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ON EXPANSIVE AND ANTI-EXPANSIVE TREE MAPS

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Abstract. With every self-map on the vertex set of a finite tree one can associate the directed graph of a special type which is called the Markov graph. Expansive and anti-expansive tree maps are two extremal classes of maps with respect to the number of loops in their Markov graphs. In this paper we prove that a tree with at least two vertices has a perfect matching if and only if it admits an expansive cyclic permutation of its vertices. Also, we show that for every tree with at least three vertices there exists an expansive map with a weakly connected (strongly connected provided the tree has a perfect matching) Markov graph as well as anti-expansive map with a strongly connected Markov graph.

Keywords: maps on trees, Markov graphs, Sharkovsky's theorem.

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1. INTRODUCTION

Let X be a finite tree and $\sigma : V(X) \to V(X)$ be a map from the vertex set $V(X)$ of X to itself. The Markov graph $\Gamma = \Gamma(X, \sigma)$ is a directed graph whose vertices are the edges of X, i.e. $V(\Gamma) = E(X)$ and there is an arc $e_1 \to e_2$ in Γ if e_1 "covers" e_2 under σ. In other words, if $u_2, v_2 \in [\sigma(u_1), \sigma(v_1)]_X$ for $e_i = u_i v_i, i = 1, 2$ (here $[a, b]_X$ denotes the vertex set of a unique shortest path joining a and b in a tree X). For example, let X be a path with at least two vertices and σ be a cyclic permutation of $V(X)$. In this case the corresponding Markov graph $\Gamma(X, \sigma)$ is called a periodic graph. In [2, 10] it was showed that using periodic graphs one can prove the famous Sharkovsky's theorem from one-dimensional dynamics in a purely combinatorial way.

The graph-theoretic criterion for periodic graphs was obtained in [8]. In [5] the pairs (X, σ) were characterized for several prescribed classes of Markov graphs $\Gamma(X, \sigma)$. The complete list of Markov graphs that are tournaments is presented in [7].

In this paper we study two classes of vertex maps on trees using Markov graphs. Namely, let X be a tree and $\sigma : V(X) \to V(X)$ be some map. The map σ is called expansive if each vertex in $\Gamma(X, \sigma)$ has a loop (i.e. an arc from the vertex to itself). Similarly, the map σ is called anti-expansive if $\Gamma(X, \sigma)$ does not contain vertices with loops. Using the properties of anti-expansive maps in [4] it was proved that Markov graphs satisfy Seymour's Second Neighborhood Conjecture as well as Caccetta-Häggkvist Conjecture. Also, one can show that a tree has a fixed point property for expansive maps if and only if it does not have a perfect matching (again, see [4]). We will strengthen this result by showing that the existence of a perfect matching in a tree X is equivalent to the existence of expansive cyclic permutation of $V(X)$. We also prove that for every tree with at least three vertices there exists an expansive map with a weakly connected (strongly connected provided the tree has a perfect matching) Markov graph as well as anti-expansive map with a strongly connected Markov graph.

2. PRELIMINARIES

2.1. BASIC DEFINITIONS

The symbols $\operatorname{Im}\sigma$ and $\operatorname{fix}\sigma$ denote the image and the set of all fixed points of a map σ. Also, having a map σ with $|\operatorname{Im}\sigma| = 2$ we define the corresponding map σ^* as the composition $f \circ \sigma$, where f is unique non-identity permutation of $\operatorname{Im}\sigma$.

A *graph* G is a pair of sets (V, E), where $V = V(G)$ is the set of its *vertices* and $E = E(G)$ is the set of its *edges* which are unordered pairs of vertices. The *neighborhood* of a vertex $u \in V(G)$ is the set

$$N_G(u) = \{v \in V(G) : uv \in E(G)\}.$$

Similarly, $N_G[u] = N_G(u) \cup \{u\}$ is called the *closed neighborhood* of u. We also put $E_G(u) = \{e \in E(G) : u \text{ is incident to } e\}$. The *degree* $d_G(u)$ of a vertex $u \in V(G)$ is the number of its neighbors, i.e. $d_G(u) = |N_G(u)|$. The vertex $u \in V(G)$ is a *leaf* provided $d_G(u) = 1$. The set of all leaf vertices in G is denoted by $L(G)$. For any set of vertices $A \subset V(G)$ we define $N_G(A) = \{u \in V(G) \setminus A : uv \in E(G) \text{ for some } v \in A\}$ and $E(A) = \{uv \in E(G) : u, v \in A\}$. By $G[A]$ we denote the subgraph of G induced by A.

A set of edges $E' \subset E(G)$ is called *dominating* if for every vertex $u \in V(G)$ we have $E_G(u) \cap E' \neq \emptyset$. Similarly, E' is called *weakly dominating* if for every non-leaf vertex $u \in V(G) \setminus L(G)$ it holds $E_G(u) \cap E' \neq \emptyset$. Further, E' is a *matching* if no two edges from E' share a common vertex. The dominating matching is called *perfect*.

On the vertex set $V(G)$ of every connected graph G one can define the "shortest paths" metric $d_G(u, v) = \min\{|E(P)| : P \text{ is a path joining } u \text{ and } v\}$ for all pairs $u, v \in V(G)$. The set

$$[u, v]_G = \{x \in V(G) : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$$

is called *metric interval* between u and v in G. For every edge $uv \in E(G)$ we put

$$A_G(u, v) = \{x \in V(G) : d_G(u, x) \leq d_G(v, x)\}.$$

Also, for a vertex $u \in V(G)$ in a connected graph G and any number $k \geq 0$ we define

$$N_G^k(u) = \{v \in V(G) : d_G(u,v) = k\}$$

(thus, $N_G^1(u) = N_G(u)$ for all $u \in V(G)$).

A *tree* is a connected acyclic graph. It is well known that each tree with at least two vertices has at most one perfect matching.

Lemma 2.1 ([6])**.** *For every tree X with $|V(X)| \geq 2$ there exists a set of leaf vertices $A \subset L(X)$ such that $X - A$ has a perfect matching.*

For every tree X and a pair of its vertices $u, v \in V(X)$ the metric interval $[u,v]_X$ induces a path in X. Also, for every connected set of vertices $A \subset V(X)$ in a tree X one can define the *projection map* $\mathrm{pr}_A : V(X) \to V(X)$ in the following way: put $\mathrm{pr}_A(u) = v$, where v is the unique vertex from A with

$$d_X(u,v) = \min\{d_X(u,x) : x \in A\}.$$

A *directed graph* or just a *digraph* D is a pair of sets (V, A), where $V = V(D)$ is the set of its *vertices* and $A = A(D) \subset V \times V$ is the set of its *arcs*. Sometimes we will write $u \to v$ for an arc (u,v). An arc of the form $u \to u$ is called a *loop* at u. A pair of vertices $u, v \in V(D)$ is a *digon* provided $(u,v), (v,u) \in A(D)$. A vertex v is *reachable* from the vertex u if there is a directed walk from u to v in D. If two digraphs D_1 and D_2 are isomorphic, then we write $D_1 \simeq D_2$.

A digraph is called *strongly connected* if each pair of its vertices lie on a cycle. Similarly, digraph is called *unilaterally connected* if for every pair of its vertices there is a (directed) path joining them. A digraph is called *weakly connected* if its underlying graph (which is obtained by "forgetting" orientations of arcs and ignoring loops and digons) is connected. Finally, digraph is *disconnected* if it is not weakly connected. The next lemma is clear.

Lemma 2.2. *A digraph D is weakly* (*strongly*) *connected if and only if for every proper* (*i.e.* $V' \notin \{\emptyset, V(D)\}$) *set of its vertices $V' \subset V(D)$ there exist two vertices $u \in V'$ and $v \in V(D) \setminus V'$ such that $u \to v$ or $v \to u$* (*v is reachable from u*)*.*

With every linear ordering of the vertex set $V(D) = \{u_1, \ldots, u_n\}$ of a digraph D we can associate the *adjacency matrix* M_D, where $(M_D)_{ij} = 1$ if $u_i \to u_j$ in D and $(M_D)_{ij} = 0$ otherwise. Note that for any number $m \geq 1$ an element $(M_D^m)_{ij}$ equals the number of walks from u_i to u_j in D of length m.

For every map of the form $f : X \to X$ we can define its *functional graph* as a digraph with the vertex set X and the arc set $\{(x,y) \in X \times X : y = f(x)\}$. A digraph is called *functional* if the out-degree of each of its vertices equals one. Similarly, digraph is *partial functional* provided out-degrees of its vertices are at most one. Clearly, each partial functional digraph D corresponds to a partial map on $V(D)$.

A *preordering* is reflexive and transitive binary relation. Having a preordered set $(X, \leq)$ an element $x \in X$ is called *minimal* (*maximal*) if from $y \leq x$ ($x \leq y$) it follows $x \leq y$ ($y \leq x$) for all $y \in X$. Further, x is called the *least element* if for all $y \in X$ we have $x \leq y$ and $x = y$ whenever $y \leq x$. Similarly, one can define the *greatest element* in a preordered set.

2.2. MARKOV GRAPHS FOR TREE MAPS

Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. The *Markov graph* $\Gamma = \Gamma(X, \sigma)$ is a digraph with the vertex set $V(\Gamma) = E(X)$ and there is an arc $e_1 \to e_2$ in Γ if $u_2, v_2 \in [\sigma(u_1), \sigma(v_1)]_X$ for $e_i = u_i v_i, i = 1, 2$.

Lemma 2.3 ([5])**.** *Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. Then for every pair of vertices $u, v \in V(X)$ and an edge $xy \in E([\sigma(u), \sigma(v)]_X)$ there exists an edge $wz \in E([u, v]_X)$ with $wz \to xy$ in $\Gamma(X, \sigma)$.*

Lemma 2.4 ([7])**.** *Let X be a tree, $A \subset V(X)$ be some connected set of vertices, $\sigma : V(X) \to V(X)$ be a map and $\Gamma = \Gamma(X, \sigma)$. Then $\Gamma(X[A], \mathrm{pr}_A \circ \sigma) = \Gamma[E(A)]$.*

Using Markov graphs one can define the natural preordering of the set of all maps $V(X)^{V(X)}$ in the following way: put $\sigma_1 \leq_m \sigma_2$ whenever $\Gamma(X, \sigma_1) \subset \Gamma(X, \sigma_2)$. Clearly, relation $\leq_m$ is reflexive and transitive, thus a preordering. This preordering is called the *Markov preordering.*

It is easy to see that minimal elements for $\leq_m$ are precisely constant maps (as they have empty Markov graphs). Thus, for a tree X with $|V(X)| \geq 2$ the Markov preordering $\leq_m$ has no least element. Similarly, in this case $\leq_m$ does not have the greatest element. To see this fix any map $\sigma \in V(X)^{V(X)}$ and consider its Markov graph $\Gamma = \Gamma(X, \sigma)$. If Γ is not a complete digraph, then there exists two edges $e_i = u_i v_i \in E(X), i = 1, 2$ such that $(e_1, e_2) \notin A(\Gamma)$. Putting $\sigma'(x) = u_2$ for all $x \in A_X(u_1, v_1)$ and $\sigma'(x) = v_2$ for all $x \in A_X(v_1, u_1)$ we obtain the map $\sigma' \in V(X)^{V(X)}$ with $A(\Gamma(X, \sigma')) = \{(e_1, e_2)\}$. This would imply $\sigma' \not\leq_m \sigma$ which is a contradiction. Otherwise, let Γ be a complete digraph. In this case $|\operatorname{Im} \sigma| = 2$. Thus, $\sigma^* \leq_m \sigma$ and $\sigma \leq_m \sigma^*$, however $\sigma^* \neq \sigma$. A contradiction again. For results on maximal elements in $(V(X)^{V(X)}, \leq_m)$ see [4].

Proposition 2.5 ([5])**.** *Let X be a tree with $n \geq 1$ vertices. Suppose that some linear ordering of the edge set $E(X)$ is fixed. Then the correspondence $\sigma \to M_{\Gamma(X,\sigma)}$ defines a homomorphism from the full transformation semigroup T_n to the matrix semigroup $\mathrm{Mat}_{n-1}(\mathbb{F}_2)$.*

2.3. EDGE LABELINGS AND NEIGHBORHOOD MAPS

Having a graph G, a map $\tau : E(G) \to V(G)$ is called an *orientation* of the edges in G provided $\tau(e)$ is incident to e for all $e \in E(G)$. The following construction is from [4]. Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. Define the next edge labeling

$$\tau_\sigma(e) = \begin{cases} u, & \text{if } \sigma(u), \sigma(v) \in A_X(u, v), \\ v, & \text{if } \sigma(u), \sigma(v) \in A_X(v, u), \\ 1, & \text{if } \sigma(u) \in A_X(u, v) \text{ and } \sigma(v) \in A_X(v, u), \\ -1, & \text{if } \sigma(u) \in A_X(v, u) \text{ and } \sigma(v) \in A_X(u, v) \end{cases}$$

for all edges $e = uv \in E(X)$. Thus, if $\tau_\sigma(e) = u$, then the edge e gets an orientation $v \to u$ (similarly, for $\tau_\sigma(e) = v$ we have the orientation $u \to v$). Otherwise, the edge e

is *σ-positive* or *σ-negative* depending on the sign of $\tau_\sigma(e)$. The corresponding *mixed tree* is denoted by $X(\tau_\sigma)$. A labeling $\tau : E(X) \to V(X) \cup \{1, -1\}$ is called *admissible* if $\tau = \tau_\sigma$ for some map σ.

Theorem 2.6 ([4])**.** *Let X be a tree and $\tau : E(X) \to V(X) \cup \{1, -1\}$ be an edge labeling such that the restriction $\tau|_{\tau^{-1}(V(X))}$ is orientation of the edges in X. Then τ is admissible if and only if*

1. *each vertex in $X(\tau)$ has out-degree at most one;*
2. *each vertex in $X(\tau)$ is incident to at most one σ-negative edge;*
3. *if the vertex from $X(\tau)$ is incident to a negative edge, then its out-degree equals zero.*

If the labeling τ satisfies the conditions of Theorem 2.6, then one can construct the map σ_τ in the following way. Put $\sigma_\tau(u) = v$ if there is an arc $u \to v$ in $X(\tau)$ or uv is a σ-negative edge and $\sigma_\tau(u) = u$ otherwise for all $u \in V(X)$. Clearly, $\tau_{\sigma_\tau} = \tau$. However, generally speaking, $\sigma_{\tau_\sigma} \neq \sigma$.

A map $f : V(G) \to V(G)$ is called a *neighborhood map* if $f(u) \in N_G[u]$ for all vertices $u \in V(G)$. Clearly, for connected graphs G (in particular, for trees) a map f is a neighborhood map if and only if $d_G(u, f(u)) \leq 1$ for every $u \in V(G)$.

Proposition 2.7 ([6])**.** *For a map σ on a tree X it holds $\sigma_{\tau_\sigma} = \sigma$ if and only if σ is a neighborhood map.*

The number of arcs in Markov graphs for neighborhood maps can be calculated explicitly. Denote by $p(X, \sigma)$ and $n(X, \sigma)$ the number of σ-positive and σ-negative edges in X, respectively.

Theorem 2.8 ([6])**.** *For a tree X and its neighborhood map $\sigma : V(X) \to V(X)$ we have*

$$|A(\Gamma(X, \sigma))| = |E(X)| + 2p(X, \sigma) - \sum_{u \in \operatorname{fix} \sigma} d_X(u).$$

It should be noted that neighborhood permutations on trees were studied in [11–13] under the name of *compatible permutations*. Proposition 2.7 asserts that each edge transposition is a compatible permutation. This means that the class of compatible permutations on a tree is a generating set for the group of all permutations. Moreover, the minimal number k such that each permutation of the vertex set of an n-vertex tree X can be obtained as a composition of at most k compatible permutations, is at least n, [12] (the equality $k = n$ holds for n-vertex path P_n, [11]).

Theorem 2.9 ([4])**.** *For a tree X and its map $\sigma : V(X) \to V(X)$ we have*

$$n(X, \sigma) + |\operatorname{fix} \sigma| = p(X, \sigma) + 1.$$

2.4. EXPANSIVE AND ANTI-EXPANSIVE MAPS

A map σ on a tree X is called *expansive* if in $\Gamma(X, \sigma)$ each vertex has a loop. In other words, σ is expansive if $\mathrm{id}_{V(X)} \leq_m \sigma$ (here $\mathrm{id}_{V(X)}$ denotes the identity map on $V(X)$). Note that maps σ with $\sigma \leq_m \mathrm{id}_{V(X)}$ were characterized in [5].

Theorem 2.10 ([5])**.** *Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. Then each arc in $\Gamma(X, \sigma)$ is a loop if and only if $\sigma = \mathrm{pr}_e^*$ for some edge $e \in E(X)$, or σ is a projection on a connected set of vertices in X.*

The following result is a direct corollary of Theorem 2.6.

Proposition 2.11 ([4])**.** *Let X be a tree and $\tau : E(X) \to \{1, -1\}$. Then τ is admissible if and only if $\tau^{-1}(-1)$ is a matching in X.*

Conversely, for every matching $E' \subset E(X)$ in a tree X put $\tau_{E'}(e) = 1$ for $e \notin E'$ and $\tau_{E'}(e) = -1$ for $e \in E'$. Clearly, the resulting labeling $\tau_{E'} : E(X) \to \{1, -1\}$ satisfies the condition of Proposition 2.11 and thus the map $\sigma_{E'} = \sigma_{\tau_{E'}} : V(X) \to V(X)$ is correctly defined. This means that there is a one-to-one correspondence between labelings τ_σ for expansive maps σ and matchings in X.

Observe that for an admissible edge labeling τ on a tree X the map σ_τ is a permutation if and only if σ_τ is expansive. Thus, the number of admissible labelings τ with σ_τ being a permutation, equals the number of matchings in X (including the empty matching). The last number is a well-known topological index, which is called the *Hosoya index*, [3].

Also, using the formula

$$W(X) = \sum_{uv \in E(X)} |A_X(u, v)| \cdot |A_X(v, u)|$$

for the *Wiener index* (originally defined as $W(X) = \sum_{\{u,v\} \subset V(X)} d_X(u, v)$) for trees X (see [1]), one can observe that the number of expansive maps σ with $n(X, \sigma) = 1$ is equal to $W(X)$.

A map σ on a tree X is called *anti-expansive* if $\Gamma(X, \sigma)$ does not contain vertices with loops. Thus, τ_σ is just an orientation of the edges in X. Having a tree X and orientation of its edges $\tau : E(X) \to V(X)$ we say that $X(\tau)$ is an *in-tree* if there is a vertex $u_0 \in V(X)$ such that $u \to v$ in $X(\tau)$ implies $v \in [u, u_0]_X$ for all edges $uv \in E(X)$. The vertex u_0 is called the *root* of $X(\tau)$.

Proposition 2.12 ([4])**.** *Let X be a tree and $\tau : E(X) \to V(X)$ be an orientation of the edges in X. Then τ is admissible if and only if $X(\tau)$ is an in-tree.*

From Proposition 2.12 we can conclude that every anti-expansive map on a finite tree has a unique fixed point (namely, the root of the corresponding in-tree). Further, if σ_1 and σ_2 are two anti-expansive maps, then $\tau_{\sigma_1} = \tau_{\sigma_2}$ if and only if $\mathrm{fix}\, \sigma_1 = \mathrm{fix}\, \sigma_2$. Therefore, there is a one-to-one correspondence between labelings τ_σ for anti-expansive maps σ and vertices in $V(X)$.

Example 2.13. Consider the tree X with the vertex set $V(X) = \{1, \ldots, 7\}$ and the edge set $E(X) = \{12, 23, 34, 45, 26, 47\}$ as well as the pair of maps

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 7 & 6 & 5 & 3 & 6 & 7 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 3 & 1 & 6 & 4 & 2 \end{pmatrix}.$$

It is easy to see that σ_1 is expansive and σ_2 is anti-expansive (see Figure 1).

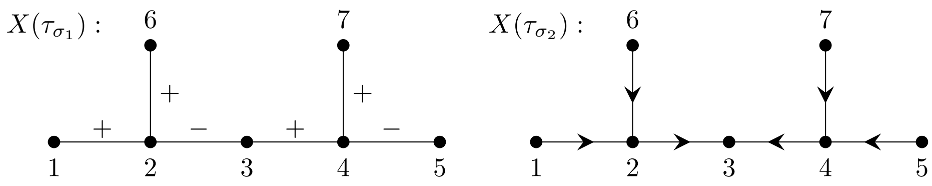

Fig. 1. Mixed trees for maps from Example 2.13 (the signs + and − denote σ_1-positive and σ_1-negative edges, respectively)

3. MAIN RESULTS

We start by showing that a tree X can be reconstructed from the Markov graph $\Gamma(X, \sigma_{\tau_\sigma})$ for any anti-expansive map $\sigma : V(X) \to V(X)$.

Theorem 3.1. *Let X be a tree and $\sigma : V(X) \to V(X)$ be an anti-expansive map with* $\operatorname{fix}\sigma = \{u_0\}$. *Then* $\Gamma(X, \sigma_{\tau_\sigma}) \simeq X(\tau_\sigma) - \{u_0\}$.

Proof. Consider the map $\varphi : V(X) \setminus \{u_0\} \to E(X)$, where $\varphi(u) = u\sigma_{\tau_\sigma}(u)$ for all $u \in V(X) \setminus \{u_0\}$. Fix an edge $e = uv \in E(X)$ and suppose that $v \in [u, u_0]_X$. Then $\sigma_{\tau_\sigma}(u) = v$ and thus $\varphi(u) = e$. Hence, φ is a surjective map. Combining this with the fact that the sets $V(X) \setminus \{u_0\}$ and $E(X)$ have the same cardinality, we can conclude that φ is bijective.

Now assume that there is an arc $u \to v$ in $X(\tau_\sigma) - \{u_0\}$. Then $uv \in E(X)$ and $\sigma_{\tau_\sigma}(u) = v$. This means that there exists an arc

$$\varphi(u) = u\sigma_{\tau_\sigma}(u) = uv \to v\sigma_{\tau_\sigma}(v) = \varphi(v)$$

in $\Gamma(X, \sigma_{\tau_\sigma})$. Therefore, φ induces an injective map

$$\varphi' : A(X(\tau_\sigma) - \{u_0\}) \to A(\Gamma(X, \sigma_{\tau_\sigma})).$$

But the equality

$$|A(\Gamma(X, \sigma_{\tau_\sigma}))| = |E(X)| - d_X(u_0) = |V(X)| - 1 - d_X(u_0) = |A(X(\tau_\sigma) - \{u_0\})|$$

implies that φ' is bijective. Therefore, φ is an isomorphism between $X(\tau_\sigma) - \{u_0\}$ and $\Gamma(X, \sigma_{\tau_\sigma})$. □

Corollary 3.2. *For a digraph Γ there exists a tree X and its anti-expansive map $\sigma : V(X) \to V(X)$ with $\Gamma(X, \sigma_{\tau_\sigma}) \simeq X(\tau_\sigma) - \{u_0\}$ if and only if Γ is acyclic partial functional digraph.*

Proof. If σ is anti-expansive, then from Theorem 3.1 it follows that $\Gamma(X, \sigma_{\tau_\sigma}) \simeq X(\tau_\sigma) - \{u_0\}$ is acyclic partial functional digraph. This proves the necessity of the condition. To prove its sufficiency assume that Γ is acyclic partial functional digraph and let f be the corresponding partial map on $V(\Gamma)$. Clearly, each component of Γ is an in-tree. Let V' denotes the set of roots of components of Γ (thus, $f : V(\Gamma) \setminus V' \to V$).

Consider the graph X with $V(X) = V(\Gamma) \sqcup \{u_0\}$ and $E(X) = \{uf(u) : u \in V(\Gamma)\} \cup \{u_0 v : v \in V'\}$. Clearly, X is a tree.

Construct the map σ on $V(X)$ in the following way:

$$\sigma(u) = \begin{cases} f(u), & \text{if } u \in V(\Gamma) \setminus V', \\ u_0, & \text{if } u \in V' \text{ or } u = u_0 \end{cases}$$

for all $u \in V(X)$. It is easy to see that σ is an anti-expansive neighborhood map on X, thus by Proposition 2.7 we have $\sigma_{\tau_\sigma} = \sigma$. Moreover, $\Gamma \simeq \Gamma(X, \sigma)$ by construction. □

The next result shows that the map σ_{τ_σ} is the least element (with respect to $\leq_m$) in $\{\sigma' \in V(X)^{V(X)} : \tau_{\sigma'} = \tau_\sigma\}$ provided σ is expansive.

Proposition 3.3. *For every tree X and its expansive map $\sigma : V(X) \to V(X)$ it holds $\sigma_{\tau_\sigma} \leq_m \sigma$.*

Proof. Fix an edge $e = uv \in E(X)$. If e is σ-negative, then

$$N^+_{\Gamma(X,\sigma_{\tau_\sigma})}(e) = \{e\} \subset N^+_{\Gamma(X,\sigma)}(e).$$

Further, let e be σ-positive. If $u, v \in \operatorname{fix} \sigma$, then

$$N^+_{\Gamma(X,\sigma_{\tau_\sigma})}(e) = \{e\} = N^+_{\Gamma(X,\sigma)}(e).$$

If $u \in \operatorname{fix} \sigma$ and $v \notin \operatorname{fix} \sigma$, then e is adjacent to some σ-negative edge $e' = vw$ with $\sigma(v) \in A_X(w, v)$. In this case

$$N^+_{\Gamma(X,\sigma_{\tau_\sigma})}(e) = \{e, e'\} \subset N^+_{\Gamma(X,\sigma)}(e).$$

Finally, if $u, v \notin \operatorname{fix} \sigma$, then there exist two σ-negative edges $e_1 = uw_1$ and $e_2 = vw_2$ with $\sigma(u) \in A_X(w_1, u)$ and $\sigma(v) \in A_X(w_2, v)$. We have

$$N^+_{\Gamma(X,\sigma_{\tau_\sigma})}(e) = \{e, e_1, e_2\} \subset N^+_{\Gamma(X,\sigma)}(e).$$

Therefore, $\Gamma(X, \sigma_{\tau_\sigma}) \subset \Gamma(X, \sigma)$ and the desired is proved. □

Corollary 3.4. *For every expansive map σ on a tree X it holds*

$$|A(\Gamma(X,\sigma))| \geq |E(X)| + 2p(X,\sigma) - \sum_{u \in \operatorname{fix} \sigma} d_X(u).$$

Proof. Follows from Theorem 2.8 and Proposition 3.3. □

Now we prove that the Markov graph $\Gamma(X, \sigma)$ for an expansive map σ almost always contains a digon.

Proposition 3.5. *Let X be a tree with $|V(X)| \geq 2$ and $\sigma : V(X) \to V(X)$ be its expansive map. Then $\Gamma(X, \sigma)$ has no digons if and only if $\sigma = \sigma_{\tau_\sigma}$.*

Proof. First, we prove the necessity of this condition. If $|V(X)| = 2$, then for each of the two different expansive maps σ on $V(X)$ we have $\sigma = \sigma_{\tau_\sigma}$. Therefore, let $|V(X)| \geq 3$. Fix a vertex $u \in V(X)$. If each edge $e \in E_X(u)$ is σ-positive, then $u \in \operatorname{fix}\sigma$ and therefore $\sigma(u) = \sigma_{\tau_\sigma}(u)$. Hence, suppose that there exists σ-negative edge $e = ux \in E_X(u)$. If $\sigma(u) \neq x = \sigma_{\tau_\sigma}(u)$, then there exists a vertex $y \in [x, \sigma(u)]_X$ adjacent to x. Clearly, the edge xy is σ-positive. This means that there is a digon $xy \leftrightarrow ux$ in $\Gamma(X, \sigma)$ which is a contradiction.

Now we prove the sufficiency of this condition. Fix an arbitrary arc $e_1 \to e_2$ in $\Gamma(X, \sigma)$ which is not a loop. Since $\sigma = \sigma_{\tau_\sigma}$, then σ is a neighborhood map on X. This means that the edges e_1 and e_2 are adjacent in X. In other words, $e_1 = u_1 v$ and $e_2 = u_2 v$ for some $u_1, u_2, v \in V(X)$. We have $\sigma(v) = u_2$ implying $e_2 \not\to e_1$ in $\Gamma(X, \sigma)$. Therefore, the Markov graph $\Gamma(X, \sigma)$ has no digons. □

Proposition 3.6. *Let X be a path and $\sigma : V(X) \to V(X)$ be its anti-expansive map with $\operatorname{fix}\sigma = \{u_0\}$. Then $\sigma_{\tau_\sigma} \leq_m \sigma$ if and only if $\sigma_{\tau_\sigma}(u) = \sigma(u)$ for all $u \in V(X) \setminus N_X(u_0)$.*

Proof. First, we prove the necessity of this condition. Fix a vertex $u \in V(X) \setminus N_X(u_0)$. If $u = u_0$, then clearly $\sigma_{\tau_\sigma}(u) = \sigma(u)$. Otherwise, $d_X(u, u_0) \geq 2$ implying the existence of two vertices $x, y \in [u, u_0]_X$ with $ux, xy \in E(X)$ (it can be $u_0 = y$). Since $\tau_\sigma(ux) = x$ and $\tau_\sigma(xy) = y$, there is an arc $ux \to xy$ in $\Gamma(X, \sigma_{\tau_\sigma})$. But $\sigma_{\tau_\sigma} \leq_m \sigma$ means that there is an arc $ux \to xy$ in $\Gamma(X, \sigma)$ as well. Therefore, $x, y \in [\sigma(u), \sigma(x)]_X$. Further, $\sigma(x), \sigma(y) \in A_X(y, x)$ as σ is anti-expansive. Hence, $\sigma(x) \notin A_X(x, y)$ implying $\sigma(u) \in A_X(x, y)$. But X is a path which means $d_X(x) = 2$. Combining these facts with $\sigma(u) \in A_X(x, u)$, we can conclude that $\sigma(u) = x$. Thus, $\sigma_{\tau_\sigma}(u) = \sigma(u)$.

To prove the sufficiency of this condition fix an edge $e \in E(X)$. If e is incident to u_0, then $N^+_{\Gamma(X,\sigma_{\tau_\sigma})}(e) = \emptyset$ trivially implying $N^+_{\Gamma(X,\sigma_{\tau_\sigma})}(e) \subset N^+_{\Gamma(X,\sigma)}(e)$. Thus let $e = ux$ is not incident to u_0. Without loss of generality, we can assume that $x \in [u, u_0]_X$. Since $x \neq u_0$, there exists a vertex $y \in [x, u_0]_X$ adjacent to x. Clearly, $N^+_{\Gamma(X,\sigma_{\tau_\sigma})}(e) = \{xy\}$. But since $u \notin N_X(u_0)$, we have $\sigma(u) = \sigma_{\tau_\sigma}(u) = x$. Combining this with $\sigma(x) \in A_X(y, x)$, we obtain $x, y \in [\sigma(u), \sigma(x)]_X$. In other words, there is an arc $e \to xy$ in $\Gamma(X, \sigma)$ which means that $N^+_{\Gamma(X,\sigma_{\tau_\sigma})}(e) \subset N^+_{\Gamma(X,\sigma)}(e)$ again. Therefore, $\Gamma(X, \sigma_{\tau_\sigma}) \subset \Gamma(X, \sigma)$ and the desired is proved. □

Theorem 3.7. *Let X be a tree and $A \subset V(X)$ be some set of its vertices. A map $\sigma : A \to V(X)$ can be extended to an expansive map on $V(X)$ if and only if the following conditions hold:*

1. *the map $\operatorname{pr}_B \circ\sigma$ is expansive on $X[B]$ for every maximal connected subset $B \subset A$;*
2. *for each vertex $u \in V(X) \setminus A$ there is at most one vertex $v \in A \cap N_X(u)$ with $\sigma(v) \in A_X(u, v)$.*

Proof. Let $\sigma' : V(X) \to V(X)$ be an expansive extension of σ. Clearly, $\operatorname{pr}_B \circ\sigma'$ is expansive on $X[B]$ for every connected set $B \subset V(X)$. Thus, the first condition holds. To prove the second condition assume that there is a vertex $u \in V(X)$ and a pair of vertices $v_1, v_2 \in A \cap N_X(u)$ with $\sigma(v_i) \in A_X(u, v_i), i = 1, 2$. Since σ' is expansive,

$\sigma'(u) \in A_X(v_i, u), i = 1, 2$. This means that the edges uv_1 and uv_2 are both σ'-negative which contradicts Proposition 2.11.

Conversely, assume both conditions hold. Consider two vertex sets

$$V_0 = \{u \in N_X(A) : \sigma(v) \in A_X(v, u) \text{ for all } v \in A \cap N_X(u)\}$$

and

$$V_1 = \{u \in N_X(A) : \sigma(v) \in A_X(u, v) \text{ for some } v \in A \cap N_X(u)\}.$$

Put

$$\sigma'(u) = \begin{cases} \sigma(u), & \text{if } u \in A, \\ u, & \text{if } u \in V_0 \cup (V(X) \setminus (A \cup N_X(A))), \\ v, & \text{if } u \in V_1 \end{cases}$$

for all $u \in V(X)$. By the second condition, the map σ' is correctly defined. Moreover, from the first condition and the construction of σ' it follows that σ' is an expansive extension of σ. □

Note that the first condition in Theorem 3.7 can be omitted provided the set A is independent.

Proposition 3.8. *Let X be a tree and $A \subset V(X)$ be some set of its vertices. A map $\sigma : A \to V(X)$ can be extended to an anti-expansive map on $V(X)$ if and only there exists a vertex $u_0 \in V(X)$ such that*

1. *for all $v \in A \setminus \{u_0\}$ the vertices $\sigma(v)$ and u_0 lie in the same connected component of $X - \{v\}$;*
2. *if additionally $u_0 \in A$, then $\sigma(u_0) = u_0$.*

Proof. Assume that $\sigma' : V(X) \to V(X)$ is an anti-expansive extension of σ. Then $|\operatorname{fix} \sigma'| = 1$. Let u_0 be the unique vertex from $\operatorname{fix} \sigma'$. To the contrary, suppose there exists $v \in A \setminus \{u_0\}$ such that $\sigma(v)$ and u_0 lie in different connected components of $X - \{v\}$. Consider the (connected) set of vertices $B = [u_0, v]_X$ and the map $\sigma'' = \mathrm{pr}_B \circ \sigma'$ on B. We have $\sigma''(u_0) = u_0$ and $\sigma''(v) = v$, i.e. $|\operatorname{fix} \sigma''| \geq 2$. Using Theorem 2.9, we obtain

$$p(X[B], \sigma'') = n(X[B], \sigma'') + |\operatorname{fix} \sigma''| - 1 \geq 1.$$

Thus, $X[B]$ contains a σ''-positive edge e. By Lemma 2.4, e is also a σ'-positive edge which is a contradiction. Conversely, suppose both conditions hold. Then the map

$$\sigma'(v) = \begin{cases} \sigma(v), & \text{if } v \in A, \\ u_0, & \text{if } v \in V(X) \setminus A \end{cases}$$

for all $v \in V(X)$, is an anti-expansive extension of σ. □

Theorem 2.9 implies that for any tree X with an odd number of vertices each expansive map $\sigma : V(X) \to V(X)$ has a fixed point. Indeed, if σ is expansive, then the sum $p(X, \sigma) + n(X, \sigma) = |E(X)| = |V(X)| - 1$ is even. Clearly, the difference $p(X, \sigma) - n(X, \sigma)$ is also even. By Theorem 2.9, $|\operatorname{fix} \sigma| = p(X, \sigma) - n(X, \sigma) + 1$ is odd. This result can be easily generalized as follows.

Proposition 3.9. *Let X be a tree and $A \subset V(X)$ be some set of its vertices. Then there exists an expansive map $\sigma : V(X) \to V(X)$ with $\operatorname{fix} \sigma = A$ if and only if $X - A$ has a perfect matching.*

Proof. For an expansive map $\sigma : V(X) \to V(X)$ we have $u \in \operatorname{fix} \sigma$ if and only if each edge from $E_X(u)$ is σ-positive. Thus, all σ-negative edges form a perfect matching in $X - \operatorname{fix} \sigma$. Conversely, given a set of vertices $A \subset V(X)$ and a perfect matching $E' \subset E(X - A)$ in $X - A$ the map $\sigma_{E'}$ is expansive and $\operatorname{fix} \sigma_{E'} = A$. □

We can also show that if a tree X has a perfect matching, then we can construct not only an expansive map on $V(X)$ without fixed points but rather an expansive cyclic permutation of $V(X)$.

Theorem 3.10. *For a tree X with $|V(X)| \geq 2$ there exists an expansive cyclic permutation σ of $V(X)$ if and only if X has a perfect matching.*

Proof. If σ is an expansive cyclic permutation of $V(X)$, then since $|V(X)| \geq 2$, σ does not have fixed points. This means that the set of all σ-negative edges in X forms a perfect matching.

To prove the converse we use induction on $|V(X)|$. Clearly, if X has a perfect matching, then X has an even number of vertices. If $|V(X)| = 2$, then X is a path with two vertices and the unique cyclic permutation of $V(X)$ is expansive. Thus the induction basis holds. Now let $|V(X)| \geq 4$ and $E' \subset E(X)$ be the perfect matching in X. Since $|V(X)| \geq 4$, the tree $X' = X - L(X)$ has at least one edge.

Fix a leaf vertex $v \in L(X')$. Since each leaf edge in X belongs to E', there exists a unique leaf vertex $u \in L(X)$ with $uv \in E'$. It is easy to see that $E' \setminus \{uv\}$ is a perfect matching in a tree $X'' = X - \{u, v\}$. By induction assumption there exists an expansive cyclic permutation $\sigma_0 : V(X'') \to V(X'')$. Since $v \in L(X')$, there is a unique vertex $w \in V(X'')$ with $vw \in E(X')$. Consider the map

$$\sigma(x) = \begin{cases} v, & \text{if } x = \sigma_0^{-1}(w), \\ u, & \text{if } x = v, \\ w, & \text{if } x = u, \\ \sigma_0(x), & \text{otherwise} \end{cases}$$

for all $x \in V(X)$. Then σ is an expansive cyclic permutation of $V(X)$. □

Example 3.11. Consider the tree X with $V(X) = \{1, \ldots, 6\}$ and $E(X) = \{12, 23, 34, 45, 36\}$. Then $E' = \{12, 45, 36\}$ is a perfect matching in X. The corresponding expansive cyclic permutation is $\sigma = (136452)$.

Theorem 2.9 implies that a tree with at least two vertices does not admit an anti-expansive cyclic permutation of its vertices. We will call a permutation σ of $V(X)$ *almost cyclic* if $|\operatorname{fix} \sigma| = 1$ and the restriction $\sigma|_{V(X) \setminus \operatorname{fix} \sigma}$ is a cyclic permutation. In [12] it was proved that for any tree X with a singleton centroid u_0 there exists almost cyclic permutation σ of $V(X)$ such that v and $\sigma(v)$ lie in different components of $X - \{u_0\}$ for all $v \in V(X) \setminus \{u_0\}$. Clearly, such a permutation σ is anti-expansive.

Proposition 3.12. *For every tree X and its non-leaf vertex $u_0 \in V(X) \setminus L(X)$ there exists an anti-expansive almost cyclic permutation σ of $V(X)$ with* $\operatorname{fix}\sigma = \{u_0\}$.

Proof. Let $N_X(u_0) = \{v_1, \ldots, v_d\}$, where $d = d_X(u_0) \geq 2$. For all $1 \leq i \leq d$ and $j \geq 0$ fix a linear ordering of $N_X^j(v_i) \cap A_X(v_i, u_0) = \{x_1^{i,j}, \ldots, x_k^{i,j}\}$, where $k = k_{ij}$ (for example, $k_{i0} = 1$ and $x_1^{i,0} = v_i$ for all $1 \leq i \leq d$). Put

$$m_i = \max\{d_X(v_i, w) : w \in A_X(v_i, u_0)\}$$

for $1 \leq i \leq d$. Consider the map

$$\sigma(y) = \begin{cases} u_0, & \text{if } y = u_0, \\ x_{l+1}^{i,j}, & \text{if } y = x_l^{i,j} \text{ and } l \neq k_{ij}, \\ x_1^{i,j-1}, & \text{if } y = x_k^{i,j} \text{ for } k = k_{ij}, j \geq 2, \\ v_i, & \text{if } y = x_k^{i,1} \text{ for } k = k_{i1}, \\ x_1^{i+1,m_{i+1}}, & \text{if } y = v_i, i \neq d, \\ x_1^{1,m_1}, & \text{if } y = v_d \end{cases}$$

for all $y \in V(X)$. Then σ is anti-expansive almost cyclic permutation of $V(X)$ with $\operatorname{fix}\sigma = \{u_0\}$. $\square$

Example 3.13. Consider the tree X with $V(X) = \{1, \ldots, 8\}$ and $E(X) = \{12, 23, 34, 45, 26, 37, 48\}$. Let $u_0 = 3$. Then $d = 3$ and $N_X(u_0) = \{2, 4, 7\}$. Let $v_1 = 2$, $v_2 = 4$ and $v_3 = 7$. Then $m_1 = m_2 = 1$ and $m_3 = 0$. We have $N_X^1(v_1) \cap A_X(v_1, u_0) = \{1, 6\}$ and $N_X^1(v_2) \cap A_X(v_2, u_0) = \{5, 8\}$. Suppose $x_1^{1,1} = 1, x_2^{1,1} = 6$ and $x_1^{2,1} = 5, x_2^{2,1} = 8$. Then the map σ has a cycle

$$x_1^{1,1} \to x_2^{1,1} \to v_1 \to x_1^{2,1} \to x_2^{2,1} \to v_2 \to v_3 \to x_1^{1,1}.$$

In other words, $\sigma = (3)(1625847)$.

Theorem 3.14. *For a neighborhood map σ on a tree X with $|V(X)| \geq 2$ its Markov graph $\Gamma(X, \sigma)$ is weakly connected if and only if* $\operatorname{fix}\sigma \subset L(X)$.

Proof. Assume that $\Gamma(X, \sigma)$ is weakly connected, but there is a non-leaf fixed vertex $u \in \operatorname{fix}\sigma \setminus L(X)$. Fix its neighbor $v \in N_X(u)$ and consider an edge set

$$E' = E(A_X(v, u)) \cup \{uv\}.$$

Clearly, $E' \neq \emptyset$. Moreover, the inequality $d_X(u) \geq 2$ implies $E(X) \setminus E' \neq \emptyset$ as well. Now observe that

$$A(\Gamma(X, \sigma)) \subset (E' \times E') \cup ((E(X) \setminus E') \times (E(X) \setminus E'))$$

which contradicts to the weak connectedness of $\Gamma(X, \sigma)$ (see Lemma 2.2).

Conversely, suppose $\operatorname{fix}\sigma \subset L(X)$. Fix a proper edge subset $E' \subset E(X)$. Since X is connected, there exists a vertex $u \in V(X)$ with $E_X(u) \cap E' \neq \emptyset$ and $E_X(u) \cap (E(X) \setminus E') \neq \emptyset$. Trivially, $u \notin L(X)$. Thus, $u \notin \operatorname{fix}\sigma$. Further, since σ is a neighborhood map, $u\sigma(u) \in E(X)$. Without loss of generality, assume $u\sigma(u) \in E'$. Fix a vertex $v \in N_X(u)$ with $uv \in E(X) \setminus E'$. Then $uv \to u\sigma(u)$ in $\Gamma(X, \sigma)$ and thus by Lemma 2.2 the Markov graph $\Gamma(X, \sigma)$ is weakly connected. $\square$

Corollary 3.15. *Every expansive tree map without fixed points has a weakly connected Markov graph.*

Proof. Follows from Theorem 3.14 and Proposition 3.3. □

Corollary 3.16. *Let X be a tree and $E' \subset E(X)$ be a matching in X. Then the Markov graph $\Gamma(X, \sigma_{E'})$ is weakly connected if and only if E' is a weakly dominating set of edges.*

Proof. Follows from Theorem 3.14 and the fact that and edge e is $\sigma_{E'}$-negative if and only if $e \in E'$. □

Corollary 3.17. *For every tree X with $|V(X)| \geq 2$ there exists an expansive map on $V(X)$ with a weakly connected Markov graph.*

Proof. From Lemma 2.1 it follows that there exists a set $A \subset L(X)$ such that $X - A$ has a perfect matching $E' \subset E(X - A)$. Clearly, E' is a matching in X. Moreover, for every vertex $u \in V(X) \setminus L(X)$ there exists an edge $e \in E'$ incident to u. Thus, E' is a weakly dominating matching in X. By Corollary 3.16, the Markov graph $\Gamma(X, \sigma_{E'})$ is weakly connected. □

Note that the result of Corollary 3.17 is best possible since stars $K_{1,n}$ for $n \geq 3$ do not permit expansive maps with unilaterally (and therefore with strongly) connected Markov graphs.

Remark 3.18. Consider a tree X together with the linear ordering of its edge set $E(X) = \{e_1, \dots, e_{n-1}\}$. If there is an arc $e_i \to e_j$ in $\Gamma(X, \sigma^m)$ for some $m \geq 1$, then by Proposition 2.5, $(M^m_{\Gamma(X,\sigma)})_{ij} = (M_{\Gamma(X,\sigma^m)})_{ij} = 1 \bmod 2$. Thus, $(M^m_{\Gamma(X,\sigma)})_{ij} \geq 1$ which means that e_j is reachable from e_i in $\Gamma(X, \sigma)$.

Theorem 3.19. *If a tree X with $|V(X)| \geq 2$ has a perfect matching, then there exists an expansive map on $V(X)$ with a strongly connected Markov graph.*

Proof. Fix a cyclic permutation σ' of $L(X)$. For every $u \in V(X) \setminus L(X)$ let $e_u = uv_u$ be the unique edge from the perfect matching in X. Since $A_X(v_u, u) \cap L(X) \neq \emptyset$ for all $u \in V(X)$, we can fix a vertex $x_u \in A_X(v_u, u) \cap L(X)$. Put

$$\sigma(u) = \begin{cases} \sigma'(u), & \text{if } u \in L(X), \\ x_u, & \text{if } u \in V(X) \setminus L(X) \end{cases}$$

for all $u \in V(X)$. Clearly, the map σ is expansive.

Let us show that $\Gamma(X, \sigma)$ is strongly connected. Fix a proper edge subset $E' \subset E(X)$. Since X is connected, there is a vertex $v \in V(X)$ with $E_X(v) \cap E' \neq \emptyset$ and $E_X(v) \cap (E(X) \setminus E') \neq \emptyset$. If $vx_v \in E(X) \setminus E'$, then $e \to vx_v$ in $\Gamma(X, \sigma)$ for all $e \in E_X(v)$. Thus, assume that $vx_v \in E'$ and fix an edge $uv \in E(X) \setminus E'$. Since the restriction $\sigma|_{L(X)}$ is a cyclic permutation, there exists $m \geq 1$ such that $\sigma^m(\sigma(v)) \in A_X(u, v)$. Let m_0 be the smallest such an m. Then $\sigma^{m_0 - 1}(\sigma(v)) = \sigma^{m_0}(v) \in A_X(v, u)$. Therefore, $uv \in E([\sigma^{m_0}(v), \sigma^{m_0}(\sigma(v))]_X)$. By Lemma 2.3, this implies the existence of an edge $e \in E([v, \sigma(v)]_X)$ with $e \to uv$ in $\Gamma(X, \sigma^{m_0})$. By Remark 3.18, the edge uv is reachable from e in $\Gamma(X, \sigma)$. Finally, the construction of σ yields $[v, \sigma(v)]_X \subset [\sigma(x_v), \sigma(v)]_X$ which means that uv is reachable from vx_v in $\Gamma(X, \sigma)$. Thus, Lemma 2.2 asserts that $\Gamma(X, \sigma)$ is strongly connected. □

Theorem 3.20. *For every tree X with $|V(X)| \geq 3$ there exists an anti-expansive map on $V(X)$ with a strongly connected Markov graph.*

Proof. First, let X be a star with $V(X) \setminus L(X) = \{u_0\}$. Consider an almost cyclic permutation on $V(X)$ with $\operatorname{fix} \sigma = \{u_0\}$. Clearly, σ is anti-expansive. Moreover, $\Gamma(X, \sigma)$ is a cycle thus a strongly connected digraph.

Further, we use induction on $|V(X)|$. If $|V(X)| = 3$, then $X \simeq P_3 \simeq K_{1,2}$ is a star. Therefore, induction basis trivially holds. Now suppose $|V(X)| \geq 4$. Without loss of generality, we can assume that X is not a star. Then there exists a vertex $u_0 \in L(X - L(X))$. Obviously, $L(X) \cap N_X(u_0) \neq \emptyset$. Hence, we can consider the tree $X' = X - (L(X) \cap N_X(u_0))$. Trivially, $|V(X')| < |V(X)|$. By induction assumption there exists an anti-expansive map $\sigma' : V(X') \to V(X')$ with a strongly connected Markov graph $\Gamma' = \Gamma(X', \sigma')$. Note that $\sigma'(u_0) \neq u_0$ as $u_0 \in L(X')$ and $|E(X')| \geq 2$ (as X is not a star).

Since $u_0 \in L(X')$, there exists a unique vertex x_0 adjacent to u_0 in X'. Also, we have $|V(\Gamma')| = |E(X')| \geq 2$. Combining this fact with the strong connectedness of Γ', we obtain the inequality $d^-_{\Gamma'}(u_0x_0) \geq 1$. But u_0 is a leaf vertex in X'. Therefore, there exists an edge $y_0y \in E(X')$ with $\sigma'(y_0) = u_0$ and $\sigma'(y) \neq u_0$.

Let $L(X) \cap N_X(u_0) = \{x_1, \ldots, x_m\}$, $m \geq 1$. Consider the following map:

$$\sigma(u) = \begin{cases} x_1, & \text{if } u = y_0, \\ x_{i+1}, & \text{if } u = x_i \text{ for } i \neq m, \\ u_0, & \text{if } u = x_m, \\ \sigma'(u), & \text{otherwise} \end{cases}$$

for all $u \in V(X)$. Then σ is anti-expansive map and Γ' is a subgraph of $\Gamma(X, \sigma)$. Moreover, $\Gamma(X, \sigma)$ contains the path

$$P = \{y_0y \to u_0x_1 \to \cdots \to u_0x_m \to u_0x_0\}.$$

Since both edges y_0y and u_0x_0 lie in Γ' and Γ' is strongly connected, we can conclude that $\Gamma(X, \sigma)$ is strongly connected as well (as it contains the spanning strongly connected subgraph $\Gamma' \cup P$). This proves the theorem. □

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