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ON DIFFERENT REPRESENTATIONS OF RISK IN BAYESIAN DECISIONS

This paper points out a relationship between differentforms ofrisk in statistical decision problems and different methods ofobtaining a Bayesian decisionfunction. The paper reflects and develops some ideas presented in [I].

Let us consider the following decision-making problem under uncertainty: we want to choose an action a, $a \in A$, that minimizes losses defined by the function $L(a,\theta)$. This function depends on the unknown parameter θ, $\theta \in \Theta$ (so-called «state of nature»). Although we do not know the true value of θ, we know probability distribution of θ. Assume that this probability distribution is absolutely continuous and has a density denoted by $\xi(\theta)$. This so-called *prior density* characterizes some our beliefs about the likely values of θ. Let x, $x \in S$, denote the data obtained from observations or experiment. These observations are correlated with the unknown parameter θ and the correlation is defined by the conditional density $f(x \mid \theta)$ of x given θ.

Let us apply Bayes theorem to the joint density $p(x,\theta) = f(x \mid \theta)\xi(0)$ of x and θ to obtain the conditional density of θ, given x:

$$\xi(\theta \mid x) = \frac{f(x \mid \theta)\xi(\theta)}{\int\limits_{\Theta} f(x \mid \theta')\xi(\theta')d\nu(\theta')}. \qquad (1)$$

This is called the *posterior density* of θ, given the data x. Using this posterior density, one can calculate the conditional expected losses for an action a, given observed data x:

$$\mathbf{E}_{\theta|x} L(a,\theta) = \int\limits_{\Theta} L(a,\theta)\xi(\theta \mid x)d\nu(\theta). \qquad (2)$$

Because $\mathbf{E}_{\theta|x} L(a,\theta)$ depends on the random variable x, it is also a random variable, and expected losses $\rho(a)$ or the risk of action a can be obtained as follows:

$$\rho(a) = \mathbf{E}_x \mathbf{E}_{\theta|x} L(a,\theta) =$$
$$= \int\limits_{S}\int\limits_{\Theta} L(a,\theta)\xi(\theta \mid x) f(x) d\nu(\theta) d\mu(x), \qquad (3)$$

where $f(x)$ denotes the non-conditional density of x; $\mu(x)$ and $\nu(\theta)$ are some probability measures on Θ and S respectively.

Due to the fact that $f(x)=\int\limits_{\Theta} f(x\mid\theta)\xi(\theta)d\nu(\theta)$ the equality (3) yields the *triple sum form* of the risk $\rho(a)$:

$$\rho(a)=$$
$$=\int\limits_{\Theta}\int\limits_{S}\int\limits_{\Theta} L(a,\theta)\xi(\theta\mid x)f(x\mid\theta')\xi(\theta')d\nu(\theta)d\mu(x)d\nu(\theta'). \quad (4)$$

At the same time the above expression implies that we are choosing the action a regardless the value of the observed data x. Thus $\rho(a)$ equals to the risk based on the prior density of parameter θ, i. e.:

$$\rho(a)=\int\limits_{\Theta} L(a,\theta)\xi(\theta)d\nu(\theta). \quad (5)$$

Let us estimate the lower bound of $\rho(a)$:

$$\inf_{a\in A}\rho(a)=$$
$$=\inf_{a\in A}\int\limits_{\Theta}\int\limits_{S}\int\limits_{\Theta} L(a,\theta)\xi(\theta\mid x)f(x\mid\theta')\xi(\theta')d\nu(\theta)d\mu(x)d\nu(\theta')$$
$$\geq\int\limits_{\Theta}\xi(\theta')\left\{\inf_{a\in A}\int\limits_{S}\int\limits_{\Theta} L(a,\theta)\xi(\theta\mid x)f(x\mid\theta')d\mu(x)d\nu(\theta)\right\}d\nu(\theta')$$
$$\geq\int\limits_{\Theta}\int\limits_{S} f(x\mid\theta')\xi(\theta')\left\{\inf_{a\in A}\int\limits_{\Theta} L(a,\theta)\xi(\theta\mid x)d\nu(\theta)\right\}d\mu(x)d\nu(\theta'). \quad (6)$$

As can be seen from the latter inequality, we can reduce the risk by finding an action a, that minimizes the conditional expected losses $\mathbf{E}_{\theta|x}L(a,\theta)$ for an action a, given x. In doing so, we will compose a mapping function $\delta^*(\cdot)$ that associates with each observed value x an action a, minimizing the conditional expected losses (2). This function $\delta^*(\cdot)$ is called the *Bayesian decision function*. According to Raiffa and Schlaifer [2] such a way to obtain the Bayesian decision function is called the *extensive form of analysis*.

Now we can introduce the notion of decision rule or *decision function* $\delta(\cdot)$ – as a mapping from the data x into the set A of possible actions. Let us denote $\rho(\xi,\delta)$ *the risk* (expected losses) corresponding to the decision function $\delta(\cdot)$. This risk may be written down as follows:

$$\rho(\xi,\delta)=\int\limits_{\Theta}\int\limits_{S} L(\delta(x),\theta)p(x,\theta)d\mu(x)d\nu(\theta), \quad (7)$$

where $p(x,\theta)$ stands for the joint density of x and θ. Since $p(x,\theta)=f(x\mid\theta)\xi(\theta)$, we obtain the following representation for the risk:

$$\rho(\xi,\delta)=\int\limits_{\Theta}\int\limits_{S} L(\delta(x),\theta)f(x\mid\theta)\xi(\theta)d\mu(x)d\nu(\theta), \quad (8)$$

or, changing order of integration (throughout this paper we assume that all density functions as well as function $L(\cdot,\cdot)$ meet the Fubini's theorem conditions and the integration order can be changed):

$$\rho(\xi,\delta)=\int\limits_{S}\int\limits_{\Theta} L(\delta(x),\theta)f(x\mid\theta)\xi(\theta)d\nu(\theta)d\mu(x). \quad (9)$$

This is the *double sum form* of the risk. As can be seen from the latter expression, the (Bayesian) decision function $\delta^*(\cdot)$ that minimizes the risk (9) can be obtained by finding an action a minimizing the integral

$$\int\limits_{\Theta} L(a,\theta)f(x\mid\theta)\xi(\theta)d\nu(\theta)$$

for each $x\in S$.

According to Raiffa and Schlaifer [2] this method of Bayesian decision function composing is called the *normal form of analysis*.

1. *Ivanenko V. I., Labkovsky V. A.* Uncertainty Issues in Decision Making Problems.- K.: Naukova dumka, 1990. (Russian)

2. *Raiffa //., Schlaifer R.* Applied Statistical Decision Theory. Division of Research Graduate School of Business Administration.- Boston: Harvard University, 1961.

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ПРО РІЗНІ ПРЕДСТАВЛЕННЯ РИЗИКУ У БАЙЄСІВСЬКИХ РІШЕННЯХ

Стаття вказує на зв 'язок між різними формами представлення ризик)> у статистичних задачах рішення та різними методами побудови байєсівської вирішуючої функції. Стаття відображає та роз виває деякі ідеї, представлені у [1].