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DETERMINATION OF GROUPS ALL OF WHOSE PROPER SUBGROUPS HAVE A COMMUTATOR SUBGROUP OF ORDER EQUAL OR LESS THEN ρ ($p \geq 3$)

The authors are grateful to Prof. Z. Janko for proposition to regard the p-groups all of whose proper subgroups are either abelian or have a commutator subgroup of order equal or less then ρ It is easy to see that abelian p-groups and p-groups with a commutator subgroup of order ρ satisfy this condition. L. Szekeres and V. Sergejchuk have studied the p-groups with a commutator subgroup of order p. The aim of this paper is to investigate of the p-groups G $(p \geq 3)$ with a commutator subgroup of order > p, in which every proper subgroup is abelian or has a commutator subgroup of order p. We prove that G' is an abelian group and it is either an elementary abelian group or a cyclic group of order p^2. We give a description all such group by the relations. The isoclinism families containing these groups are pointed. Our determination will follow in five propositions covering all cases: is G' a cyclic or an elementary abelian, is G' in $Z(G)$ or not, has G an abelian maximal subgroup or not.

Introduction

It is well known that the properties of a group are connected to its subgroup structure. Sometimes to obtain the answer to the question whether the group has a subgroup of a given kind is more difficult than to obtain the list of all groups without subgroups of this kind. For example, Miller and Moreno described nonabelian groups all of whose proper subgroups are abelian ([1]). Due to this result one can practically answer the question whether a finite nonabelian group has a proper nonabelian subgroup or not?

In the same way, to find out if a finite nonabelian p-group contains a proper subgroups whose commutator subgroups have order greater than p we should describe all groups whose proper subgroups are abelian or have the commutator subgroup of order p.

It is clear that all abelian p-groups and all p-groups with the commutator subgroup of order p satisfy this condition.

The groups with the commutator subgroup of order p were studied by L. Szekeres ([2]) and V. Sergejchuk ([3]). They have shown that the group G with a commutator subgroup of order p is the central product of 2-generated subgroups $B_i = \langle g_i, h_i \rangle$. It has following relations for $p \geq 3$:

$$G = \langle g_i, h_i, z \mid g_i^{p^{m_1-1}} = z,$$
$$z^p = 1,\ g_i^{p^{m_i}} = 1,\ h_i^{p^{n_i}} = 1,\ [g_i, h_i] = z^{\varepsilon_i},$$
$$[g_i, g_j] = [g_i, h_j] = [h_i, h_j] = 1$$
$$(i, j = 1 \ldots n,\ i \neq j)\rangle$$

where m_i, n_i – integers: $m_i > 0$; $n_i \geq 0$, max $n_i > 0$; $m_i \geq n_i$ for $i > 1$, $m_1 > 1$ if $n_1 > 0$; $\varepsilon_i = 1$ if $n_i > 0$ and $\varepsilon_i = 1$ if $n_i \geq 0$.

It is easy to see that $|G| = p^{\Sigma(m_i + n_i)}$. The method to obtain all p-groups of given order with a commutator subgroup of order p up to isomorfism was described by V. Sergejchuk in [3].

Our aim is to describe all groups with the commutator subgroup of order greater than p all of whose proper subgroups are either abelian or have a commutator subgroup of order p.

Further we shell denote such group by G.

We show that the commutator subgroup G' is an elementary abelian or a cyclic group of the order p^2. We describe all such groups.

Our determination will follow in five theorems covering all such groups. We consider the following cases:

G' is a cyclic group;

G' is an elementary abelian, G' belongs to a center $Z(G)$ and G has a maximal subgroup, which is abelian;

G' is an elementary abelian, G' in $Z(G)$, but G has not any abelian maximal subgroup,

G' is an elementary abelian, $G' \not\subset Z(G)$ and G has a maximal subgroup, which is abelian,

G' is an elementary abelian, $G' \not\subset Z(G)$ and G has not any abelian maximal subgroup.

Here we determine these groups in terms of generators and relationship up to *isoclinism*. We can determine such groups up to isomorphism, but the full list of such groups is too large to be published in a journal article.

Two groups G, H with centers $Z(G)$, $Z(H)$ and commutator groups G_2, H_2 are said to be *isoclinic* (written $G \approx H$), if there exist isomorphisms $\theta : G/Z(G) \to H/Z(H)$ and $\varphi : G_2 \to H_2$ such that $\varphi([a,b]) = [a', b']$ for all $a, b \in G$, where $a'Z(H) = \theta(aZ(G))$ and $b'Z(H) = \theta(bZ(G))$.

It is easy to show that this relation is well defined and is, in fact, equivalence relations. The equivalence classes are called (isoclinism) *families*. The isoclinism relation is weaker than isomorphism relation. P. Hall ([4]) has showed that every family Φ has the groups of minimal order, called stem groups of Φ. We show all investigated groups belong to families with stem groups of order less or equal to p^6, so we can use the list of isoclinism families obtained by P. Hall ([4]) (for families of rank 5), T. Easterfield ([5]) and R. James ([6]) (for rank ≤ 6).

1. The main properties

We denote by G the group all of whose proper subgroups have a commutator subgroup of order equal or less then p, but the group itself has a commutator group of order greater than p.

We need the following technical lemmas.

Lemma 1. *Let M be a proper normal subgroup of G and M' is its commutator subgroup. Then $M' \subseteq Z(G)$ if $|M'|=p$.*

It is clear that M' is characteristic subgroup of M, so M' is normal subgroup of G. Every normal subgroup of p-group G has a nontrivial intersection with $Z(G)$, so $M' \subseteq Z(G)$. #

Lemma 2. *For each $a,d \in G$ we have*

$$[a^p,b] = [a,b^p] = [a,b]^p. \qquad (1)$$

Proof. Let be $g = [a,b]$. If $[a,g] = 1$ then $[a^p,b] = [a,b]^p$ is clear.

Suppose now $[a,g] = z \neq 1$. Consider $H' = \langle a, \Phi(G)\rangle$. It is a proper normal subgroup of G. We have $g \in G' \subseteq \Phi(G) \subseteq H$. From Lemma 1 we have $H' \subseteq Z(G)$. So $z \in Z(G)$, $\exp(z) = p$. From $g^a = a^{-1}ga = g[g,a] = gz$ the next follows:

$$[a^p, b] = [a, b]^{a^{p-1}} [a, b]^{a^{p-2}} \ldots [a,b]^a [a,b] =$$
$$= g^{a^{p-1}} g^{a^{p-2}} \ldots g^a g = (gz)^{a^{p-2}} (gz)^{a^{p-3}} \ldots (gz)g = \ldots$$
$$\ldots = g^p z^{\frac{p(p-1)}{2}} = g^p.$$

Analogously $[a,b^p] = [a,b]^p = g^p$. Lemma has been proved. #

Lemma 3. $\exp(G') \leq p^2$.

Proof. Let $\exp(G') \geq p^3$. Then there is an element $g \in G'$ such that $|g| = p^3$, and there are $a,b \in G$ such that $[a,b] = g$. The subgroup $A = \langle a,b^p\rangle$ is a proper subgroup of G. From Lemma 2 we have $g^p = [a,b^p] \in A'$, but g^p has the order p^2. This contradicts the assumption that every proper subgroup of G has a commutator subgroup of order $\leq p$. Lemma has been proved. #

Theorem 1. *Let G be a group, all of whose proper subgroup have a commutator subgroup of order $\leq p$ and G itself has the commutator subgroup of order bigger then p. The following conditions are equivalent:*

(*a*) $\exp(G') = p^2$;

(*b*) *all maximal subgroups of G have the same commutator subgroup;*

(*c*) *G' is a cyclic group.*

Proof. $(a) \Rightarrow (b)$. Assume that the element $g \in G'$ is such that $|g| = p^2$, and elements $a,b \in G$ are such that $[a,b] = g$. Consider the subgroup $A = \langle a,b\rangle$ of G. Since $\exp(A') = p^2$ thus $A = G$. So, we have $g = \langle a,b\rangle$ and $\Phi(G) = \langle a^p, b^p, G'\rangle$. Since the subgroups $\langle ab^\alpha, \Phi(G)\rangle$, $\alpha = 0,\ldots, p-1$, and $\langle b, \Phi(G)\rangle$ exhaust all maximal subgroups of $G = \langle a,b\rangle$. From Lemma 2 we have $[a^p,b] = [a,b^p] = g^p \neq 1$. Thus every maximal subgroup M of $G = \langle a,b\rangle$ is nonabelian and contains $g^p \neq 1$ in their commutator subgroup. From the definition of G we have $|M'| = p$ and $M' = \langle g^p\rangle_p$ for every maximal subgroup M.

$(b) \Rightarrow (c)$. Now suppose that for every maximal subgroup A, B of G we have $A' \cap B' = A' = B' = D$, where $|D| = p$. If G/D is abelian then $G' = D$. This contradicts the condition $|G'| > p$.

If G/D is nonabelian, then it is a minimal nonabelian group because its every proper subgroup is abelian. Every maximal subgroup of G/D is abelian thus intersection of two maximal subgroups coincides with center of G/D. Therefore the center of G/D has index p^2, $G/D = \langle \overline{a},\overline{b},Z(G/D)\rangle$, and it follows that $\overline{(G/D)}$ is the cyclic group generated by the $[\overline{a}, \overline{b}]$. The commutator subgroup G' of G is the extension of the cyclic group of order p by another cyclic group of order p. So the order of G' is equal to p^2.

We will prove that G has exactly 2 generators in this case too. Let a, b are the preimages of $\overline{a}$, $\overline{b}$ and $[a, b] = g$. We have $g \notin D$, so the subgroup $H = \langle a,b\rangle$ has a commutator subgroup, which is unequal to D. According to the condition (*b*) of the theorem H may not be the proper subgroup of G, so $G = \langle a,b\rangle$.

Suppose that G' is noncyclic; then there is $c \in G'$, $c \notin \langle g\rangle$. Since $G = \langle a,b\rangle$, we get $G' = \langle g,G_3\rangle$ and $c \in G_3$. So, there are elements d_1, d_2 such that $d_1 \in G'$, $d_2 \in G \backslash G'$, $[d_1, d_2] = c$. Let $M = \langle d_2, \Phi(G)\rangle$ is a maximal subgroup of $G = \langle a,b\rangle$. M contains d_1, d_2, thus $c = [d_1, d_2] \in M'$. Since $M' = \langle g^p\rangle_p$ then $c \in \langle g^p\rangle_p$. The contradiction with assumption $c \notin \langle g\rangle$ proves that G' can not be a noncyclic group. We get G' is the cyclic group of order p^2.

$(c) \Rightarrow (a)$. It follows from the determinations of G and Lemma 3.

That completes the proof.

Corollaries. 1) *If G' is the cyclic group of order p^2 then G has two generators, $G = \langle a,b\rangle$.*

2) *If G' is the cyclic group of order p^2, then all maximal subgroups of G are nonabelian.*

3) *G' is noncyclic group if and only if there are (at least) two maximal subgroups A, B of G such that $A' \cap B' = 1$.*

Theorem 2. *G' is abelian.*

Proof. At first let us suppose that G possesses (at least) two maximal subgroups A, B such that $A' \cap B' = 1$. Thus $A \cap B$ is abelian, so $G' \subseteq \Phi(G) \subseteq$ $\subseteq A \cap B$ is abelian too.

Now suppose that for every maximal subgroups A, B of G holds $A' \cap B' = A' = B' = D$, where $|D| = p$. If G/D is abelian then $G' = D$. We get G' is the cyclic group of order p, so G' is abelian.

In case G/D is nonabelian, we have proved that G' is cyclic of order p^2 and thus it is abelian too. Theorem has been proved.

Corollary. *If the commutator subgroup G' is a noncyclic group, then it is an elementary abelian group.*

This proposition follows immediately from Theorem 1 and Theorem 2.

Now we will investigate the lower central series of G. Set $G_2 = G' = [G, G]$, $G_3 = [G_2, G]$, $G_4 = [G_3, G]$ and so on.

Lemma 4. $G_3 \subseteq Z(G)$, $G_4 = 1$.

Proof. We will fix a system of generators of G, $G = \langle g_1, g_2, \ldots, g_n\rangle$ such that $G/G_2 = \langle \overline{g}_1\rangle \times \langle \overline{g}_2\rangle \times \ldots \times \langle \overline{g}_n\rangle$ and $\overline{g}_i = g_iG_2$ are the generators of abelian group G/G_2. Thus $G_3 = [G_2, G] = \langle [g_1, G_2], \ldots, [g_n, G_2]\rangle$. Consider the normal subgroups $M_i = \langle g_i, G_2\rangle$, where $i = 1, \ldots, n$. Since $M_i' = \langle [g_i, G_2]\rangle$ and from Lemma 1 $M_i' \subseteq Z(G)$ we get $[g_i, G_2] \in Z(G)$ for each $i = 1, \ldots, n$, and finally we obtain $G_3 \subseteq Z(G)$ and $G_4 = 1$. That completes the proof.

Theorem 3. *If $G_3 \neq 1$ then G has exactly two generators, $G = \langle a, b\rangle$.*

Proof. Let $g \in G_2 - G_3$ and $a, b \in G$ are such that $g = [a, b]$. If $a \notin Z(g)$ or $b \notin Z(g)$, then subgroup $\langle a, b\rangle$ has commutator subgroup with order larger than p, so $\langle a, b\rangle$ coincide with G.

We will regard the case $[a, g] = [b, g] = 1$.

Suppose that G has more then 2 generators. If $G_3 \neq 1$, then there are elements $c \in G$, $g \in G'$ such that $[g, c] = z$, $z \neq 1$, $z \in G_3$. From Lemma 4 we have $G_3 \subseteq Z(G)$, so $z \in Z(G)$. The assumption $[a, c] = [b, c] = 1$ contradicts with $[a, b] = g, [g, c] = z$. Thus we may assume that $[a, c] = d \neq 1$. Regard the proper subgroup $\langle a, c, G_2\rangle$. The assumption $d \notin \langle z\rangle$, where $z = [g, c]$, contradicts with the condition that every proper subgroup of G has a commutator subgroup of order $\leq p$. Thus $d = z^{\alpha} \in Z(G)$. In a similar way we have $[b, c] \in Z(G)$. From Witt's identity $[[a,b^{-1}], c]^b [[b,c^{-1}], a]^c [[c,a^{-1}], b]^a = 1$ we have that $z^{-1} = 1$. The contradiction with condition $z \neq 1$ implies that our supposition is wrong.

Theorem has been proved.

2. The groups with the cyclic commutator subgroup of order p^2

Theorem 4. *Let G be a group all of whose proper subgroups have a commutator subgroup of order $\leq p$. Suppose that the commutator subgroup G' of G is the cyclic group of order p^2. We have one of the two following possibilities.*

(a) Let be $G' \subset Z(G)$. Then G is 2-generated group from the family Φ_{14}, G has the following relations:

$$G = \langle a, b, g \mid [a, b] = g, [a, g] = [b, g] = 1, g^{p^2} = 1,$$
$$a^{p^m} = g^{\alpha}, b^{p^n} = g^{\beta}\rangle, (m \geq n \geq 2)$$

where α, β are equal to 1, 0 or p.

(b) Let be $G' \not\subset Z(G)$. Then G is 2-generated group from the family Φ_8, G has following relations:

$$G = \langle a, b, g \mid [a, b] = g,\ [a, g] = g^p,$$
$$[b, g] = 1,\ g^{p^2} = 1,$$
$$a^{p^m} = g^{p\alpha_1},\ b^{p^n} = g^{\beta}\rangle,$$
$$(m \geq 2,\ \ n \geq 1)$$

where in case $n = 1$ we have that $\beta = 1$ and $\alpha = 0$ for $m = 2$, $\alpha = 0$ or p for $m > 2$;

and in case $n > 1$ we have $\beta \equiv 0 \bmod p$, $\alpha = 0$ or p.

Warning: This is not the determination up to isomorfism.

Proof. We have proved that $G = \langle a, b\rangle$. So $G/G' = \langle \overline{a}\rangle_{p^m} \times \langle \overline{b}\rangle_{p^n}$. Let g be a generator of the cyclic group G' of order p^2, $g = [a, b]$.

Then G has following relations:

$$G = \langle a, b, g \mid [a, b] = g,$$
$$[a, g] = g^{\lambda}, [b, g] = g^{\mu}, \tag{2}$$
$$g^{p^2} = 1,\ a^{p^m} = g^{\alpha},\ b^{p^n} = g^{\beta}\rangle,$$

where $0 \leq \lambda, \mu, \alpha, \beta < p^2$.

(a) Suppose that $G' \subset Z(G)$. Thus $a^{p^m}, b^{p^n} \in Z(G)$. According (1) we have $1 = [a^{p^m}, b] = [a, b]^{p^m} = g^{p^m}$, thus $m \geq 2$. Analogously $n \geq 2$. So $G/Z(G) = \langle \hat{a}\rangle_{p^2} \times \langle \hat{b}\rangle p^2$. We have obtained the set of all two-generated groups of family Φ_{14}.

Every above mentioned group is determined by the relations:

$$G = \langle a, b, g \mid [a, b] = g, [a, g] = [b, g] = 1, g^{p^2} = 1,$$
$$a^{p^m} = g^{\alpha}, b^{p^n} = g^{\beta}\rangle, (m \geq n \geq 2).$$

We may choose the generators a, b such that α, β are equal to 1, 0 or p.

(b) Suppose that $G' \not\subset Z(G)$. We may assume that $[a, g] \neq 1$, $[a, g] \in \langle g^p\rangle_p$. We may choose a, b such that $[b, g] = 1$, $[a, g] = g^p$. Thus $a^{p^m} \in Z(G)$ and $b^{p^n} = g^{\beta}$ may be an element of $G' - Z(G)$. $b^{p^n} \in G - Z(G)$ if and only if $\beta \neq 0 \bmod p$.

According to (1) we have $1 = [a^{p^m}, b] = [a, b]^{p^m} = g^{p^m}$ thus $m \geq 2$; $g^{p^n} = [a, b]^{p^n} = [a, b^{p^n}] = [a, g^{\beta}] = g^{\beta p}$ thus $n \geq 1$. It is easy to see that $n = 1$ if and only if $\beta \neq 0 \bmod p$. So the case $n = 1$ must be considered separately. In this case we have $g^p = [a, b]^p = [a, b^p] = [a, g^{\beta}] = g^{\beta p}$ thus $\beta = 1$. From the relations $b^p = g$, $g = [a, b]$, $[a, g] = g^p$ we have that $b^p g^{-1} \in Z(G)$ and $b^p = gz$, where $z \in Z(G)$.

So, we have $G/Z(G) = \langle \hat{a}, \hat{b}, \hat{g} \mid [\hat{a}, \hat{b}] = \hat{g}, [\hat{a}, \hat{g}] = [\hat{b}, \hat{g}] = 1, \hat{a}^{p^2} = 1, \hat{b}^p = \hat{g}, \hat{g}^p = 1 \rangle$ for every such group. All groups with given factor-group $G/Z(G)$ and cyclic commutator subgroup $G' = \langle g \rangle_{p^2}$ form the family Φ_8. Thus we have obtained all 2-generated metacyclic groups of family Φ_8.

We have seen that $a^{p^m} = g^{\alpha_1 p}$. If $m = 2$ then replacing a with $a' = ab^{-\alpha p}$ we obtain $a'^{p^m} = 1$. If $m > 2$ then we may choose the generators a, b such that α_1 is equal to 1 or 0, so α is equal to p or 1. The following relations determine every such group:

$$G = \langle a, b, g \mid [a, b] = g, [a, g] = g^p, [b, g] = 1, g^{p^2} = 1,$$
$$a^{p^m} = g^{p^{\alpha_1}}, b^p = g \rangle, (m \geq 2, n = 1),$$

where $\alpha_1 = 0$ in the case $m = 2$, and α_1 may be equal to 0 or 1 for $m > 2$.

Let be $n > 1$. Thus we have $b^{p^n} = g^{\beta_1 p}$. We may choose the generators a, b such that α_1, β_1 are equal 1 or 0. From the relations $g = [a, b]$, $[a, g] = g^p$ we obtain that $b^p g^{-1} \in Z(G)$ and $b^p = gz$, where $z \in Z(G)$.

Thus for $n > 1$ we have the same factor-group: $G/Z(G) = \langle \hat{a}, \hat{b}, \hat{g} \mid [\hat{a}, \hat{b}] = \hat{g}, [\hat{a}, \hat{g}] = [\hat{b}, \hat{g}] = 1, \hat{a}^{p^2} = 1, \hat{b}^p = \hat{g}, \hat{g}^p = 1 \rangle$ and the same cyclic commutator subgroup $G' = \langle g \rangle_{p^2}$. Therefore the groups obtained for $n > 1$ are from family Φ_8 too.

The following relations determine every such group:

$$G = \langle a, b, g \mid [a, b] = g, [a, g] = g^p, [b, g] = 1, g^{p^2} = 1,$$
$$a^{p^m} = g^{p^{\alpha_1}}, b^{p^n} = g^{p^{\beta_1}} \rangle, (m \geq 2, n \geq 2).$$

Thus we have obtained the all 2-generated groups of the family Φ_8.

From the other hand, if G is a 2-generated group of Φ_8 then $\Phi(G) = \langle a^p, b^p, g \rangle$. Subgroups $\langle ab^{\alpha}, a^p, b^p, g \rangle$, $\alpha = 0, \ldots, p-1$, and $\langle b, a^p, g \rangle$ exhaust all maximal subgroups of G. It is easy to see that every maximal subgroup has a commutator subgroup $\langle g^p \rangle$ of order p. Thus every proper subgroup of G is abelian or has the commutator subgroup of order p.

Theorem has been proved.#

3. The groups with all of whose proper subgroup have a commutator subgroup of order $\leq p$, but the group itself has noncyclic commutator subgroup of order $\geq p^2$ and $G' \subseteq Z(G)$

Suppose G' is a noncyclic group. We have proved that G' is an elementary abelian in this case.

Assume in addition that $G' \subseteq Z(G)$. The next two theorems give all possibilities for such groups.

Theorem 5. *Let G' be a group all of whose proper subgroups have a commutator subgroup of order $\leq p$, but G' has a commutator subgroup of order $\geq p^2$. Then G' has a maximal subgroup A which is abelian, the commutator subgroup G' is noncyclic group and $G' \subseteq Z(G)$ if and only if G is 3-generated group from the family Φ_4.*

G has following relations:

$$G = \langle g_1, g_2, g_3, z_1, z_2 \mid [g_2, g_1] = z_1,$$
$$[g_3, g_1] = z_2, [g_2, g_3] = 1, \tag{2}$$
$$[g_i, z_j] = 1, [z_1, z_2] = 1, z_j^p = 1,$$
$$g_i^{p^{n_i}} = z_1^{\alpha_i} z_2^{\beta_i}, (i = 1, 2, 3; j = 1, 2) \rangle.$$

Proof. Fix a system of generators of G: $G = \langle g_1, g_2, \ldots, g_n \rangle$ such that abelian group $G/\Phi(G)$ is generated by the $\overline{g}_i = g_i G^2$: $G/\Phi(G) = \langle \overline{g}_1 \rangle \times \langle \overline{g}_2 \rangle \times \ldots \times \langle \overline{g}_n \rangle$.

We have shown that if G' is noncyclic then it is an elementary abelian group. Assume $G' = \langle z_1 \rangle \times \langle z_2 \rangle \times H$, where H is an elementary abelian or trivial group. Suppose that the maximal subgroup $A = \langle g_1^p, g_2, \ldots, g_n \rangle$ is abelian, thus $[g_i, g_j] = 1$ for all $i, j = 2, \ldots, n$. Let us set $[g_2, g_1] = z_1$, $[g_3, g_1] = z_2$, $[g_i, g_1] \in H$ for $i = 3, \ldots, n$. Consider the maximal subgroup $M = \langle g_1, g_2^p, g_3, \ldots, g_n \rangle$. We have $M' = \langle z_2 \rangle \times H$. The condition $|M'| = p$ implies that $H = 1$.

If G has more than 3 generators $g_1, g_2, \ldots, g_n$, then subgroup $B = \langle g_1, g_2, g_3 \rangle$ is a proper subgroup of G. It has a commutator subgroup of order p^2. This is a contradiction since every proper subgroup of G has a commutator subgroup of order $\leq p$. Thus G has exactly 3 generators g_1, g_2, g_3 and has the relations (2). It is easy to verify that $G/Z(G) \approx \langle \overline{g}_1 \rangle_p \times \langle \overline{g}_2 \rangle_p \times \langle \overline{g}_3 \rangle_p$, $G' = \langle z_1 \rangle \times \langle z_2 \rangle$ and $[g_2, g_1] = z_1$, $[g_3, g_1] = z_2$, $[g_2, g_3] = 1$. All groups with such relations belong to the family Φ_4.

From the other side, the conditions $G/Z(G) \approx \langle \overline{g}_1 \rangle_p \times \langle \overline{g}_2 \rangle_p \times \langle \overline{g}_3 \rangle_p$, $G' = \langle z_1 \rangle_p \times \langle z_2 \rangle_p$, $G \subseteq Z(G)$ hold for each group G with 3 generators from Φ_4, so we can choose the generators thus that $[g_2, g_1] = z_1$, $[g_3, g_1] = z_2$, $[g_2, g_3] = 1$, where $g_i Z(G) = \overline{g}_i$ for $i = 1, 2, 3$. Thus every group from Φ, with 3 generators may be determined by the relations (2). It is easy verify that every such group satisfies the conditions of theorem. #

Theorem 6. *Let G be a group all of whose proper subgroup have a commutator subgroup of order $\leq p$, but the group itself has commutator subgroup of order $\geq p^2$. Then G' is noncyclic group, $G' \subseteq Z(G)$, and all maximal subgroups of G are nonabelian if and only if one of the following conditions holds:*

(a) *G is 4-generated group from the family Φ_{12} and G has following relations:*

$$G = \langle g_1, g_2, g_3, g_4, z_1, z_2 \mid [g_2, g_1] = z_1, [g_4, g_3] = z_2,$$
$$[g_1, g_3] = [g_1, g_4] = [g_2, g_3] = [g_2, g_4] = 1,$$
$$[g_i, z_j] = 1, [z_1, z_2] = 1, z_j^p = 1, g_i^{p^{n_i}} = z_1^{\alpha_i} z_2^{\beta_i},$$
$$(i = 1, 2, 3, 4; j = 1, 2; \alpha_i, \beta_i = 0 \ldots p-1)\rangle$$

(b) *G is 3-generated group from the family Φ_1 and G has following relations:*

$$G = \langle g_1, g_2, g_3, z_1, z_2, z_3 \mid [g_1, g_2] = z_3,$$
$$[g_1, g_3] = z_2, [g_2, g_3] = z_1,$$
$$[g_i, z_j] = 1, [z_i, z_j] = 1, z_j^p = 1, g_i^{p^{n_i}} = z_1^{\alpha_i} z_2^{\beta_i} z_3^{\gamma_i},$$
$$(\alpha_i, \beta_i, \gamma_i = 0..p-1; i = 1, 2, 3; j = 1, 2, 3)\rangle.$$

Proof. Suppose that G is a group in which every proper subgroup has a commutator subgroup of order $\le p$, but the group itself has a noncyclic commutator subgroup of order $\ge p^2$. Suppose that $G' \subseteq Z(G)$ and all maximal subgroups of G are nonabelian. We have shown that there are maximal subgroups M_1 and M_2 such that $M_1' \neq M_2'$.

Let $G/\Phi(G) = \langle \bar{g}_1 \rangle \times \langle \bar{g}_2 \rangle \times \ldots \times \langle \bar{g}_n \rangle$. We fix the generators $G = \langle g_1, g_2, \ldots, g_n \rangle$ such that $\bar{g}_i = g_i\Phi(G)$ $(i = 1, \ldots, n)$ are the generators of $G/\Phi(G)$. We may assume that $M_1 = \langle g_1^p, g_2, \ldots, g_n \rangle$, $M_2 = \langle g_1, g_2^p, g_3, \ldots, g_n \rangle$. Since $(M_1 \cap M_2)' \subseteq M_1' \cap M_2' = 1$ then $M_1 \cap M_2 = \langle g_1^p, g_2^p, g_3, \ldots, g_n \rangle$ is abelian. Thus $[g_i, g_j] = 1$ for all $i, j = 3, \ldots, n$. From Lemma 2 we have $g_i^p \in Z(G)$. Therefore the relations between g_1, g_2 and between g_1, g_2 and other generators exhaust all nontrivial commutator relations in G.

a) Suppose that $[g_1, g_2] = 1$. Set $M_1' = \langle z_2 \rangle_p$, $M_2' = \langle z_1 \rangle_p$. We may choose the generators $g_3, g_4, \ldots, g_n$ such that $[g_1, g_3] = z_1$ and $[g_1, g_i] = 1$ for $i = 4, \ldots, n$. If additionally $[g_2, g_i] = 1$ for all $i = 4, \ldots, n$, then the maximal subgroup $A = \langle g_1, g_2, g_3^p, g_4, \ldots, g_n \rangle$ is abelian. This contradicts that all maximal subgroups of G are nonabelian.

So there exists an element $h \in \langle g_1, \ldots, g_n, \Phi(G) \rangle$, which does not commute with g_2.

Taking g_4 for h (if necessary) we may assume $[g_2, g_4] = z_2$. We may redefine the generators $g_4, \ldots, g_n$ such way that $[g_1, g_i] = 1$ for $i = 4, \ldots, n$. Thus the subgroup $A = \langle g_1, g_2, g_3^p, g_4, \ldots, g_n \rangle$ has a commutator subgroup $\langle z_2 \rangle_p$. If $[g_2, g_i] \neq 1$ for $i = 5, \ldots, n$ then $[g_2, g_i] \in \langle z_2 \rangle_p$ and we may redefine the generators $g_5, \ldots, g_n$ such way that $[g_2, g_i] = 1$ for $i > 4$. If $[g_2, g_3] \neq 1$ then regarding subgroup $M_1 = \langle g_1^p, g_2, \ldots, g_n \rangle$ we obtain $[g_2, g_3] = z_2^i$ since $g_2, g_4 \in M_1'$ and $M_1' = \langle z_2 \rangle_p$. Thus $g_3' = g_3 g_4^{-i}$ commutes with g_2, and the set of the elements $\langle g_1, g_2, g_3', g_4, \ldots, g_n \rangle$ is a set of the generators of G too. Writing g_3 instead of g_3' we obtain the set of generators of G satisfying the following relations: $[g_1, g_3] = z_1$, $[g_1, g_i] = 1$ for $i = 4, \ldots, n$, $[g_2, g_4] = z_2$, $[g_2, g_i] = 1$ for $i > 4$ and $[g_2, g_3] = 1$. The subgroup $\langle g_1, g_2, g_3, g_4 \rangle$ has a commutator subgroup of order p^2, thus it is not a proper subgroup of G. Therefore $G = \langle g_1, g_2, g_3, g_4 \rangle$. Regarding G as an extension of the elementary abelian group $G' = \langle z_1 \rangle_p \times \langle z_2 \rangle_p$ by the abelian group G/G' generated by $\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4$, where $\bar{g}_i = g_i G'$, we obtain a group with relations (a).

All these groups are 4-generated groups of the family Φ_{12}.

From the other hand, each group H from Φ_{12} has an elementary abelian commutator subgroup of order p^2 and an elementary abelian group $H/Z(H)$ of order p^4. If H is contained in the family Φ_{12} then $H' \subseteq Z(H)$. For every 4-generated group H of this family we may choose the generators $\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4$ of $H/Z(H)$ and the generators z_1, z_2 of H' such that $[g_1, g_2] = 1$, $[g_1, g_3] = z_1$, $[g_2, g_4] = z_2$, $[g_1, g_4] = 1$, $[g_2, g_3] = 1$, $[g_3, g_4] = 1$, where $g_i Z(H) = \bar{g}_i$ for $i = 1, 2, 3, 4$. It is easy to see that all maximal subgroups of 4-generated group H have a commutator subgroup of order p so all proper subgroups of 4-generated group H have a commutator subgroup of order equal or less then p.

b) Now suppose that $[g_1, g_2] \neq 1$. We may assume that $[g_2, g_3] = z_1$, $[g_2, g_i] = 1$ for $i = 4, \ldots n$. If additionally $[g_1, g_3] = 1$, $[g_1, g_4] = z_2$, $[g_1, g_i] = 1$ $(i > 4)$, then either subgroup $\langle g_1, g_2, g_3 \rangle$ or subgroup $\langle g_1, g_2, g_4 \rangle$ has a commutator subgroup of order p^2. Assume $[g_1, g_3] = z_2$, $[g_1, g_i] = 1$ for $i = 4, \ldots n$. In this case if $[g_1, g_2] \in \langle z_1 \rangle \times \langle z_2 \rangle$, $[g_1, g_2] = z_1^{\alpha} z_2^{\beta}$ then the maximal subgroup $A = \langle g_1 g_3^{\beta}, g_2 g_3^{-\alpha}, g_4, \ldots, g_n \rangle$ is abelian. This contradicts to the condition of theorem that all maximal subgroups of G are nonabelian, and hence $[g_1, g_2] \notin \langle z_1 \rangle \times \langle z_2 \rangle$. Set $[g_1, g_3] = z_3$. The subgroup $\langle g_1, g_2, g_3 \rangle$ has a commutator subgroup of order p^3, so it coincides with G. Thus G has 3 generators. Regarding G as an extension of the elementary abelian group $G' = \langle z_1 \rangle_p \times \langle z_2 \rangle_p \times \langle z_3 \rangle_p$ by the abelian G/G', we obtain the groups with relations (b). All groups of this kind belong the family Φ_{11}.

From the other hand, each group H from Φ_{11} has an elementary abelian commutator subgroup of order p^3 and an elementary abelian group $H/Z(H)$ of order p^3. Each group H from Φ_{11} satisfies condition $H' \subseteq Z(H)$. For every 3-generated group H of this family we can choose the generators $\bar{g}_1, \bar{g}_2, \bar{g}_3$ of $G/Z(G)$ and the generators z_1, z_2, z_3 of G' such that $[g_1, g_2] = z_3$, $[g_1, g_3] = z_2$, $[g_2, g_3] = z_1$, where $g_i Z(G) = \bar{g}_i$ for $i = 1, 2, 3$. It is easy to see that all maximal subgroups have commutator subgroups of order p for every 3-generated group H of Φ_{11}.

We have proved that all 3-generated groups of Φ_{11} are the groups all of whose proper subgroups have a commutator subgroup of order equal or less then p, but the group itself has a commutator group of order greater than p.

Theorem has been proved. #

The groups determined here exhaust all groups G with noncyclic commutator subgroup, which is contained in $Z(G)$.

4. The groups all of whose proper subgroup have a commutator subgroup of order $\le p$, but the group itself has noncyclic commutator subgroup of order $\ge p^2$ and $G' \not\subset Z(G)$

We have proved that all such groups with $G' \not\subset Z(G)$ are 2-generated.

Theorem 7. *Let G be a group all of whose proper subgroup have a commutator subgroup of order $\le p$, but the group itself has noncyclic commutator subgroup of order $\ge p^2$ and $G' \not\subset Z(G)$.*

G possesses an abelian maximal subgroup A if and only if

$$G = \langle a, b, g, z \mid [b, a] = g, [g, a] = z,$$
$$[g, b] = 1, [g, z] = [a, z] = [b, z] = 1, z^p = g^p = 1,$$
$$a^{p^n} = z^\alpha, b^{p^m} = z^\beta \ (\alpha, \beta = 0, 1, \ldots, p-1)\rangle$$

so G is the 2-generated group of the family Φ_3.

Proof. Let be $G' \not\subset Z(G)$. From the theorem 3 we get that G has exactly 2 generators, $G = \langle a, b\rangle$. Set $g = [b, a]$. Thus we may assume that $G' = \langle g\rangle_p \times \langle z\rangle_p \times H$, where $[g, a] = z$. Assume the subgroup $A = \langle a^p, b, G'\rangle$ is abelian. Then we have $b \in C_G(G')$. Consider the maximal subgroup $M = \langle a, b^p, G'\rangle$. The commutator subgroup M' is generated by all comutators, which generate G' except for $g = [b, a]$. Therefore $M' = \langle z\rangle_p \times H$. The condition $|M'| = p$ implies that $H = 1$. Thus $G' = \langle g\rangle \times \langle z\rangle$, where $\langle z\rangle \subseteq G_3 \subseteq Z(G)$. Consider G as the extension G' by the G/G'. We have $G/G' = \langle \overline{a}\rangle_{p^n} \times \langle \overline{b}\rangle_{p^m}$, where $\overline{a} = aG'$, $\overline{b} = bG'$. From $a^{p^n} \in Z(G)$ we get $a^{p^n} = z^\alpha$, where $\alpha = 0, 1, \ldots, p-1$. Suppose $b^{p^n} \in Z(G)$. Then for each $h \in G$ we have $[b^{p^m}, h] = [b, h]^{p^m} = 1$ according to (1) and since G' has exponent p. This contradiction implies that $b^{p^m} \in Z(G)$. So we may assume $b^{p^m} = z^\beta$. Thus G may be determined by the relations

$$G = \langle a, b, g, z \mid [b, a] = g, [g, a] = z,$$
$$[g, b] = 1, [g, z] = [a, z] = [b, z] = 1, z^p = g^p = 1,$$
$$a^{p^n} = z^\alpha, b^{p^m} = z^\beta \ (\alpha, \beta = 0, 1, \ldots, p-1)\rangle.$$

All groups with these relations are the 2-generated groups of the family Φ_3.

From the other side, every group H from Φ_3 has the nonabelian group $H/Z(H)$ of order p^3 and exponent p, a commutator subgroup H' of order p^2, which is an elementary abelian and is not contained in the center. For every 2-generated group H of this family we can choose the generators $\overline{a}, \overline{b}$ of $H/Z(H)$ such that $[\overline{b}, \overline{a}] = \overline{g}$ and the generators g, z of H', where $z \in Z(H) \cap H'$ and $g \in G' - Z(H) \cap H'$, such that $[b, a] = g$, $[g, a] = z$, $[b, g] = 1$, where $aZ(H) = \overline{a}$, $bZ(H) = \overline{b}$, $gZ(H) = \overline{g}$. So we may determine every such group with 2 generators by the relations given above. Every such group has an abelian maximal subgroup $A = \langle a^p, b, H'\rangle$. The subgroup A and the subgroups $\langle ab^\alpha, \Phi(H)\rangle$, where $\alpha = 1, \ldots, p-1$, exhaust all maximal subgroups of 2-generated group $H = \langle a, b\rangle$. It is easy to see that every maximal subgroup of H has a commutator subgroup of order $\le p$. Thus every proper subgroup of H has a commutator subgroup of order $\le p$.

Theorem has been proved.

Theorem 8. *Let G be a group all of whose proper subgroup have a commutator subgroup of order $\le p$, but the group itself has noncyclic commutator subgroup of order $\ge p^2$.*

Then $G' \not\subset Z(G)$ and all maximal subgroup of G are nonabelian if and only if G is the 2-generated group of the family Φ_6 and it may be determined by the relations

$$G = \langle a, b, c, z_1, z_2 \mid [a, b] = c, [a, c] = z_2,$$
$$[b, c] = z_1, [a, z_i] = [b, z_i] = [c, z_i] =$$
$$= [z_1, z_2] = 1, a^{p^n} = z_1^{\alpha_1} z_2^{\alpha_2},$$
$$b^{p^m} = z_1^{\beta_1} z_2^{\beta_2}, c^p = z_i^p = 1 \ (i = 1, 2)\rangle$$

where $\alpha_i, \beta_i = 0, 1, \ldots, p-1, i = 1, 2$.

Proof. According to the Theorem 3 we have that G is 2-generated group, $G = \langle a, b\rangle$. The commutator $c = [a, b]$ does not belong to center: $c \notin Z(G)$, but $G' \cdot Z(G)$ is abelian since $G' = \langle c, G_3\rangle$, $G_3 \subseteq Z(G)$. So $G' \cdot Z(G) = Z(G) \cdot \langle c\rangle$.

According to the Corollary 3 to the Theorem 1 we have maximal subgroups A, B of G, which have different commutator subgroups. We may choose the generators a, b such that $A = \langle a^p, b, G'\rangle$ and $B = \langle a, b^p, G'\rangle$. Let us set $A' = \langle z_1\rangle$, $B' = \langle z_2\rangle$, $z_1 \ne z_2$. From Lemma 2 we have $h^p \in Z(G)$ for every $h \in G$, thus $A = Z(G) \cdot G' \cdot \langle b\rangle = Z(G) \cdot \langle b\rangle \cdot \langle c\rangle$ and $B = Z(G) \cdot G' \cdot \langle a\rangle = Z(G) \cdot \langle a\rangle \cdot \langle c\rangle$. Therefore $A' = \langle [b, c]\rangle$ and $B' = \langle [a, c]\rangle$. Thus we may assume that $[a, c] = z_2$, $[b, c] = z_1$, where $z_1, z_2 \in Z(G)$.

Considering G as an extension of $G_3 = \langle z_1\rangle \times \langle z_2\rangle$ by the group

$$G/G_3 = \langle \overline{a}, \overline{b} \mid [\overline{a}, \overline{b}] = \overline{c},$$
$$[\overline{a}, \overline{c}] = [\overline{b}, \overline{c}] = 1, \overline{a}^{p^n} = \overline{b}^{p^m} = \overline{c}^p = 1\rangle$$

we obtain the relations of G:

$$G = \langle a, b, c, z_1, z_2 \mid [a, b] = c, [a, c] = z_2, [b, c] = z_1,$$
$$[a, z_i] = [b, z_i] = [c, z_i] = [z_1, z_2] = 1,$$
$$a^{p^n} = z_1^{\alpha_1} z_2^{\alpha_2}, b^{p^m} = z_1^{\beta_1} z_2^{\beta_2}, c^p = z_i^p = 1 \ (i = 1, 2)\rangle,$$

where $\alpha_i, \beta_i = 0, 1 \ldots, p-1, i = 1, 2$.

Every such group is a 2-generated group of the family Φ_6.

From the other side, every group H from Φ_6 has a nonabelian factor-group $H/Z(H)$ of order p^3 and exponent p and an elementary abelian commutator subgroup of order p^3, which is not contained in the center. For every 2-generated group of this family we can choose the generators $\overline{a}$, $\overline{b}$ of $H/Z(H)$ and the generators c, z_1, z_2 of H', where $z_1, z_2 \in Z(H) \cap H'$ and $c \in H' - Z(H) \cap H'$, such that

$[a, b] = c$, $[a, c] = z_2$, $[b, c] = z_1$, where $aZ(H) = \overline{a}$, $bZ(H) = \overline{b}$, $cZ(H) = [\overline{a}, \overline{b}]$. So, we may determine every such group by the relations given above. The subgroups $A = \langle a^p, b, H'\rangle$ and $\langle ab^\alpha, b^p, H'\rangle$, where $\alpha = 1, \ldots, p-1$ exhaust all maximal subgroups of 2-generated group $H = \langle a, b\rangle$. It is easy to see that every maximal subgroup of H is nonabelian and has a commutator subgroup of order p. Thus every proper subgroup of H has the commutator subgroup of order $\leq p$.

Theorem has been proved. #

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ГРУПИ, В ЯКИХ КОЖНА ВЛАСНА ПІДГРУПА МАЄ КОМУТАНТ ПОРЯДКУ НЕ БІЛЬШЕ *p*

Автори висловлюють подяку професору Звонимиру Янку, який запропонував дослідити p-групи G (p ≥ 3), в яких кожна власна підгрупа або абелева, або має комутант порядку p. Очевидно, що такими групами будуть абелеві групи і групи, комутант яких є циклічною групою порядкур. Групи з комутантом порядку p досліджено Л. Секерешем і В. Сергейчуком. Мета цієї праці - дослідження груп G, у яких кожна власна підгрупа або абелева, або має комутант порядку p, але сама група має порядок комутанта більший за p. Показано, що в цьому випадку комутант G' групи G є абелевою групою і або є циклічною групою порядку p^2, або елементарною абелевою групою. Для груп з комутантом, порядок якого більший за p, вказано визначаючі співвідношення, а також визначено родини ізоклінності, яким належать ці групи.

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ЧИСЕЛЬНИЙ РОЗВ'ЯЗОК КРАЙОВОЇ ЗАДАЧІ ТЕОРІЇ ФІЛЬТРАЦІЙНОЇ КОНСОЛІДАЦІЇ З УРАХУВАННЯМ НАСИЧЕНОСТІ МАСИВУ СОЛЬОВИМ РОЗЧИНОМ ТА ПОВЗУЧОСТІ ҐРУНТОВОГО СКЕЛЕТА

Запропоновано чисельний метод розв'язування одновимірної нестаціонарної крайової задачі фільтраційного ущільнення ґрунтового масиву, розміщеного на непроникній основі та насиченого сольовим розчином, за умови повзучості ґрунтового скелета.

1. Вступ

Актуальність досліджень процесів фільтраційного ущільнення грунтів, насичених сольовими розчинами, зумовлена важливістю вивчення умов екологічно безпечного функціонування накопичувачів промислових стоків (зокрема шламо- та хвостосховищ [1, 2]).Часто вказані накопичувачі заповнюють відходами хімічної та гірничої промисловості, які є концентрованими сольовими розчинами. За таких умов некоректно використовувати для розрахунків процесів ущільнення в цих інженерних спорудах класичну теорію фільтраційної консолідації [1], яка базується на припущенні, що фільтрат у масиві є чистою водою. Як показано в роботах [3-5], насиченість масиву сольовим розчином суттєво впливає на розподіл надлишкових напорів у ньому, і врахування цього факту є обов'язковим для одержання адекватного прогнозу перебігу процесу ущільнення. У цій роботі одержано розв'язок одновимірної задачі фільтраційного ущільнення масиву, наси-