Locally-zero binary operations as elements of $\mathrm{Bin}(X)$

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Let X be a nonempty set. Denote by $\mathrm{Bin}(X)$ the collection of all binary operations on X. In [1] the binary operation $\square$ on $\mathrm{Bin}(X)$ was considered: for $\circ, * \in \mathrm{Bin}(X)$ put $a(\circ\square*)b = (a\circ b)*(b\circ a)$ for all $a, b \in X$. It can be proved that $\square$ is associative (see [1]).

Denote by $\circ_{lz}$ and $\circ_{rz}$ the *'left-zeroes' operation* (defined as $a \circ_{lz} b = a$ for all $a, b \in X$) and the *'right-zeroes' operation* (defined as $a \circ_{lz} b = b$ for all $a, b \in X$), respectively. It is easy to see that $\circ_{lz}$ is the neutral element for $\square$. An operation $\circ \in Bin(X)$ on X is called *locally-zero* if the restriction of $\circ$ on any two-element subset of X equals $\circ_{lz}$ or $\circ_{rz}$. Clearly, both $\circ_{lz}$ and $\circ_{rz}$ are locally-zero operations.

It was proved in [2] that $\circ \in \mathrm{Bin}(X)$ lies in the center of the semigroup $(\mathrm{Bin}(X), \square)$ if and only if $\circ$ is a locally-zero operation. As a corollary, we obtain that the set of all locally-zero operations is closed under the $\square$.

With each locally-zero operation $\circ \in \mathrm{Bin}(X)$ we naturally associate an (undirected) graph $G(\circ)$ on X defined as follows: $V(G(\circ)) = X$, $E(G(\circ)) = \{ab : \circ \text{ is left-zero on } \{a, b\}\}$. It is easy to see that this correspondence between locally-zero operations and graphs on X is bijective. Denote by $\overline{G}$ the complement of G and by $G\triangle H$ the symmetric difference of two graphs G, H.

Proposition 1. *For a pair of locally-zero operations $\circ, * \in \mathrm{Bin}(X)$ it holds $G(\circ\square*) = \overline{G(\circ)\triangle G(*)}$.*

It was also shown in [2] that a locally-zero operation $\circ$ is associative if and only if $\circ \in \{\circ_{lz}, \circ_{rz}\}$. In fact, it is possible to obtain a characterization of associative triples for such operations $\circ$ in terms of their graphs $G(\circ)$. For a graph G and $A \subset V(G)$ by $E_G(A)$ we denote the set of edges in G whose vertices lie in A.

Proposition 2. *Let $\circ \in \mathrm{Bin}(X)$ be a locally-zero operation and $a, b, c \in X$. Then the triple (a, b, c) is* not *associative for $\circ$ if and only if a, b, c are pairwise distinct, and $E_{G(\circ)}(\{a, b, c\}) = \{ac\}$ or $E_{G(\circ)}(\{a, b, c\}) = \{ab, bc\}$.*

One can observe that $\circ\square\circ = \circ_{lz}$ for each locally-zero operation $\circ \in \mathrm{Bin}(X)$. Hence, any such $\circ$ is invertible in $(\mathrm{Bin}(X), \square))$. Moreover, a locally-zero operation $\circ$ does not have other inverses under $\square$.

Proposition 3. *Let $\circ \in \mathrm{Bin}(X)$ be a locally-zero operation and $*$ be its left or right inverse under $\square$. Then $* = \circ$.*

An operation $\circ \in \mathrm{Bin}(X)$ is called *graph operation* if $a \circ b \in \{a, b\}$ for all $a, b \in X$. Trivially, each locally-zero operation is also a graph operation.

Theorem 1. *For a graph operation $\circ \in \mathrm{Bin}(X)$ the next conditions are equivalent:* (1) *$\circ$ is left-invertible under $\square$;* (2) *$\circ$ is right-invertible under $\square$;* (3) *$\circ$ is locally-zero.*

1. Kim H. S., Neggers J. The semigroups of binary systems and some perspectives. Bull. Korean Math. Soc., 2008, 45, No. 4, 651 − 661.
2. Fayoumi H. Locally-zero groupoids and the center of *Bin*(X). Comm. Korean Math. Soc., 2011, 26, 163 − 168.