Infiniteness of groups of automata over a binary alphabet

A. Russyev

Let X be a finite nonempty set. This set is called an alphabet and its elements are called letters. An automaton over alphabet X is a tuple $A = \langle X, Q, \varphi, \lambda \rangle$, where Q denotes the set of states, $\varphi : Q \times X \to Q$ is the transition function and $\lambda : Q \times X \to X$ is the output function. An automaton is said to be finite if the set of states is finite.

Consider the set $X^* = \bigcup_{n \geq 1} X^n \cup \{\Lambda\}$ of all words over alphabet X. On this set one can define an operation of concatenation. The transition and output functions of an automaton $A = \langle X, Q, \varphi, \lambda \rangle$ can be extended to the set $Q \times X^*$ by the next formulas. For all $q \in Q$, $w \in X^*$ and $x \in X$

$$\begin{aligned} \varphi(q, wx) &= \varphi(\varphi(q, w), x), & \varphi(q, \Lambda) &= q, \\ \lambda(q, wx) &= \lambda(q, w)\lambda(\varphi(q, w), x), & \lambda(q, \Lambda) &= \Lambda. \end{aligned}$$

Every state $q \in Q$ defines maps $\pi_q = \lambda(q, \cdot) : X \to X$ and $f_q = \lambda(q, \cdot) : X^* \to X^*$. The automaton A is called invertible if all maps f_q are bijections, or equivalently, all maps π_q are permutations of the set X.

The group $G(A)$ of an invertible automaton $A = \langle X, Q, \varphi, \lambda \rangle$ is the group generated by the set $\{f_q : q \in Q\}$ ([1]).

Let A be a finite automaton over the binary alphabet $X = \{0, 1\}$. Consider the sets of states

$$\begin{aligned} Q_e &= \{\varphi(q, x) : q \in Q, x \in X, \pi_q = e\}, \\ Q_\sigma &= \{\varphi(q, x) : q \in Q, x \in X, \pi_q = \sigma\}, \end{aligned}$$

where σ is the transposition $(0, 1)$ of the alphabet X and e is the identity permutation.

Theorem. *If $Q_e \cup Q_\sigma = Q$ and $Q_e \cap Q_\sigma \neq \varnothing$, then the group $G(A)$ is infinite.*

If the automaton A does not satisfy condition $Q_e \cup Q_\sigma = Q$ one can "reduce" it to the automaton A' with the set of states $Q_e \cup Q_\sigma$ (the group $G(A)$ is finite if and only if the group $G(A')$ is finite [2]). After finite number of "reductions" the first condition of the theorem will become true.

References

[1] *Nekrashevych V. V.* Self-similar groups, volume 117 of Mathematical Surveys and Monographs. – American Mathematical Society: Providence, RI, 2005. – 231 p.

[2] *Russyev A. V.* On finite and Abelian groups generated by finite automata, Matematychni Studii, 24 (2005) 139–146.

▷ *Contact Information*

A. Russyev National University of Kyiv-Mohyla Academy
✉ andrey.russev@gmail.com