Міністерство освіти і науки України
Національний Університет «Києво-Могилянська Академія»
Кафедра математики

Coherence in the coupled oscillators for the case of financial time series
Когерентність у об'єднаних у мережу осциляторах для випадку фінансових часових рядів

Курсова робота
за спеціальністю 113 "Прикладна математика"

Керівник курсової роботи
К.ф.-м.н., доц. Щестюк Н.Ю.

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(підпис)
"__"____________ 2020 р.

Виконала студентка
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"__"____________ 2020 р.

Київ 2020

Міністерство освіти і науки України
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Кафедра математики факультету інформатики

Затверджую
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"__"_____________ 2020 р.

Індивідуальне завдання
на курсову роботу

студентці прикладної математики факультету інформатики 4 курсу
ТЕМА: "Coherence in the coupled oscillators for the case of financial time series"
Зміст курсової роботи:

Individual task

Calendar plan

Introduction

1. Nonlinear dynamical systems
2. Dynamical chaos
3. Oscillators
4. Financial time series
5. Coherence in coupled oscillators

Conclusions

References

Дата видачі "__"_____________ 2020 р.   Керівник ________________
(підпис)
Завдання отримала ________________
(підпис)

TEMA: "Coherence in the coupled oscillators for the case of financial time series"

№ п/п	Назва етапу курсової роботи	Термін виконання	Примітка
1	Отримання завдання на курсову роботу	10.10.2019	
2	Отримання довідкового теоретичного матеріалу від наукового керівника	10.10.2019	
3	Вивчення матеріалів за темою	20.03.2020	
4	Визначення змісту теоретичної та практичної частини курсової роботи з науковим керівником	27.03.2020	
5	Написання курсової роботи	19.04.2020	
6	Підготовка презентації	20.04.2020	

Студентка ___________________

Науковий керівник ___________________

"__"_________________ 2019 р.

Introduction

The analysis in natural science leads to spreading the ideas of chaos theory and non-linear dynamics to financial mathematics and creating the new researches to consider similar models and procedures for financial time series. Also, the irregular fluctuations in these series are sometimes considered as an outcome from chaotic systems.[1] This can be used, for example, to forecast the value of an investment portfolio, which is the combination of different financial assets, for example, stocks, bonds, cash. One of the ways to think about a successful portfolio is when the chosen equities have the high expected returns and synchronized in time for bottom moments.[2] Then the dynamics of these financial assets can be described as oscillators connected in the network.

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Chapter 1

Nonlinear dynamical systems

1.1 Dynamical system

Dynamical system is the mathematical object for which the state is defined as the totality of some values in the current moment of time and defined the operator, which describes the evolution of initial state in time.

The strict definition of dynamical system can be shown as the combination of three components [8]:

- metric space D, which is called the phase space. The phase space can coincide with the whole $n-$dimensional euclidean space R^n
- time t, which can be continuous ($t \in R^1$) or discrete ($t \in Z$ - all integers)
- the rule of evolution operator - the map of any point in phase space D for any meaning of time t in the defined state $\phi(t, x) \in D$, which has the next properties:

$$\phi(0, x) = x$$
$$\phi(t_1, \phi(t_2, x)) = \phi(t_1 + t_2, x)$$
$$\phi(t, x) \text{ is continuous of } (t, x)$$

1.2 Modelling of the dynamical system

The system with the pre-set of initial parameters or coordinates and the evolution operator, which describe the state of system in any moment of future time is known as the mathematical model of the dynamical system. The states of the system can be described with the discrete time moments of with the probability distribution for the set of states.

1.3 Difference equations

Difference equations are the type of equations which are used in the case of discrete modelling of the dynamical system. Such equations connects the meaning of the function (which in fact is unknown) in any point of the system with its meaning in one or more points, which are far away on the specific time term. The evolution operator in this case is the recurrent formula:

$$x_{n+1} = f(x_n) \tag{1.1}$$

.

For the case of one-dimensional phase space , the right part of this equation gives the meaning of how the systems evolutes in time. It can be easily shown with the graphics.

Chapter 2

Dynamical chaos and discrete models of dynamical chaos

2.1 Definition of dynamical chaos

Dynamical chaos is the occurance when the behavior of nonlinear system seems to be random, nevertheless it is described using deterministic rules. The deterministic system is the system in which all process are interconnected, and knowing the initial rules and conditions, all the future states can be predict. But it is not the same for the dynamical chaos. The formal definition of chaotic function is:

1. f has dense periodic points.
2. f is sensitive for initial conditions.
3. f performs topological mixing.

The topological mixing means that the system is forgetting the initial conditions with some time, and it is no matter how well the system knows this condition, as far as the time goes, this point become more dense.
From this definition follows the definition of dynamical sequences: the sequence is called chaotic if it is generated with chaotic function.

2.2 Logistic map

The logistic map is one of the chaotic functions, which is described as:

$$xTx \equiv \lambda x(1 - x), \tag{2.1}$$

which creates the one-dimensional nonlinear dynamical system:

$$\begin{cases} x_{n+1} = \lambda x_n(1 - x_n) \quad \lambda \in R, n \geq 0 \\ x_0 \in (0,1) \end{cases} \tag{2.2}$$

The meaning of λ , as denoted can be any real number. But let take a close look of some cases.

When the $\lambda \leq 1$ the solution will be $x_n = x_n(\lambda) \to 0$ when $n \to \infty$ for all $0 < x < 1$. From this 0 is the only stable state for which coincide the meaning of x_n.

When the $\lambda = 2$, then $x_n \to \frac{1}{2}$. For this case $\frac{1}{2}$ is the only stable state for x_n.

Even by increasing λ till the meaning of 3, the system still has only one stable state. From 3 there are two states of stability, and system are in on or other with the specific change. With increasing the meaning of λ till 4 there are more and more states of stability. But even with $\lambda = 3.6$ is known that there are so many such states, that the system is no longer stable and shows the chaotic behaviour. There is no longer any predefined way of changing the states of stability. It means, that we cannot predict where the system will be in some time. [7]

Chapter 3

Oscillators

3.1 The dynamic of oscillators

The definition of dynamical system was described in the Chapter 1. From there also known that there are two types of dynamical systems:

1. Differential equation - stands for continuous case:

   $\frac{dx_i}{dt} = f(x_i) +$ the conditions for connection

2. Difference equation - stands for discrete case:

   $x_i^{t_1} = f(x_i^t) +$ the conditions for connection,
   where $f(x) : R^M \to R^M$ - non-linear function.

In this work I have worked with the discrete case.

3.2 Coupled oscillators

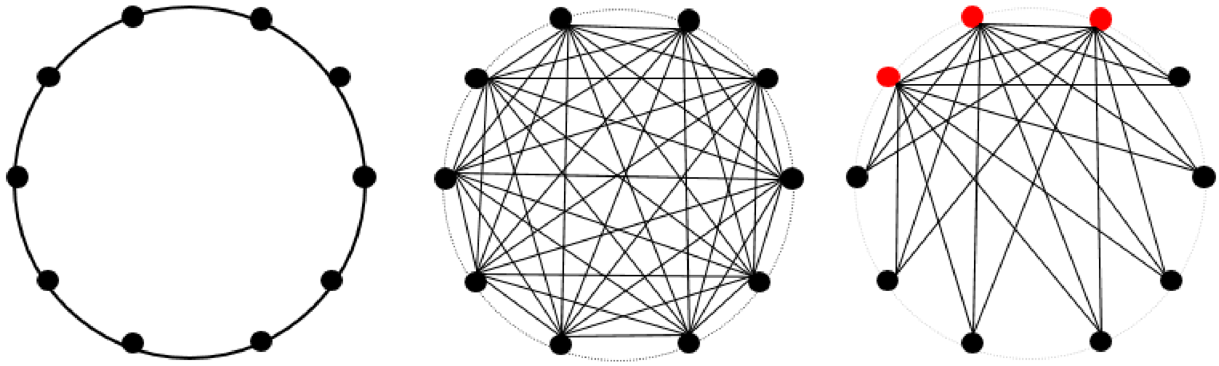
The set of dynamical equations, which describes the map of N oscillators are:

$$S_i^{t+1} = f(S_i^t) + \frac{\sigma}{P} \sum_{j=i-P}^{i+P} [f(S_j^t) - f(S_i^t)] \tag{3.1}$$

Here the S_i is the states of the oscillator,
f is one-dimensional mapping,
σ is the strength of connections,
P is the number of connections in each direction.

3.3 Different coupling in oscillators networks

The coupling in the oscillators can have three types:
Local coupling - each connected to the two neighbours.
Global coupling - all to all.
Non-local coupling - any other type of connection, for example, one to all or three to four.



Coupling radius is defined by this equation $r = \frac{P}{N}$

Chapter 4

Financial time series

4.1 Definition of financial time series

Let $y_1, y_2, ..., y_t$ - the meanings of observation of some process during T periods. This sequence has the numerical meanings with the specific index for each, which depends on the period number in which it was observed. Time series is this sequence with the increasing index. Time series will be denoted as S_t.

Any time series can be presented as the sum of determined and random components:

$$S_t = d_t + r_t, t = \overline{1, T}$$

The determined component is also the sum op three parts - trend(tr_t), seasonal(s_t) and cyclic(c_t) components. Then the first equation can be rewritten as:

$$S_t = tr_t + s_t + c_t + r_t, t = \overline{1, T}$$

The analysis of time series begins with plotting the series and studying it. The components described above take place here.

4.2 Chaotic approach for the time series modelling

The autoregressive process is one of the way of modelling the time series. The forecasting variable in this process is the linear combination of the past values and random process with zero mean and constant variation. Let ε_t is the white noise and p is non-negative integer. The autoregressive model of order p:

$$S_t = \phi_1 S_{t-1} + \phi_2 S_{t-2} + ... + \phi_p S_{t-p} + \varepsilon_t \tag{4.1}$$

where $\phi_1 ... \phi_p$ are some constants. For modelling the time series using this model there should be at least p sequential initial meanings of S_t.

In this work the AR(2) model was used. The general formula for AR(2) process is:

$$S_t = \phi_1 S_{t-1} + \phi_2 S_{t-2} + \varepsilon_t \tag{4.2}$$

But to consider the chaotic approach of modelling some time series, there are some changes that can be done with the AR model. Generally the white noise in this model is changed with logistic map, and from this moment the ε means equation (2.2) with the λ coefficient 3.87.

Chapter 5

The practical application

5.1 Modelling the financial time series

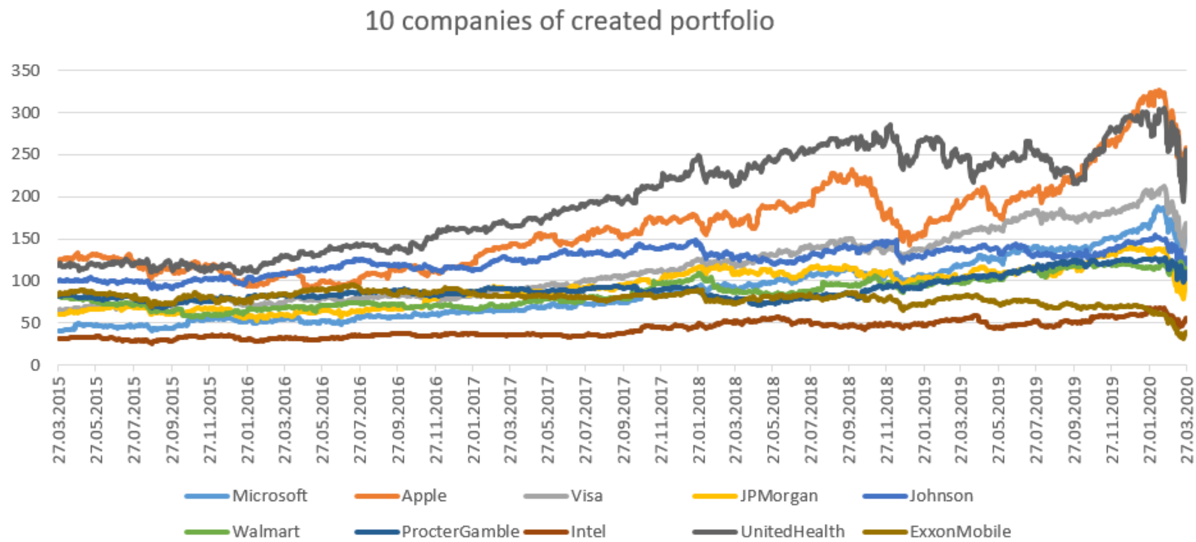
Basket in finance is the group of some securities which are bought or sold simultaneously. There are several types of baskets:

- index funds
- market baskets
- currency baskets
- portfolio

In this work I decided to work with the financial portfolio. It can be denoted as the group of different financial assets, which can be stocks, commodities, currencies, bonds and other parts. [4] In fact, the investment portfolio is usually the combination of stocks and bonds, which is created for mixing the securities that way that the financial goals and risk tolerance of investor was satisfied. [6] For this work with some research aim I has chosen the portfolio which consists only of stocks. The stocks were chosen using the Dow Jones Index.
Short overview: the Dow Jones Index or the Dow Jones Industrial Average is a stock index, which consists of 30 companies, which mostly are the largest on the USA stock market. The brief information about any of this companies can be read in [5]
The created portfolio consists of 10 best companies of this index on the March 2020. Here is this list:

1. Microsoft
2. Apple
3. Visa

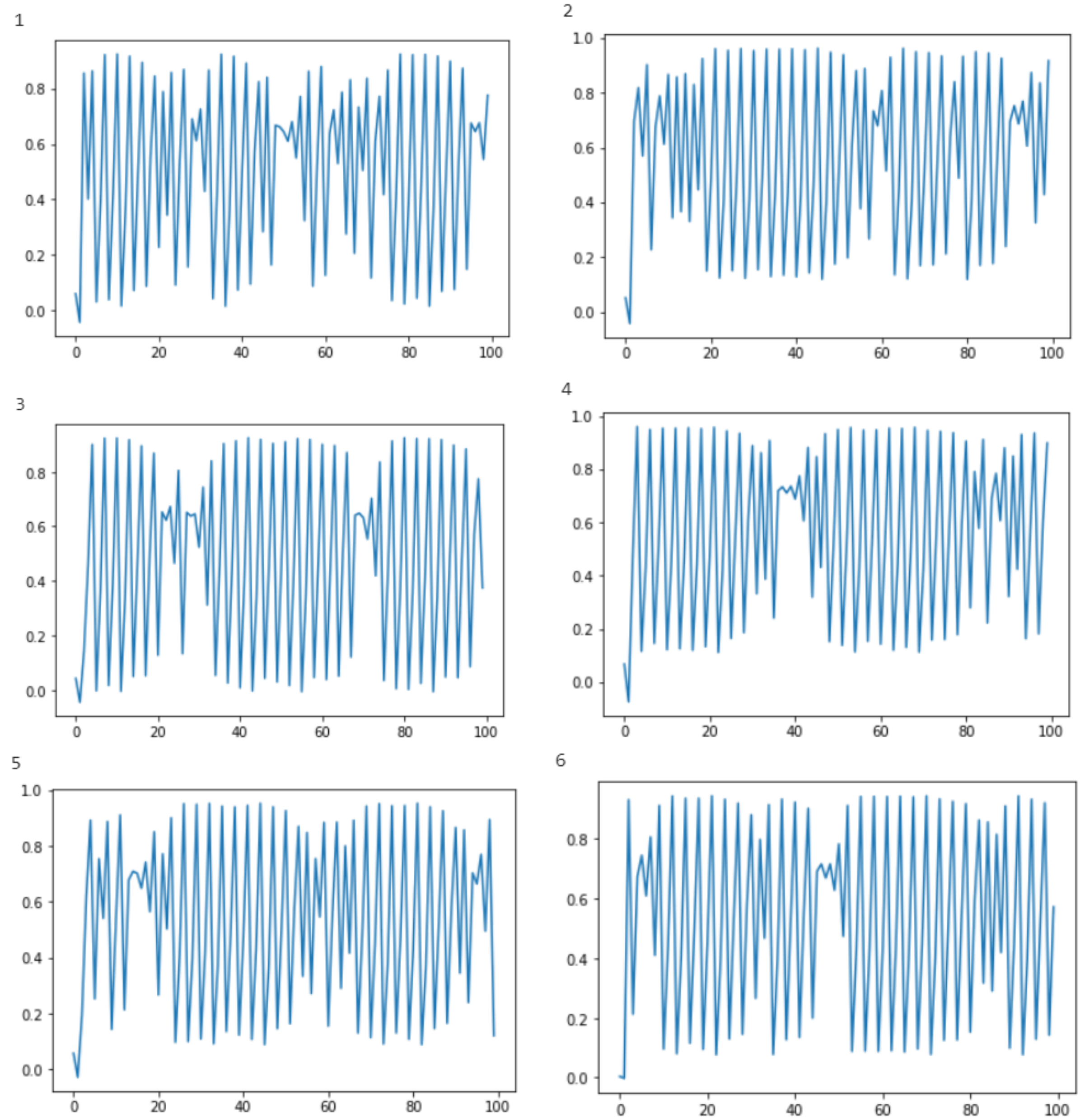
4. JPMorgan Chase&Co.
5. Johnson & Johnson
6. Walmart
7. Procter & Gamble
8. Intel
9. UnitedHealth
10. ExxonMobil



As it can be seen from the graphic, the time series of this companies are taken from the last five years and shows pretty similar fluctuations over the years, especially the companies, which has the highest prices.

Then for each of this companies was built the log-returns series, which can be done from initial series S_t using this equation $H_t = log\frac{S_{t+1}}{S_t}$, but for the easing the next equations, let me redefine the H_t as S_t for the next parts of the work.
Then for each companies there is the need to define the coefficients of the AR(2) model. There are the set of two coefficients for each companies, and they were chosen using the typical AR(2) model with the white noise. There are the coefficients for each of the companies: Microsoft - $[-0.0894, -0.0745]$, Apple - $[0.0088, -0.0425]$, Visa - $[0.0068, -0.0135]$, JPMorgan Chase&Co. - $[-0.1005, -0.1098]$, Johnson & Johnson - $[-0.0106, -0.0189]$, Walmart - $[-0.0108, -0.0718]$, Procter & Gamble - $[-0.0376, -0.0413]$,
Intel - $[-0.0317, -0.0594]$, UnitedHealth - $[-0.0456, 0.0044]$, ExxonMobil - $[0.0016, -0.0054]$.

Then for each of the companies was prepared 4000 points of forecasting data using the modified AR(2) model with the one-dimensional mapping. For the researches' purpose, the initial coefficients of the logistic map was fixed too. This way is was easier to combine the results of the experiments. That's why there is the array of initial meanings for logistic map - [0.777418078954809, 0.5310272826146695, 0.03941861705563554, 0.42867904415420643, 0.8804906059061588, 0.006410991194767179, 0.6572306930357502, 0.2990225166816993, 0.827517158747358]. The $\lambda = 3.87$ was chose as the coefficient of the map for the purpose of chaotic behaviour (was explained in the 2.3). Here are the graphics of the first 100 points for all companies.



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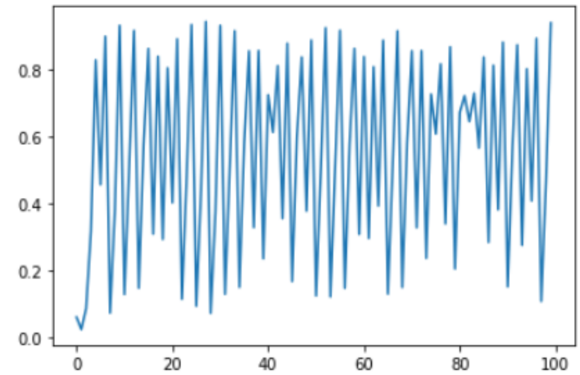

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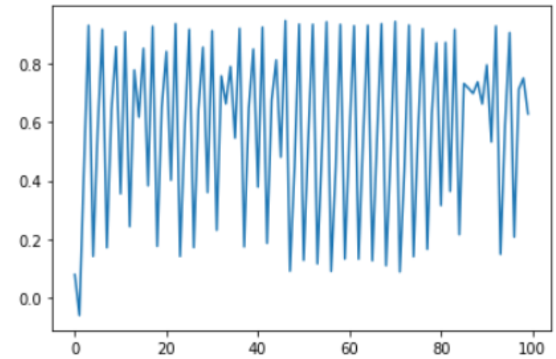

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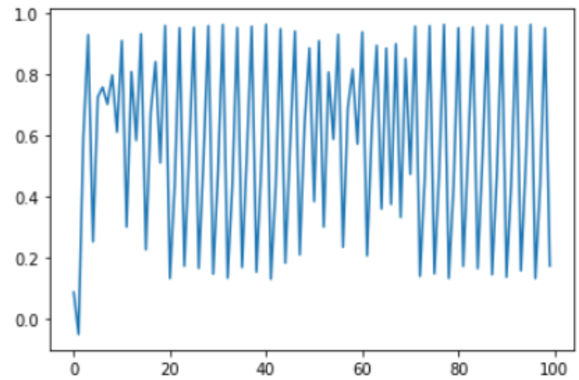

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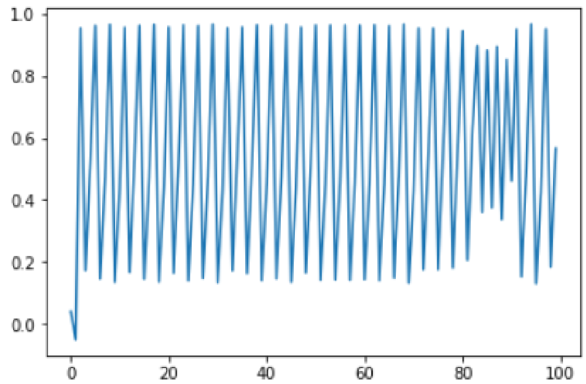


5.2 The cases of different couplings

For this work I have chosen three types of couplings. First of them is all to all. For our case it means that each of ten companies is connected to other nine. The second type is each to two. It means that each of ten companies is connected to the two closest neighbours. The last type of connection is three to all. It means that there are three main companies which are connected to other nine, and all other companies are connected only with those ones. Let me show the oscillator dynamics for this cases. I will present here the detailed equation for the five points in each cases.

1. All-to-all

$$\begin{aligned}S_1^{t+1} = f(S_1^t) + \frac{\sigma}{18}\Big(\big(f(S_{10}^t) - f(S_1^t)\big) + \big(f(S_9^t) - f(S_1^t)\big) + \big(f(S_8^t) - f(S_1^t)\big) + \big(f(S_7^t) - f(S_1^t)\big) + \\ + \big(f(S_6^t) - f(S_1^t)\big) + \big(f(S_5^t) - f(S_1^t)\big) + \big(f(S_4^t) - f(S_1^t)\big) + \\ + \big(f(S_3^t) - f(S_1^t)\big) + \big(f(S_2^t) - f(S_1^t)\big)\Big)\end{aligned}$$

$$\begin{aligned}S_2^{t+1} = f(S_2^t) + \frac{\sigma}{18}\Big(\big(f(S_{10}^t) - f(S_2^t)\big) + \big(f(S_9^t) - f(S_2^t)\big) + \big(f(S_8^t) - f(S_2^t)\big) + \big(f(S_7^t) - f(S_2^t)\big) + \\ + \big(f(S_6^t) - f(S_2^t)\big) + \big(f(S_5^t) - f(S_2^t)\big) + \big(f(S_4^t) - f(S_2^t)\big) + \big(f(S_3^t) - f(S_2^t)\big) + \\ + \big(f(S_1^t) - f(S_2^t)\big)\Big)\end{aligned}$$

$$
\begin{aligned}
S_5^{t+1} = f(S_5^t) + \frac{\sigma}{18}\Big(&\big(f(S_{10}^t) - f(S_5^t)\big) + \big(f(S_9^t) - f(S_5^t)\big) + \big(f(S_8^t) - f(S_5^t)\big) + \big(f(S_7^t) - f(S_5^t)\big) + \\
&+ \big(f(S_6^t) - f(S_5^t)\big) + \big(f(S_4^t) - f(S_5^t)\big) + \big(f(S_3^t) - f(S_5^t)\big) + \big(f(S_2^t) - f(S_5^t)\big) + \\
&+ \big(f(S_1^t) - f(S_5^t)\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
S_7^{t+1} = f(S_7^t) + \frac{\sigma}{18}\Big(&\big(f(S_{10}^t) - f(S_7^t)\big) + \big(f(S_9^t) - f(S_7^t)\big) + \big(f(S_8^t) - f(S_7^t)\big) + \big(f(S_6^t) - f(S_7^t)\big) + \\
&+ \big(f(S_5^t) - f(S_7^t)\big) + \big(f(S_4^t) - f(S_7^t)\big) + \big(f(S_3^t) - f(S_7^t)\big) + \big(f(S_2^t) - f(S_7^t)\big) + \\
&+ \big(f(S_1^t) - f(S_7^t)\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
S_9^{t+1} = f(S_9^t) + \frac{\sigma}{18}\Big(&\big(f(S_{10}^t) - f(S_9^t)\big) + \big(f(S_8^t) - f(S_9^t)\big) + \big(f(S_7^t) - f(S_9^t)\big) + \big(f(S_6^t) - f(S_9^t)\big) + \\
&+ \big(f(S_5^t) - f(S_9^t)\big) + \big(f(S_4^t) - f(S_9^t)\big) + \big(f(S_3^t) - f(S_9^t)\big) + \big(f(S_2^t) - f(S_9^t)\big) + \\
&+ \big(f(S_1^t) - f(S_9^t)\big)\Big)
\end{aligned}
$$

2. Each-to-two

$$S_1^{t+1} = f(S_1^t) + \frac{\sigma}{4}\Big(\big(f(S_{10}^t) - f(S_1^t)\big) + \big(f(S_2^t) - f(S_1^t)\big)\Big)$$

$$S_2^{t+1} = f(S_2^t) + \frac{\sigma}{4}\Big(\big(f(S_3^t) - f(S_2^t)\big) + \big(f(S_1^t) - f(S_2^t)\big)\Big)$$

$$S_4^{t+1} = f(S_4^t) + \frac{\sigma}{4}\Big(\big(f(S_5^t) - f(S_4^t)\big) + \big(f(S_3^t) - f(S_4^t)\big)\Big)$$

$$S_5^{t+1} = f(S_5^t) + \frac{\sigma}{4}\Big(\big(f(S_6^t) - f(S_5^t)\big) + \big(f(S_4^t) - f(S_5^t)\big)\Big)$$

$$S_8^{t+1} = f(S_8^t) + \frac{\sigma}{4}\Big(\big(f(S_9^t) - f(S_8^t)\big) + \big(f(S_7^t) - f(S_8^t)\big)\Big)$$

3. Three-to-all

$$
\begin{aligned}
S_1^{t+1} = f(S_1^t) + \frac{\sigma}{18}\Big(&\big(f(S_{10}^t) - f(S_1^t)\big) + \big(f(S_9^t) - f(S_1^t)\big) + \big(f(S_8^t) - f(S_1^t)\big) + \big(f(S_7^t) - f(S_1^t)\big) + \big(\\
&+ \big(f(S_5^t) - f(S_1^t)\big) + \big(f(S_4^t) - f(S_1^t)\big) + \big(f(S_3^t) - f(S_1^t)\big) + \big(f(S_2^t) - f(S_1^t)\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
S_2^{t+1} = f(S_2^t) + \frac{\sigma}{18}\Big(&\big(f(S_{10}^t) - f(S_2^t)\big) + \big(f(S_9^t) - f(S_2^t)\big) + \big(f(S_8^t) - f(S_2^t)\big) + \big(f(S_7^t) - f(S_2^t)\big) + \\
&+ + \big(f(S_6^t) - f(S_2^t)\big) + \big(f(S_5^t) - f(S_2^t)\big) + \big(f(S_4^t) - f(S_2^t)\big) + \big(f(S_3^t) - f(S_2^t)\big) + \\
&+ \big(f(S_1^t) - f(S_2^t)\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
S_3^{t+1} = f(S_3^t) + \frac{\sigma}{18}\Big(&\big(f(S_{10}^t) - f(S_3^t)\big) + \big(f(S_9^t) - f(S_3^t)\big) + \big(f(S_8^t) - f(S_3^t)\big) + \big(f(S_7^t) - f(S_3^t)\big) + \\
&+ \big(f(S_6^t) - f(S_3^t)\big) + \big(f(S_5^t) - f(S_3^t)\big) + \big(f(S_4^t) - f(S_3^t)\big) + \big(f(S_2^t) - f(S_3^t)\big) + \\
&+ \big(f(S_1^t) - f(S_3^t)\big)\Big)
\end{aligned}
$$

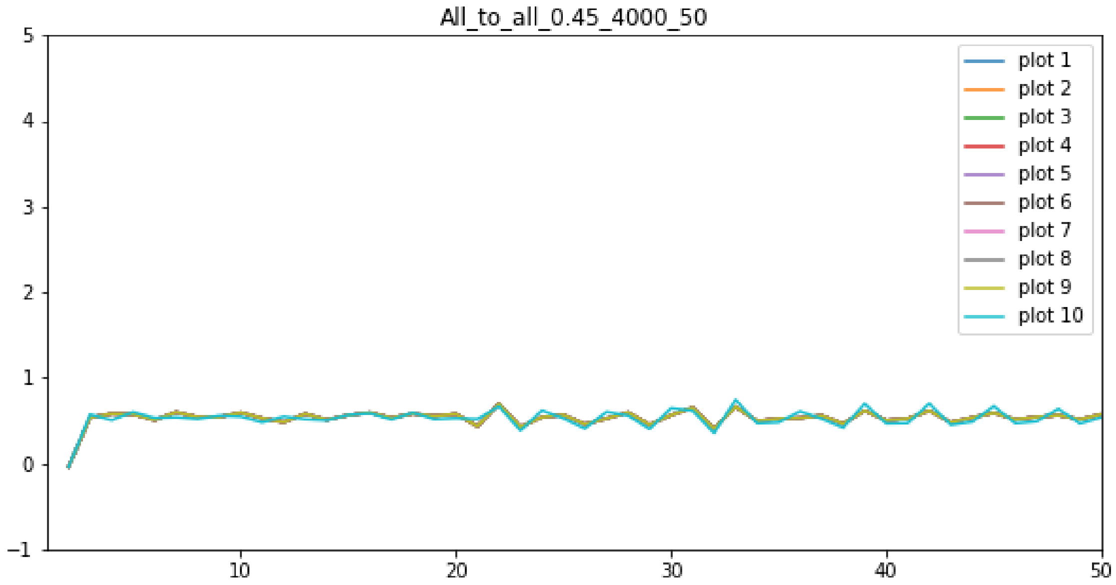
$$S_5^{t+1} = f(S_5^t) + \frac{\sigma}{6}\Big(\big(f(S_3^t) - f(S_5^t)\big) + \big(f(S_2^t) - f(S_5^t)\big) + \big(f(S_1^t) - f(S_5^t)\big)\Big)$$

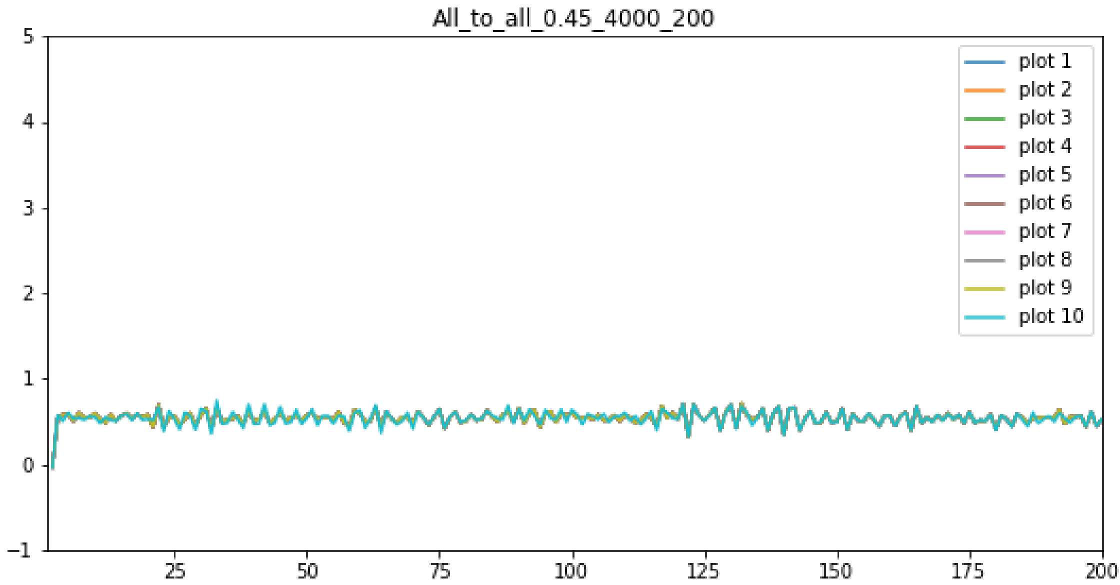
$$S_7^{t+1} = f(S_7^t) + \frac{\sigma}{6}\Big(\big(f(S_3^t) - f(S_7^t)\big) + \big(f(S_2^t) - f(S_7^t)\big) + \big(f(S_1^t) - f(S_7^t)\big)\Big)$$

The more connections the point has, the more summations are here to be done, and imagining the time series, which consists of really large set of meanings, it can be harder to describe the dynamics of this oscillators.

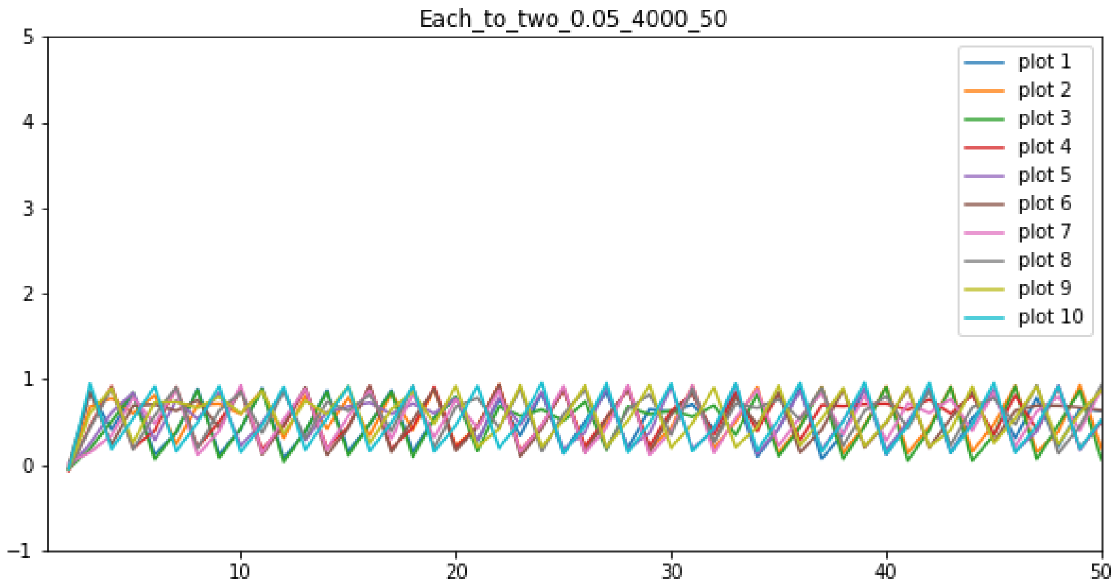
Here are the graphics of oscillators dynamics for each case of connection.

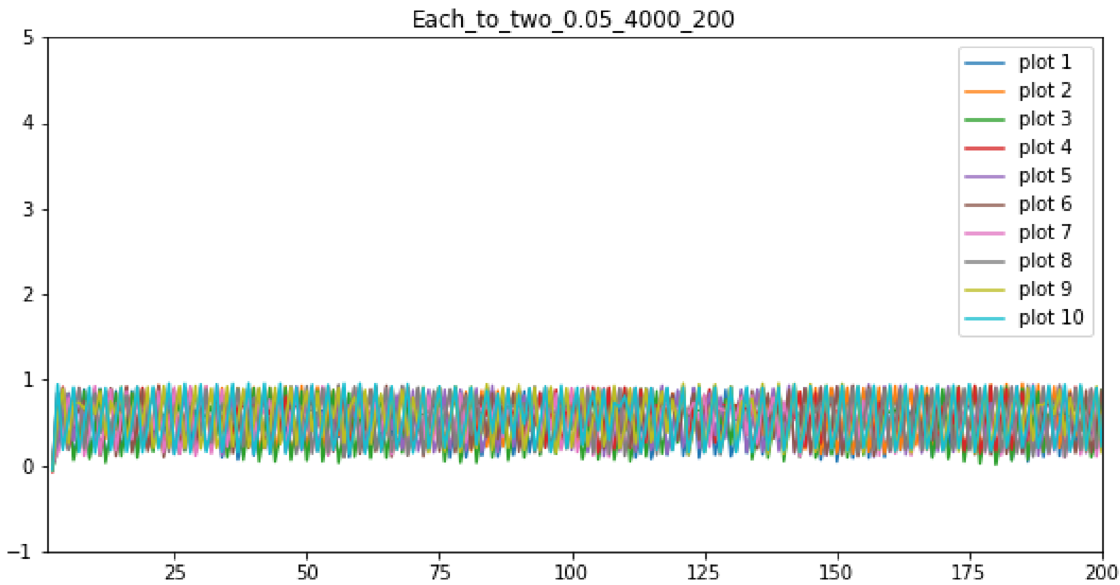
1. All-to-all



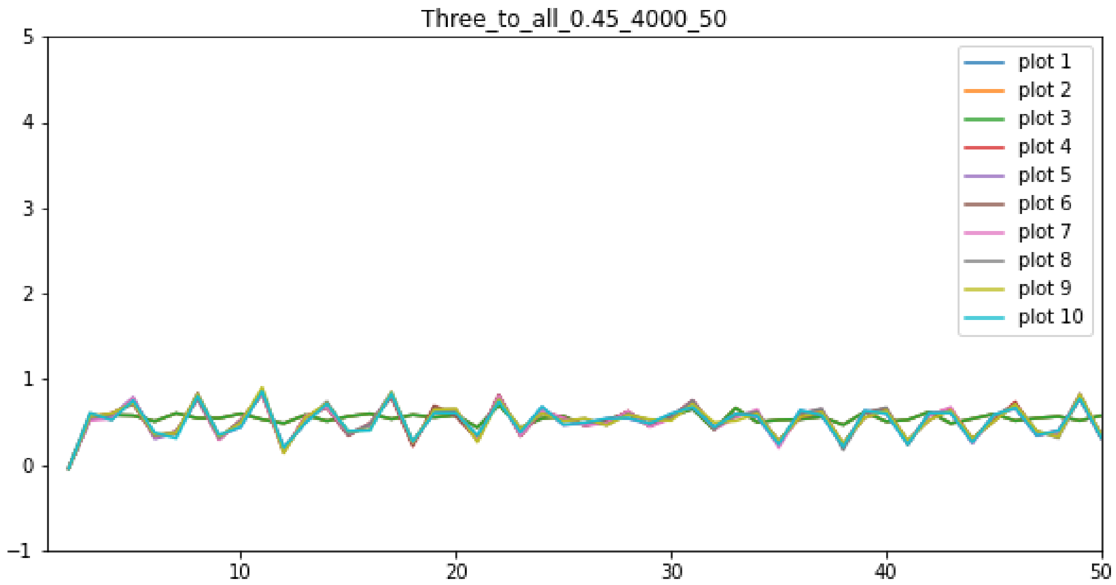


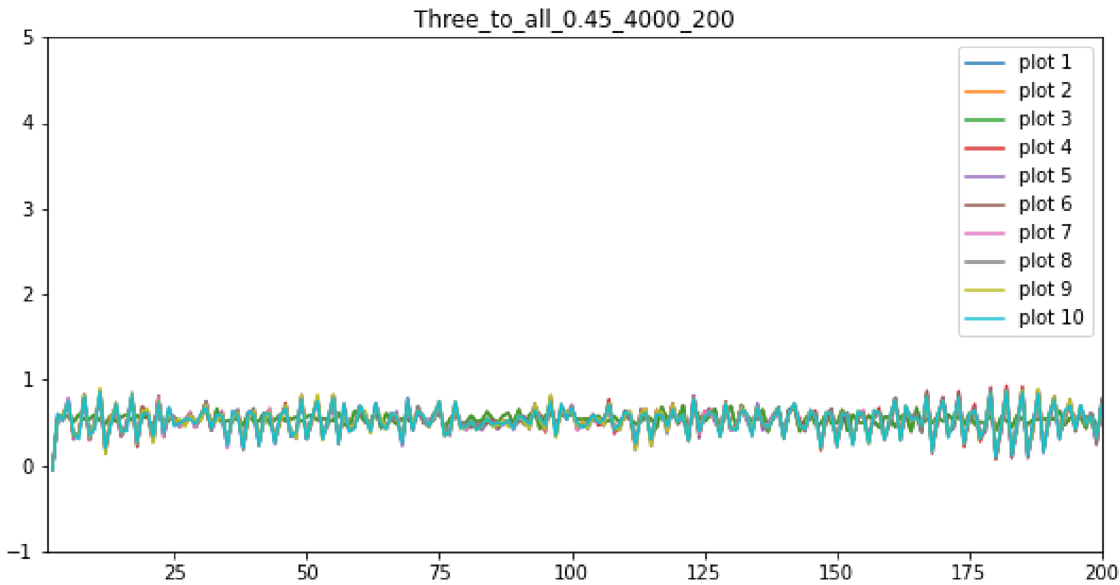
2. Each-to-two

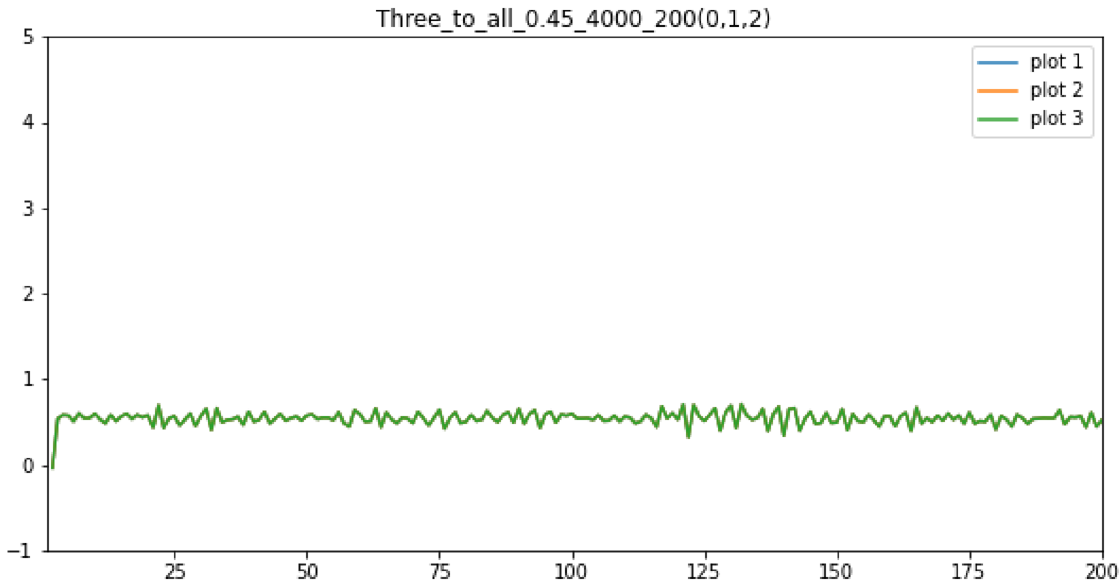




3. Three-to-all


Three_to_all_0.45_4000_50
plot 1
plot 2
plot 3
plot 4
plot 5
plot 6
plot 7
plot 8
plot 9
plot 10


Three_to_all_0.45_4000_200
plot 1
plot 2
plot 3
plot 4
plot 5
plot 6
plot 7
plot 8
plot 9
plot 10



5.3 Coherence in the networks of oscillators

There are some ways to find the coherence in the systems. The first one is the peer-seek study from Kuramoto, which is described as:

$$\lim_{t\to\infty}[S_i^{t+1} - S_j^{t+1}] \longrightarrow const \tag{5.1}$$

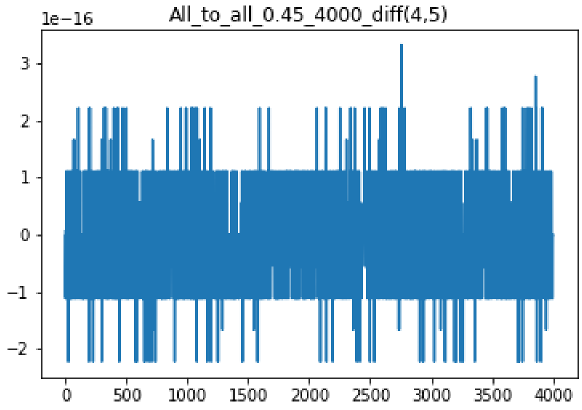
Here the S_i^{t+1} stands for 3.1 formula from Chapter 3.
The second approach is by using

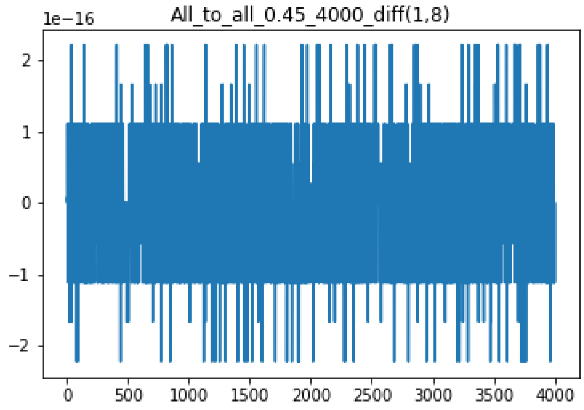
$$C_{xy} = \frac{|P_{xy}^2|}{P_{xx}P_{yy}}, \tag{5.2}$$

where P_{xy} is cross spectral density and P_{xx}, P_{yy} are power spectral density.
But in this work I will show the results only for the first approach. The second approach has the main condition to the sequences to be ergodic and can be used as the topic for the next researches: if the modified AR(2) model creates the ergodic sequence.

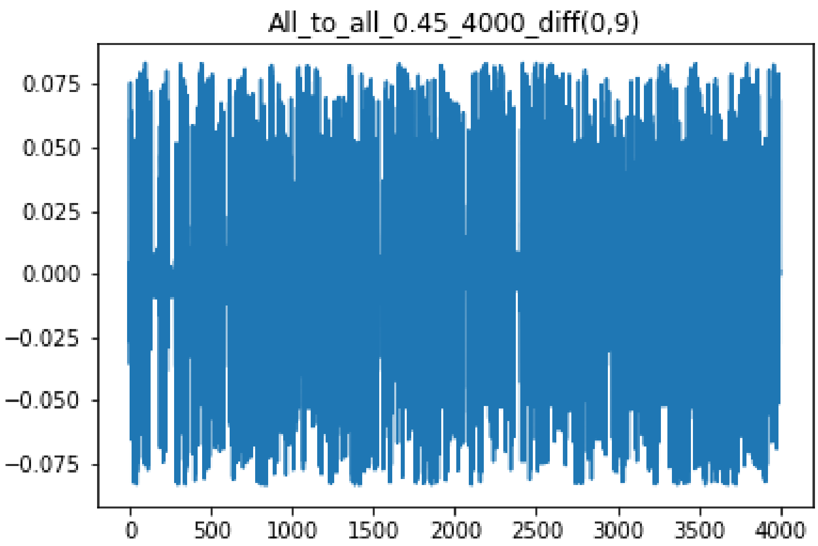
The problem which occurred with the first approach is that it is hard to create the sequence which is going to infinity and see what is happening there, even with a large set of points there is hard to say if the two oscillators are fully synchronised (which in case couldn't be at all, because there are states which are called chimera states.) But here I will show some results, which can be described as the coherence for some type of connections.

The case when I saw the coherence states for my oscillator network was all-to-all. The couple strength was denoted to be 0.45. Here is the graphic of the limit shown in 5.1.



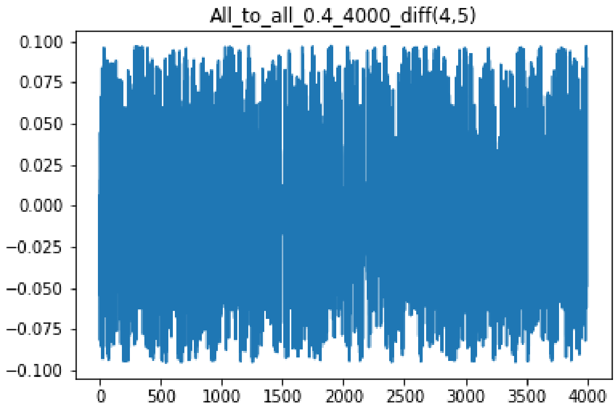


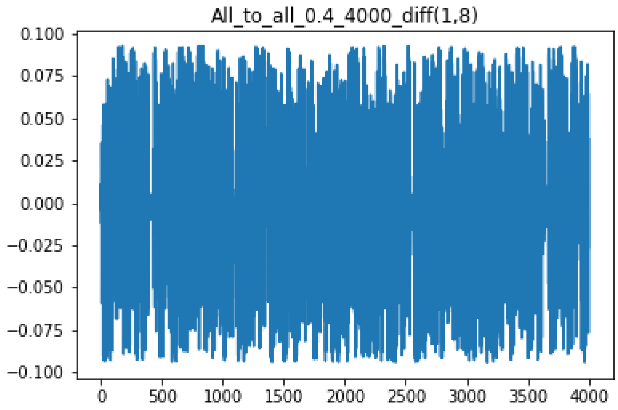
On this graphics we can see that for points 2 and 9, and for points 5 and 6, the limit of their difference is fluctuates between 1 and -1, which can be interpreted as the coherent state for this two points. Of course, they are not fully synchronised, but at least we can see, that, for example, point 1 and 10 (picture below) are close to be coherent, but they do not shows the



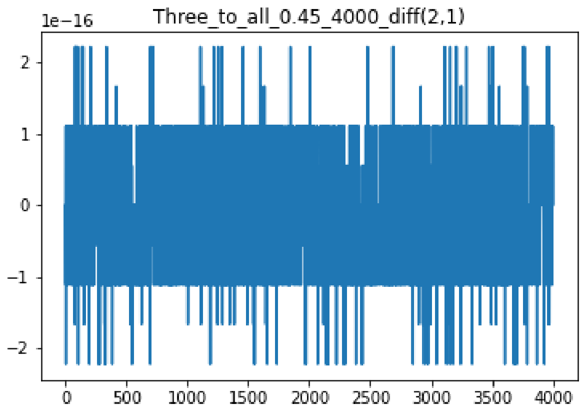
similar fluctuations as 2 and 9.

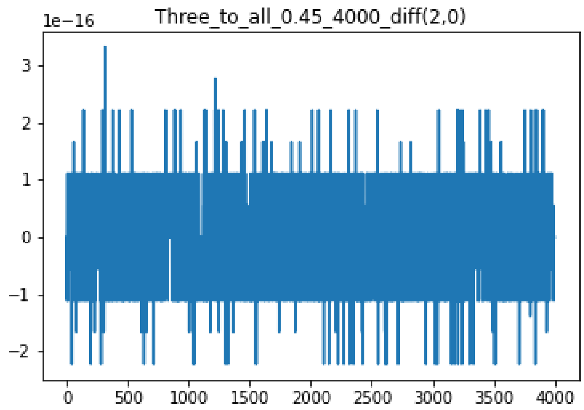
I also did the similar research for the same case, but with $\sigma = 0.40$ and $\sigma = 0.5$.





As can be seen here the same oscillators presented, but their state is far different from being coherent, and it is only 0.05 step from the previous coupling strength.
For the case of each-to-two I haven't found any coupling strength with which the created network shows the coherent oscillators.
For the case of three-to-all, the only time when I found some cases of coherent was with $\sigma = 0.45$, similar as for case all-to-all. Here are two graphics of these cases.





Conclusions

In this work I have created the networks of coupled oscillators with three different types of connections for the real financial time series, which were taken from Dow Jones Index. Then each of network was researched on the states of coherent for specific coupling strength. I have conclude that the method of searching the coherence is quite difficult if there is no enough data points in the sets, and that this method is can be used only in some cases for the graphical representation of the oscillators difference. The recommendations of this work is to try to prove that the second method, which is more accurate, can be used with the system like this - created from AR(2) sequences with logistic map, as random part.

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Appendix A

The Python code for creating the oscillators.