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Н. Ю. Щестюк , *к.ф.-м.н., доцент*
С. В. Тищенко , *магістр, студент-аспірант*

Метод Монте-Карло для оцінювання опціонів у субдифузійних моделях арифметичного броунівського руху

[1] Національний університет "Києво-Могилянська академія 04070, м. Київ, вул. Сковороди 2
e-mail: n.shchestiuk@ukma.edu.ua
[2] Національний університет "Києво-Могилянська академія 04070, м. Київ, вул. Сковороди 2
e-mail: s.tyshchenko@ukma.edu.ua

N. Y. Shchestyuk[1], *PhD, docent*
S. O. Tyshchenko[2], *master*

Monte-Carlo method for option pricing in sub-diffusion arithmetic Brownian motion models

[1] National University Of Kyiv-Mohyla Academy, Hryhoriya Skovorody St, 2, Kyiv, 04655
e-mail: n.shchestiuk@ukma.edu.ua
[2] National University Of Kyiv-Mohyla Academy, Hryhoriya Skovorody St, 2, Kyiv, 04655
e-mail: s.tyshchenko@ukma.edu.ua

Ця стаття присвячена застосуванню методу Монте-Карло до оцінювання опціонів на неліквідних ринках. Аномальна субдифузія - це добре відома модель для опису таких ринків, коли спостерігаються відносно тривалі періоди без будь-якої торгівлі. Для побудови субдифузійних моделей нам потрібно замінити календарний час t на деякий стохастичний процес $S(t)$, який називається оберненим субординатором. Обернений субординатор $S(t)$ інтерпретується як час першого досягнення деякої ціни, яка потім протягом тривалогу часу може не змінюватись і базується на використанні деякого іншого випадкового процесу, що називається субординатором. У цій роботі ми пропонуємо використати гамма-процес як субординатор для субдифузійної моделі Башельє. Використовуючи добре відомі властивості для гамма-процесів та обернених гамма-процесів, ми знаходимо коваріаційну структуру фрактальної моделі Башельє з FBM (фрактальним броунівським рухом) у якому замінюємо час на гамма-процес, а потім досліджуємо її асимптотичну поведінку. Потім ми застосовуємо метод Монте-Карло для оцінювання опціонів у цій субдифузійної моделі Башельє. Для цього ми використовуємо ітераційні схеми для моделювання N сценаріїв цін на акції для наших моделей. Також ми демонструємо числові результати.

Ключові слова: субдифузія, гамма-процес, метод Монте-Карло, оцінювання опціонів.

This paper focuses on applying Monte Carlo approach to option pricing in markets with illiquid assets. Anomalous sub-diffusion is a well-known model for describing such markets, when relatively long periods without any trading are observed. For constructing sub-diffusive models we need to replace a calendar time t with the some stochastic processes $S(t)$, which is called inverse subordinator. The inverse subordinator $S(t)$ means first hitting time and based on subordinator processes. In this paper we propose to use gamma gamma process as subordinator for Bashelie sub-diffusion model. Using well-known properties for gamma and inverse gamma processes we find the covariance structure of fractional Bachelier model with FBM time-changed by gamma process and then explore the asymptotic behavior of it. Then we apply Monte-Carlo method and propose procedure of option pricing for Bashelie sub-diffusion model. For this aim we use the iterative schemes for simulating N scenarios of stock prices for our models. Finally we demonstrate numerical results.

Key Words: sub-diffusion, gamma process, Monte-Carlo approach, option pricing.

1 Introduction

In recent decades time-changed stochastic processes are getting increasing attention.These stochastic processes are used for example in finance in fractal activity time (FAT) models to provide long memory (dependence) for log returns. In the papers [1], [2], [6]- [8] were presented option pricing for such models, which belong of the larger class of diffusion process with *"market"* time.

Time-changed stochastic processes are also used in statistical physics to model anomalous diffusion phenomena. In markets with illiquid assets, we often see the similar anomalous sub-diffusion, when relatively long periods without any trading are observed. This feature is most common for emerging markets in which the number of participants, and thus the number of transactions, is rather low.

Sub-diffusion is a well known and established phenomenon in statistical physics. It occurs if we replace the calendar time t with $S(t)$, where $S(t)$ is called the inverse subordinator. The inverse subordinator means first hitting time and it is given by the following formula:

$$S(t) = \inf\{\tau > 0 : U(\tau) > t\}. \tag{1}$$

There are several methods for finding price values of $C_\psi(X_0, K, T, \alpha)$ for subdiffusion models. One of them is to find call or put option price as solution of the forward partial integro-differential equation (PIDE) with some initial conditions [5]. The another method is by approximating integral in opting pricing formula [3]. And finally the method we use is Monte-Carlo approach.

The Monte-Carlo method is based on modeling the trajectories of the inverse subordinator $S_\psi(t)$ in the interval $[0, T]$ and calculate the fair price as the mathematical expectation. This method already was applied for two kinds of subordinators [3]: α stable and tempered α stable.

In this paper we propose to use gamma processes as subordinators.

The paper is organized as follows. In Section 2 we recall what is non-fractional Bachelier model, how to find fair price for this model and how to estimate difference between Bachelier price $C^B = C(X_0, K, T, \sigma)$ and Black-Scholes price C^{BS}.

In Section 3 we we remain what is subdiffusion and consider gamma processes as subordinators for Bashelie sub-diffusion model. Some basic well-known properties of the gamma process as a subordinator and some very useful properties of inverse gamma process as a inverse subordinator were considered. Then using these properties we find the covariance structure of fractional Bachelier model with FBM time-changed by gamma process and then explore the asymptotic behavior of it.

In Section 4 we apply Monte-Carlo method and propose procedure of option pricing for Bashelie sub-diffusion model. For this aim we use the iterative schemes for simulating N scenarios of stock prices for our models as in [9].

In Section 5 we present numerical results for evaluating fair prices for Bashelie sub-diffusion model with gamma process as a subordinator.

Last section contains conclusions.

2 Non-fractional Bachelier model

This section reviews some well-known results about option pricing in a arithmetic diffusion framework. We will consider later a time-changed version of this model.

It is worth to mention that recently in the day 2020-04-20, oil futures reached for first time in history negative values, where Bachelier model took an important role in option pricing and risk management. It demonstrates that sometimes using of Bachelier model is more reasonable than Black-Scholes.

Now we consider a financial market in which is traded a risk-free bond and a stock. The risk-free bond, noted A_t, earns a constant interest rate r and satisfies the differential equation:

$$\frac{dA_t}{A_t} = rdt \; A_0 = 1, \; t \geqslant 0. \tag{2}$$

We assume that the stock price, X_t, is ruled by a following diffusion:

$$dX_t = \mu dt + \sigma dB_t, \tag{3}$$

where $\mu \in \mathbb{R}$ is the drift parameter, $\sigma \in \mathbb{R}$ is the volatility.

It means that the stock price dynamic follows arithmetic Brownian motion:

$$X(t) = X_0 + \mu t + \sigma B(t), \; t \geqslant 0. \tag{4}$$

Here, $B(t)$ is the standard Wiener process or Brownian motion (BM) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The model (2)-(3) with solution (4) is arbitrage free and complete (see [3]). The option prices can be obtained by standard martingale method. The fair values of call and put on a stock X_t with expiry date T and strike price K are expressed as

$$C(X_0, K, T, \sigma) = \sigma\sqrt{T}\phi\left(\frac{X_0 - K}{\sigma\sqrt{T}}\right) + (X_0 - K)N\left(\frac{X_0 - K}{\sigma\sqrt{T}}\right), \tag{5}$$

$$P(X_0, K, T, \sigma) = \sigma\sqrt{T}\phi\left(\frac{K - X_0}{\sigma\sqrt{T}}\right) + (K - X_0)N\left(\frac{K - X_0}{\sigma\sqrt{T}}\right). \tag{6}$$

Here, ϕ and N are respectively the probability distribution function (PDF) and cumulative distribution functions (CDF) of the standard normal distribution.

The link between fair prices (5) and (6) for call and put options is given by the put-call parity formula:

$$P(X_0, K, T, \sigma) = C(X_0, K, T, \sigma) + K - X_0.$$

Notice, the values of fair prices computed by Bachelier and Black-Scholes approaches are very close. The difference between Bachelier price $C^B = C(X_0, K, T, \sigma)$ and Black-Scholes price C^{BS} was estimated in [3] as follows

$$0 \leqslant C^B - C^{BS} \leqslant \frac{X_0}{12\sqrt{2}}\sigma^B T^{3/2},$$

where σ^B is the implied volatility in the Bachelier model.

In the remaining of this paper, we explore the option pricing of a time-changed version of this model. We would like to extend Bachelier model for emerging market with periods, where assets price does not change. The next section introduces sub-diffusion and the stochastic clocks that we use as time-changes.

3 Sub-diffusion Bachelier model with Gamma subordinator

The usual model of sub-diffusion in physics is the celebrated Fractional Fokker-Planck equation. This equation was derived from the continuous-time random walk scheme with heavy-tailed waiting times. Equivalent description of sub-diffusion is in terms of subordination, where the standard diffusion process is time-changed by the so-called inverse subordinator.

A subordinator is a stochastic process of the evolution of time within another stochastic process, the subordinated stochastic process. In other words, a subordinator will determine the random number of "time steps"that occur within the subordinated process for a given unit of chronological time.

Thus, subordinator U_t is a stochastic process with positive, non-decreasing sample paths and taking value in $R+$.

We consider Levy subordinators for which increments are independent and homogeneously distributed. It is known that from the Levy–Khintchine formula the Laplace transform of Lévy subordinators has the following form:

$$\mathbb{E}\left(e^{-\omega U_t}\right) = e^{-tf(\omega)},$$

where $f(\omega) = b\omega + \int_0^{+\infty}(1 - e^{-\omega z})\overline{\nu}(dz)$ and $b \in \mathbb{R}^+$.

The inverse subordinator means first hitting time and it is defined by the formula (1).

This inverted Lévy subordinator is in general no more a Levy process. However S_t, $t \geqslant 0$, is positive and non-decreasing and has a [5] all requisite properties to be used as stochastic clock. By construction, the inverted process may be constant. Therefore, any process subordinated by S_t exhibits motionless periods. These random distributed motionless periods are characteristic for the subdiffusive dynamics and they represent the waiting times in which the test particle gets immobilized in the trap. In analogy with the physical description of sub-diffusion, we can consider the sub-diffusive arithmetic BM [3]:

$$X(S(t)) = X_0 + \mu S(t) + \sigma B_H(S(t)), \ t \geqslant 0, \tag{7}$$

where $S(t)$ is subordinating process and $B(t)$ and $S(t)$ are independent.

In recent years number of papers devoted to using the processes time-changed by inverse subordinators as models adequate for anomalous diffusion phenomena have grown rapidly. We only mention, for instance an inverse α - stable subordinator, the inverse tempered stable subordinator and the inverse gamma process as a time-change was examined in papers of M. Magdziarz, , A. Wyłomańska. The general case inverse subordinators based on infinite divisible processes were explored for example by T. R. Hurd and A. Kuznetsov, J..

In this paper we would like to use gamma process as a subordinator for description time periods in which price does not change. It is reasonable because gamma process is a pure-jump increasing Lévy process and thus it is a good candidate for construction waiting time.

Suppose U_t, $t \geqslant 0$, is gamma process $\Gamma(t, \alpha, \beta)$, thus it is a positive nondecreasing process such that $U_0 = 0$, and its increments $U_{t+s} - Ut$, $t \geqslant 0$, are stationary independent and have gamma distribution.

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}. \tag{8}$$

where $\alpha = s/\nu$, $\beta = 1$ and ν is a positive parameter. Some basic properties of the gamma process are well-known:

1) Multiplication of a gamma process by a scalar constant λ is again a gamma process with different mean increase rate.

$$\lambda\Gamma(t; \alpha, \beta) \simeq \Gamma(t; \alpha, \beta/\lambda).$$

3) The sum of two independent gamma processes is again a gamma process.

$$\Gamma(t; \alpha_1, \beta) + \Gamma(t; \alpha_2, \beta) \simeq \Gamma(t; \alpha_1 + \alpha_2, \beta).$$

4) The density function $f(x, t)$ for gamma process $U(t)$ is given by

$$f(x, t) = \frac{1}{\Gamma(t/v)} x^{\frac{t}{v}-1} e^{-y}, y > 0,$$

where $\Gamma(z)$ is the Gamma function.

5) The Laplace exponent for gamma process $U(t)$ is given by

$$\Psi_{U(u)} = \frac{1}{v} \log(1 + u).$$

6) The moments for gamma process $U(t)$:

$$E(U^q(t)) = \frac{\Gamma(q + t/\nu)}{\Gamma(t/\nu)} \sim \frac{t}{\nu}$$

if $t \to \infty$.

7) The correlation function is

$$Corr(U(s), U(t)) = \sqrt{\frac{s}{t}},$$

$s < t$, for any gamma process $U(t)$.

The inverse subordinator $S(t)$ defined by (1), with gamma process $U(t)$ has follows properties (see [4]):

1)The inverse subordinator $S(t)$ for gamma process $U(t)$ is inverse gamma process, and its tail behavior satisfies:

$$P(S(t) > x) \sim (\frac{2\pi x}{\nu})^{-1/2} e^{x/\nu - t} (\frac{\nu t}{x})^{x/\nu}$$

if $x \to \infty$.

2) The moments for $S(t)$ of first and second order can be computed as

$$M_1(t) = \mathbb{E}S(t) = \nu\left(t + \frac{1}{2}\right) - \nu e^{-t} \int_0^\infty \frac{e^{-yt}}{(1+y)\left[(\log(y))^2 + \pi^2\right]} dy, \tag{9}$$

$$M_2(t) = \mathbb{E}S(t)^2 = \nu^2\left(t + \frac{1}{12} + \frac{t^2}{2}\right) - 2\nu^2 e^{-t} \int_0^\infty \frac{e^{-yt}\log(y)}{(1+y)\left[((\log(y))^2 + \pi^2)^2\right]} dy. \tag{10}$$

3) The covariation function is

$$\mathrm{Cov}(S_t, S_{t+k}) = \frac{1}{2} M_2(t) + \int_0^t M_1(t + k - y) dM_1(y) - M_1(t + k) M_1(t). \tag{11}$$

It is useful to find the covariance structure of fractional Bachelier model with FBM time-changed by gamma process and then explore the asymptotic behavior of it.

Proposition 1. *If X_t is a stochastic process given by (), where $U(t)$ is Gamma process, then, for any integer $k \geqslant 0$:*

$$\mathrm{Cov}(X(t), X(t + k)) = \mu^2 \mathrm{Cov}(S(t), S(t + k)) + \sigma^2 \mathrm{Cov}(B_H S(t), B_H S(t + k)), \tag{12}$$

where covariation function between $S(t)$ and $S(t + k)$ is defined by property (11) and covariation function between $B_H S(t)$ and $B_H S(t + k)$ was explored in [5]:

$$\mathrm{Cov}(B_H S(t), B_H S(t+k)) = M_{2H}(t) + 2H \int_0^t M_{2H-1}(t+k-y) dM_1(y). \tag{13}$$

where $M_q(t) = \mathbb{E}(S(t)^q(t))$, H *- is the Hurst exponent.*

The results follows from (9)-(11) and the fact that

$$\begin{aligned}\mathrm{Cov}(X(U(t), X(U(t+k)) = \mu^2 \mathrm{Cov}(S(t), S(t+k)) + \mu\sigma \mathrm{Cov}(S(t), B_H(S(t+k))+ \\ + \mu\sigma \mathrm{Cov}(B_H(S(t)), S(t+k)) + \sigma^2 \mathrm{Cov}(B_H(S(t)), B_H(S(t+k))),\end{aligned} \tag{14}$$

where $\mathrm{Cov}(S(t), B_H(S(t+k)) = \mathrm{Cov}(B_H(U(t)), S(t+k)) = 0$ because we suppose the inverse subordinator $S(t)$ and FBM $B_H(S(t))$ are independent.

4 Monte-Carlo Option pricing in subdiffusion Bachelier model

For the sub-diffusion market described above the usual requirement for the fair option pricing is that arbitrage opportunities do not exist. For this it is enough to prove the existence of the equivalent martingale measure. In [3] was introduced the following measure

$$\mathbb{Q}(A) = \int_A \exp\left\{-\gamma B(S_\psi(T)) - \frac{\gamma^2}{2} S_\psi(T)\right\} d\mathbb{P},$$

where $\gamma = \frac{\mu}{\sigma}$ and $A \in \mathcal{F}$, and was proved that the subdiffusive arithmetic BM $X(t)$ is a martingale with respect to Q. Also in [3] was shown that the market model, in which the asset price is described by the subdiffusive arithmetic BM, is arbitrage-free. The second question is completeness of market model. The Second Fundamental theorem of asset pricing states that the model is complete if and only if there is a unique martingale measure. In the paper [3] were given two different proofs of the incompleteness of financial market model based on subdiffusive ABM. If the subdiffusive model is incomplete, then it is reasonable to obtain different prices of derivatives depending on the choice of the martingale measure. The choice of the measure Q defined in this way is the natural extension of the martingale measure from the classical Bachelier model, in the context of subordination. Moreover, measure Q minimizes the relative entropy. So, from paper [3] we know that the subdiffusive model (7) is arbitrage-free, incomplete and the fair price of the European call option with expiry date T and strike price K is given by:

$$C_\psi(X_0, K, T, \alpha) = \langle C(X_0, K, S_\psi(T), \alpha)\rangle = \int_0^\infty C(X_0, K, x, \alpha) g_\psi(x, T)\, dx \tag{15}$$

where, g_ψ is the PDF of $S_\psi(T)$. There are some ways of finding fair price. One of them is by approximating integral in (15). But this approach requires that g is known exactly. The another method is to find call or put option price as solution of the forward partial integro-differential equation (PIDE) with some initial conditions. And finally the method we use is Monte-Carlo approach. The Monte-Carlo method is based on modeling the trajectories of the inverse subordinator $S_\psi(t)$ in the interval $[0, T]$ and calculate the fair price as the mathematical expectation.

This method already was applied for α stable and tempered α stable subordinators in [3]. In this paper we apply Monte-Carlo method for gamma subordinator.

5 Numerical Results

In this section, the algorithm for using Monte-Carlo approach are demonstrated.

We considered spot price $S_0 = 277.0$ for March 14, 2020. The strike price for call options with maturity $T = 1/12$ year is set at $K = 255; 260; 265; 270$, the yearly volatility for returns of the underlying asset is computed at $\sigma = 33.7\%$, the yearly riskless interest rate is set at $r = 5.8\%$.

The main steps in a basic Monte Carlo approach to estimating fair option price in the arithmetic sub-diffusion model are as follows:

1. First we simulate N trajectories of $U(t)$, that is a process of independent stationary increments having gamma distribution. We divide the interval $[0, T]$ into subintervals of length

/*delta*, where the increments $U(t+\delta) - U(t)$, $t = 0, \delta, 2\delta, ..., T - \delta$ have gamma distribution with parameters δ/ν and 1. We simulate T/δ independent random variables from this distribution. Finally, the trajectory of $U(t)$ is obtained as the cumulative sum of the increments.

2. In order to simulate the approximate trajectory of the inverse gamma subordinator we need to use following approximation scheme:

$$S_{\psi,\delta}(t) = (\min\{n \in \mathbb{N} : T_\psi(\delta n) > t\} - 1)\delta. \quad (16)$$

Here, $\delta > 0$ is the step length and $T_\psi(\tau)$ is the subordinator.

3. For a given time of maturity T generate N values of the inverse subordinator $S(T)$ and calculate N option price values, using formula (15). See Fig. 1.

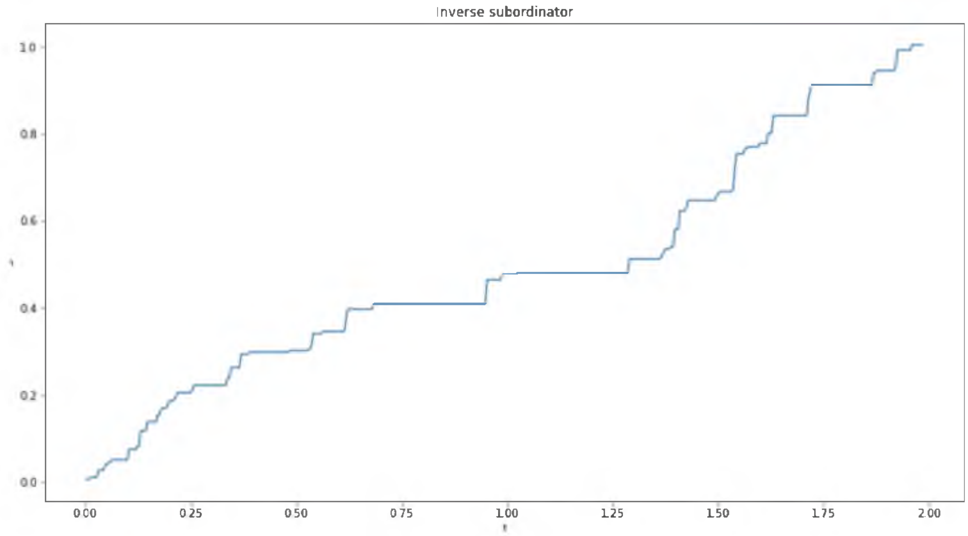

Рис. 1: The trajectories of inverse subordinator

4. Find the fair price as mean for N scenarios, obtained in the step 3.

The results of the above algorithm are presented in Fig. 2. We observe a typical trajectory of ABM with gamma waiting times. And we use the above algorithm to perform Monte Carlo simulations in order to approximate the fair option price for sub-diffusion model. Some basic results for European call options are presented in Table 1.

Табл. 1: Numerical comparisons

Strike Price	Last (Market)	Subdiffusion	Bachelier	Black-Scholes
255	32.60	31.09	30.34	26.67
260	28.67	26.82	25.80	22.76
265	27.00	24.81	23.97	19.17
270	23.00	20.53	19.35	15.92

In order to evaluate the accuracy of the proposed method and to compare it with fair pricing according to the Black-Scholes formula, we can compare their percentage errors. The percentage error for a given strike price K can be computed as a following ratio:

$$PercentageError = \frac{|c_{theoretical} - c_{market}|}{c_{market}} 100\%$$

Overall, the results show that average percentage errors for Subdiffusive pricing (7,5 %) and for

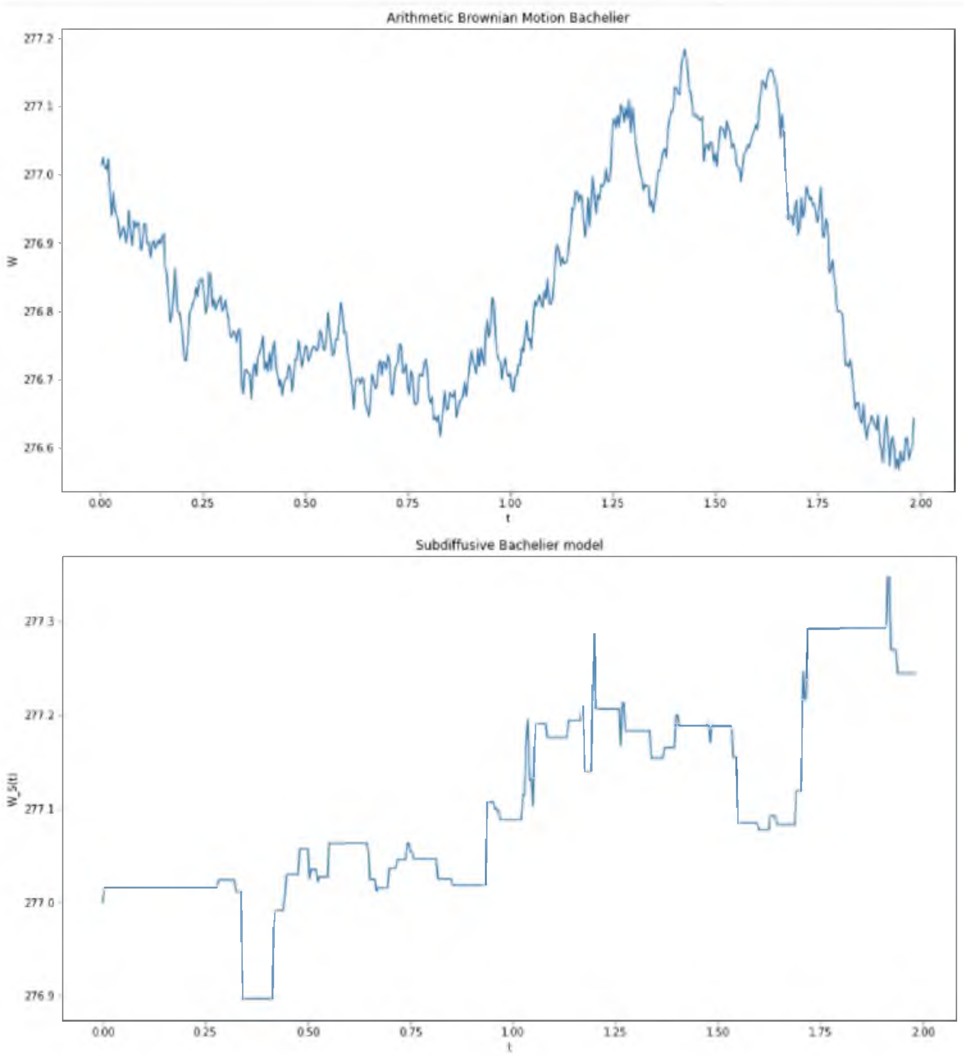

Рис. 2: Sample realizations of the standard ABM (top panel), and the corresponding subdiffusive ABM

Optimal ABM pricing (11 %) are less than for Fair Black-Scholes Pricing (24%).

However, these errors may indicate that the proposed model is very sensitive to volatility σ, and to the periods without price's changing.

Thus, we offer our approach as an alternative to the Bachelier and Black-Scholes formula for emerging markets.

6 Summary

Following the financial tsunami experiences of 2008 and the crisis of Covid19 in 2020, we can observe that some kinds of risky assets have the periods in their dynamic without change and the risk controls of derivative instruments on stocks and other financial assets have become extremely important. Thanks to sub-diffusion approach, the investor gets a tool, which allows him to take into account these features in the option pricing. Monte-Carlo approach with gamma subordinator allows evaluate option price easy.

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