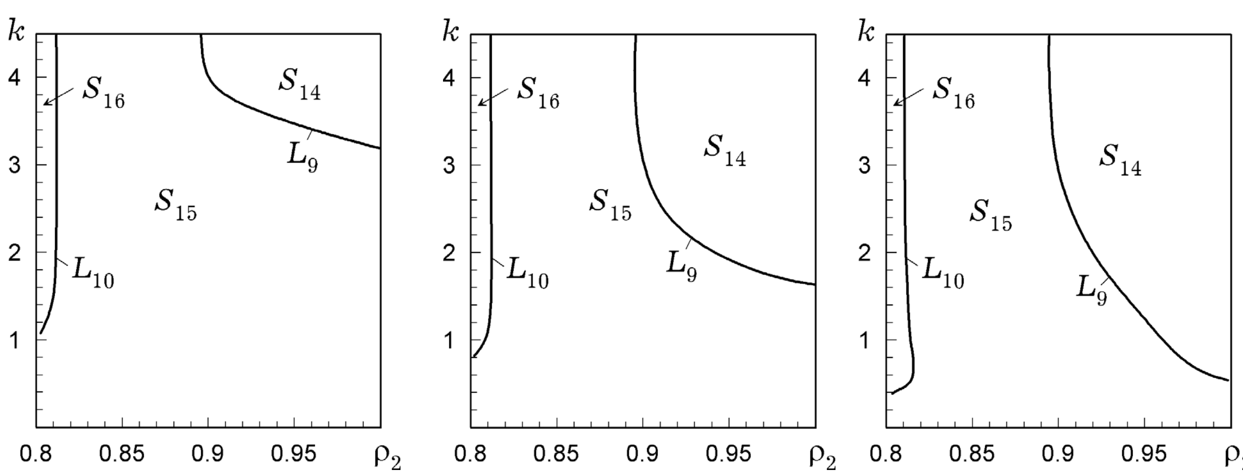
We revealed the curves L_6, L_7, and L_8 decomposing the plane (ρ_2, k) into five domains. In passing through these curves, the sign of the coefficient Λ_1 of the second harmonic on the bottom contact surface changes into the opposite. In this case, the coefficient Λ_1 is equal to zero along the curves L_7 and L_8 and takes arbitrarily large values along the curve L_6. In the domains S_9, S_{11}, and S_{13} (where the coefficient $\Lambda_1 > 0$), the waves are $\bigcup$- shaped, whereas in the domains S_8 and S_{10} (where the coefficient $\Lambda_1 < 0$), the waves are $\bigcap$- shaped.



(a) $k = 0.9$ (b) $k = 1.1$

(c) $k = 3.5$

Fig. 7. Plots of the shapes of deviations of the bottom contact surface $\eta_1(x,t)$.



(a) $h_3 = 0.5$ (b) $h_3 = 1$ (c) $h_3 = 3$

Fig. 8. Domains of sign constancy for Λ_2 in the plane (ρ_2, k).

Note that, as in the already analyzed ***Case 1*** for Λ_1 (see Fig. 2), the thickness of the top layer h_3 affects the area of the domain S_{10}: as the thickness of the top layer increases, the area of the domain S_{10} increases in its part, where the long waves are $\cap$-shaped. We also observe the presence of narrow domains with the different signs of the coefficient.

In Fig. 6a, we marked three points E, F, and G ($k = 0.9$, $k = 1.1$, $k = 3.5$, respectively) for which the plots (Fig. 7) of the shapes of deviations of the bottom contact surface $\eta_1(x,t)$ are constructed for the following parameters of the system: $\rho_1 = 1$, $\rho_2 = 0.9$, $\rho_3 = 0.85$, $h_2 = 1$, $h_3 = 0.5$, $T_1 = T_2 = 0$, and $a = 0.15$.

The plots of deviations of the bottom contact surface $\eta_1(x,t)$ in Figs. 7a, b correspond to the points E and F lying in the domains separated by the curve L_6. The point E belongs to the domain S_9, where the coefficient Λ_1 is positive. Therefore, the waves are $\bigcup$- shaped. The point F belongs to the domain S_{12} (where $\Lambda_1 < 0$). Therefore, the waves are $\bigcap$- shaped.

The plot of the shape of deviations of the surface presented in Fig. 7c reveals the shape of waves in the domain S_{11} (where the coefficient $\Lambda_1 > 0$) separated from the domain S_{12} by the curve L_7. Along this curve, the value of Λ_1 is equal to zero. Therefore, we observe the decay of the second harmonic in its neighborhood.

2.2. Shape of Deviations of the Top Contact Surface. In Fig. 8, we show the domains of sign constancy for the coefficient Λ_2 of the second harmonic on the top contact surface, for the frequency of the wave packet ω_2 and the following parameters of the system: $\rho_1 = 1$, $\rho_3 = 0.8$, $0 \le k \le 4.5$, $h_2 = 1$, and $T_1 = T_2 = 0$.

We observe two curves L_9 and L_{10} (along which $\Lambda_2 = 0$) decomposing the plane (ρ_2, k) into three domains. In the domains where Λ_2 is positive (S_{14} and S_{16}), the waves are $\bigcup$- shaped. In the domain S_{15}, Λ_2 is negative. Therefore, the waves are $\bigcap$- shaped. By analogy with ***Case 1*** for the coefficient Λ_2 (Fig. 4), the area of the domain S_{15} decreases as the thickness of the top layer h_3 increases.

CONCLUSIONS

We have studied a weakly nonlinear problem of propagation of wave packets in a "half space–layer–layer with rigid lid" system. By the method of multiscale expansions up to the third order, we have posed the problem in the second approximation, established the condition of its solvability, and found its solutions. For any frequency of the center of wave packet, we constructed the domains of sign constancy for the coefficient of the second harmonic in the bottom and top contact surfaces.

The following effects and regularities were discovered:

– In the domains of the planes (ρ_2, k) and (ρ_3, k), where the coefficients Λ_1 and Λ_2 of the second harmonics in the bottom and top contact surfaces, respectively, are positive, the waves take the $\bigcup$-like shapes. In the case where Λ_1 and Λ_2 are negative, the waves are $\bigcap$- shaped.

– The capillary waves are mostly $\bigcup$-shaped. In this case, we revealed two narrow domains with $\bigcap$-like shapes.

– The thickness of the top layer affects the domains with $\bigcup$-like and $\bigcap$-like waves. In particular, as the thickness of the top layer increases, the area of the domain in which the long waves are $\bigcap$-shaped also increases, whereas the domains with $\bigcup$-like waves become narrower;

– We revealed the domains of vanishing of the second harmonic and the domains in which the second harmonic strongly affects the shape of the contact surface.

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