The diagonal limits of Hamming spaces

Bogdana Oliynyk

Communicated by V. V. Kirichenko

Abstract. We consider a continuum family of subspaces of the Besicovitch–Hamming space on some alphabet $B$, naturally parametrized by supernatural numbers. Every subspace is defined as a diagonal limit of finite Hamming spaces on the alphabet $B$. We present a convenient representation of these subspaces. Using this representation we show that the completion of each of these subspace coincides with the completion of the space of all periodic sequences on the alphabet $B$. Then we give answers on two questions formulated in [1].

Introduction

Let $B = \{b_1, \ldots, b_q\}$ be an alphabet, $q \geq 2$. Denote by $H_n(q)$ the Hamming space of dimension $n$ on the alphabet $B$. This space consists of all $n$-tuples $(a_1, \ldots, a_n)$, $a_i \in B$, $1 \leq i \leq n$, where the distance $d_{H_n}$ between two $n$-tuples is equal to the number of coordinates where they differ. The scaled Hamming space $\hat{H}_n(q)$ have the same set of points, but the distance is defined as $\frac{1}{n}d_{H_n}$. A natural generalization of the scaled Hamming space is the Besicovitch space (in other terms, the Besicovitch–Hamming space), consisting of all infinite sequences on the alphabet $B$ ([2], [3]). This space is used since the 1960s in symbolic dynamics and ergodic theory.

In this article we consider a family of subspaces of the Besicovitch–Hamming space on alphabet $B$, naturally parametrized by infinite supernatural numbers. The space corresponding to a supernatural number
$u$ is called the $u$-periodic Hamming space. For every $u$ the $u$-periodic Hamming space is isometric to the direct limits of finite scaled Hamming spaces with respect to diagonal embeddings.

If the alphabet $B$ consists of two elements, i.e. $B = \{0, 1\}$, then $u$-periodic Hamming spaces were characterized in [1] and [5]. In [1] the $2^\infty$-periodic Hamming space was considered as the space of finite unions of half-open subintervals of the interval $[0, 1)$ with binary-rational endpoints. In [5] $u$-periodic Hamming spaces were regarded as the spaces of clopen subsets of the boundaries of spherically homogeneous rooted trees. The completion of every space from these family is isometric to the completion of the space of $2^\infty$-periodic $(0, 1)$-sequences [1] or of all periodic $(0, 1)$-sequences [5].

In this paper we consider the case $q > 2$. In [1] P. J. Cameron and S. Tarzi formulated the questions:

(A) Is there a convenient representation of $2^\infty$-periodic Hamming space on alphabet $B$ and its completion?

(B) Are completions of $u$-periodic Hamming spaces on alphabet $B$ independent of choice of $u$?

We give answers to these questions. Namely, for every infinite supernatural numbers $u$ we represent the $u$-periodic Hamming space on alphabet $B$ and its completion as the spaces of functions defined in a special way on the boundaries of spherically homogeneous rooted trees. These functions define a measurable partition in sense of [4]. Using this representation we prove that the completion of every $u$-periodic Hamming space is independent of choice of $u$ and coincides with the completion of the space of $2^\infty$-periodic sequences or of all periodic sequences.

1. Preliminaries

1. Let $\mathbb{P}$ be the set of all primes. A supernatural number (or Steinitz number) is an infinite formal product of the form

\[ \prod_{p \in \mathbb{P}} p^{k_p} \]

where $k_p \in \mathbb{N} \cup \{0, \infty\}$. Denote by $\mathbb{SN}$ the set of all supernatural numbers. The set $\mathbb{N}$ is a natural subset of $\mathbb{SN}$. The elements of the set $\mathbb{SN} \setminus \mathbb{N}$ are called infinite supernatural numbers. A supernatural number $v$ divides a supernatural number $u$ if there exists $t \in \mathbb{SN}$, such that $u = v \cdot t$. The divisibility relation $|$ transforms the set $\mathbb{SN}$ into a partially ordered set with the greatest element $I = \prod_{p \in \mathbb{P}} p^\infty$ and the least element 1. Moreover, the poset $(\mathbb{SN}, \mid)$ is a complete lattice.
A sequence of positive integers \( \tau = (m_1, m_2, \ldots) \) is called \textit{divisible} if \( k_i | k_{i+1} \) for all \( i \in \mathbb{N} \). For divisible sequence \( \tau = (m_1, m_2, \ldots) \) the supernatural number
\[
m_1 \cdot \frac{m_2}{m_1} \cdot \frac{m_3}{m_2} \ldots
\]
is called the \textit{characteristic of the sequence} \( \tau \) and denoted by \( \text{char}(\tau) \). It is easy to see that the sequence \( \tau \) is a strictly increasing divisible sequence iff \( \text{char}(\tau) \) is an infinite supernatural number.

2. Let \( X \) be a nonempty set, \( \Sigma_1 = \{X_i, 1 \leq i \leq q\} \) and \( \Sigma_2 = \{Y_i, 1 \leq i \leq q\} \) be ordered partitions of the set \( X \). These partitions decompose the set \( X \) into \( q \) (possibly empty) blocks. Introduce the \textit{symmetric difference of partitions} \( \Sigma_1 \) and \( \Sigma_2 \) as the set \( \Sigma_1 \triangle \Sigma_2 \) defined by the equation
\[
\Sigma_1 \triangle \Sigma_2 = \bigcup_{i \neq j} (X_i \cap Y_j).
\] (1)

Then \( \Sigma_1 \triangle \Sigma_2 \subset X \). We can formulate some properties of the symmetric difference of partitions which are not difficult to verify.

\textbf{Lemma 1.} Let \( \Sigma_1 = \{X_i, 1 \leq i \leq q\}, \Sigma_2 = \{Y_i, 1 \leq i \leq q\} \) and \( \Sigma_3 = \{Z_i, 1 \leq i \leq q\} \) be ordered partitions of the set \( X \). Then the following properties hold.

1) \( \Sigma_1 \triangle \Sigma_2 = \emptyset \) iff \( \Sigma_1 = \Sigma_2 \).

2) \( \Sigma_1 \triangle \Sigma_2 = \Sigma_2 \triangle \Sigma_1 \).

3) \( (\Sigma_1 \triangle \Sigma_2) \triangle \Sigma_3 = \Sigma_1 \triangle (\Sigma_2 \triangle \Sigma_3) \).

4) \( \Sigma_1 \triangle \Sigma_2 \subseteq (\Sigma_1 \triangle \Sigma_3) \cup (\Sigma_2 \triangle \Sigma_3) \).

2. \textbf{The periodic Hamming space}

Let \( \tau = (m_1, m_2, \ldots) \) be an increasing divisible sequence, \( \hat{H}_{m_1}(q), \hat{H}_{m_2}(q), \ldots \) be the corresponding infinite sequence of scaled Hamming spaces on alphabet \( B \). Denote by \( (s_1, s_2, \ldots) \) the sequence of ratios of the sequence \( \tau \), i. e.
\[
s_1 = m_1, \quad s_{i+1} = \frac{m_{i+1}}{m_i}, \quad i \geq 1.
\] (2)

For any \( i \geq 1 \) define an isometric embedding \( \psi_{s_i} : \hat{H}_{m_i}(q) \to \hat{H}_{m_{i+1}}(q) \) by the rule:
\[
\psi_{s_i}(x_1, \ldots, x_{m_i}) = (x_1, \ldots, x_{m_i}|x_{1}, \ldots, x_{m_i}| \ldots |x_1, \ldots, x_{m_i}).
\] (3)
Then the sequence \( \tau \) determines the directed system of scaled Hamming spaces on the alphabet \( B \)

\[
\langle \hat{H}_{m_i}(q), \psi_{s_i} \rangle_{i \in \mathbb{N}}
\]

with the diagonal embeddings \( \psi_{s_i}, i \geq 1 \), defined by (3).

The limit space of the directed system (4)

\[
\mathcal{H}(\tau, q) = \lim_{\rightarrow} \langle \hat{H}_{m_i}(q), \psi_{s_i} \rangle
\]

is called a diagonal limit of spaces \( \hat{H}_{m_i} \).

**Proposition 1.** Let \( \tau_1, \tau_2 \) be increasing divisible sequences. Then the spaces \( \mathcal{H}(\tau_1, q) \) and \( \mathcal{H}(\tau_2, q) \) are isometric iff \( \text{char}(\tau_1) = \text{char}(\tau_2) \).

The diagonal limit \( \mathcal{H}(\tau, q) \) admits a natural description using supernatural numbers.

The infinite sequence \( a = (a_1, a_2, \ldots), a_i \in B \) is said to be periodic if there exists a natural number \( k \) such that the equality \( a_i = a_{i+k} \) holds for all \( i \in \mathbb{N} \). In this case the number \( k \) is called a period of the sequence \( a \). A periodic sequence \( a \) is called \( u \)-periodic for some supernatural number \( u \) if its minimal period divides \( u \).

Let \( u \) be some infinite supernatural number and \( \mathcal{H}(u, q) \) be the set of all \( u \)-periodic sequences on \( B \). In particular, the space \( \mathcal{H}(I, q) \) is the space of all periodic sequences on \( B \). We can introduce a natural metric on \( \mathcal{H}(u, q) \) putting

\[
d_{\mathcal{H}(u,q)}((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \frac{1}{l} d_{\mathcal{H}_l}((x_1, \ldots, x_l), (y_1, \ldots, y_l)),
\]

where \( l \) is a common period of sequences \( (x_1, x_2, \ldots) \) and \( (y_1, y_2, \ldots) \) from \( \mathcal{H}(u, q) \). It is clear that definition (5) is independent of choice of a common period. We call the metric space \( (\mathcal{H}(u, q), d_{\mathcal{H}(u,q)}) \) the \( u \)-periodic Hamming space over the alphabet \( B \). It is not difficult to verify

**Proposition 2.** Let \( \tau \) be an increasing divisible sequence, \( u \) be some infinite supernatural number. Then the spaces \( \mathcal{H}(\tau, q) \) and \( \mathcal{H}(u, q) \) are isometric iff \( \text{char}(\tau) = u \).

### 3. Representations on boundaries of rooted trees

Let \( T \) be an infinite locally finite rooted tree with the root \( v_0 \), \( n \) be some nonnegative integer. The \( n \)-th level of the tree \( T \) is the set \( L_n \) of all vertices \( v \) of \( T \) such that the length of the unique simple path connecting \( v \) and \( v_0 \) in \( T \) equals \( n \).
A rooted tree \((T, v_0)\) is called \textit{spherically homogeneous} if for every nonnegative integer \(n\) the degrees of all vertices from \(L_n\) are equal. A spherically homogeneous rooted tree \(T\) is uniquely defined by its \textit{spherical index}, i.e. by an infinite sequence of positive integers \([s_1; s_2; \ldots]\) such that \(s_i\) is the number of edges joining a vertex of the \((i - 1)\)th level with vertices of the \(i\)th level, \(i \geq 1\). If the tree \((T, v_0)\) has the spherical index \([s_1; s_2; \ldots]\) then the sequence \(m_i = s_1 \cdot s_2 \cdot \ldots \cdot s_i, i \geq 1\), is divisible and additionally \(|L_i| = m_i, i \geq 1\).

The boundary \(\partial T\) of a tree \(T\) is the set of infinite rooted paths, i.e. the set of infinite sequences of pairwise distinct vertices \((v_0, v_1, v_2, \ldots)\) such that the vertices \(v_i, v_{i+1}\) are connected by an edge for every \(i, i \geq 0\). These paths are also called the ends of \(T\). Define a distance \(\rho\) on the set \(\partial T\) as

\[
\rho(\gamma_1, \gamma_2) = \begin{cases} 
\frac{1}{k+1}, & \text{if } \gamma_1 \neq \gamma_2 \\
0, & \text{if } \gamma_1 = \gamma_2,
\end{cases}
\]

(6)

where \(k\) is the length of the common beginning of rooted paths \(\gamma_1\) and \(\gamma_2\). The space \((\partial T, \rho)\) is an ultrametric totally disconnected compact space with diameter 1.

Let as before \(\tau = (m_1, m_2, \ldots)\) be an increasing divisible sequence with the sequence of its ratios \((s_1, s_2, \ldots)\) defined by (2). Assume that \(T_\tau\) is a spherically homogeneous rooted tree with spherical index \([s_1; s_2; \ldots]\) and \(\rho_\tau\) is a metric defined by (6) on \(\partial T_\tau\). The set of all rooted paths from \(\partial T_\tau\) passing through a vertex \(v\) is denoted

\[C_v = \{\gamma \in \partial T_\tau \mid v \in \gamma\}\]

and called the \textit{cylindrical set} \(C_v\) corresponding to \(v\).

The metric \(\rho_\tau\) induces a topology on \(\partial T_\tau\). The clopen subsets are finite unions of cylindrical sets. Denote by \(\Omega T_\tau\) the set of all clopen subsets of \(\partial T_\tau\). Define the Bernoulli measure \(\mu\) on the Borel \(\sigma\)-algebra of \(\partial T_\tau\) by the rule:

\[\mu(C_v) = \frac{1}{n_v},\]

where \(n_v\) is the number of vertices of \(T_\tau\) on the level containing the vertex \(v\). The space \((\partial T_\tau, \mu)\) is isomorphic as a measure space to the space \(([0, 1], l)\), where \(l\) is the Lebesgue measure (see [6] for instance).

Introduce the discrete metric \(g\) on the set \(B\), i.e.

\[g(b_i, b_j) = \begin{cases} 
1, & \text{if } i \neq j \\
0, & \text{if } i = j,
\end{cases}\]
for all \(1 \leq i, j \leq q\). The metric \(\rho\) induces the discrete topology on \(B\). Denote by \(C(\partial T_\tau, B)\) the set of all continuous functions from the space \(\partial T_\tau\) to the space \(B\). Note that for every \(f \in C(\partial T_\tau, B)\) the subsets \(\{f^{-1}(b_i)\}, 1 \leq i \leq q\) form an ordered partition of the set \(\partial T_\tau\). Define a mapping \(d_\mu : C(\partial T_\tau, B) \times C(\partial T_\tau, B) \to \mathbb{R}^+\) by putting

\[
d_\mu(f, g) = \mu(\Sigma_f \triangle \Sigma_g),
\]

where \(f, g \in C(\partial T_\tau, B)\) and the symmetric difference of \(\Sigma_f = \{f^{-1}(b_i)\}, 1 \leq i \leq q\) and \(\Sigma_g = \{g^{-1}(b_i)\}, 1 \leq i \leq q\) is determined by (1). The proof of the next proposition follows from Lemma 1.

**Lemma 2.** The function \(d_\mu\) is a metric on the set \(C(\partial T_\tau, B)\).

**Theorem 1.** Let \(B = \{b_1, \ldots, b_q\}\) be some alphabet, \(q > 2\), \(\tau = (m_1, m_2, \ldots)\) be an increasing divisible sequence with the sequence of its ratios \((s_1, s_2, \ldots)\). Assume that \(T_\tau\) is a spherically homogeneous rooted tree with spherical index \([s_1; s_2; \ldots]\) and \(u\) is a supernatural number with \(\text{char}(\tau) = u\). Then the \(u\)-periodic Hamming space \(H(u, q)\) on the alphabet \(B\) is isometric to the space of all continuous functions \(C(\partial T_\tau, B)\) with the metric \(d_\mu\) defined by (7).

**Proof.** Let \(n\) be a positive integer. A function \(f : \partial T_\tau \to B\) is called \(n\)-determined if for every \(v \in L_n\) there exists \(i, 1 \leq i \leq q\), such that the equality \(f(x) = b_i\) holds for any \(x \in C_v\). We write \(f(C_v) = b_i\) in this case. Note that every \(n\)-determined function is continuous. Conversely, for any continuous function \(f \in C(\partial T_\tau, B)\) there exists a level \(l\) such that for every \(n \geq l\) the function \(f\) is \(n\)-determined.

Denote by \(\text{Fun}(n)\) the set of all \(n\)-determined function. Enumerate all vertices in \(L_n\) and assume that \(L_n = \{v_1, \ldots, v_{m_n}\}\). Define a mapping \(\varphi_n : \text{Fun}(n) \to H_{m_n}(q)\) by the rule:

\[
\varphi_n(f) = (f(C_{v_1}), \ldots, f(C_{v_{m_n}})).
\]

The mapping \(\varphi_n\) is bijective. We are going to show that \(\varphi_n\) preserves distances between points. Let \(f, g \in \text{Fun}(n)\). The sets \(f^{-1}(b_i)\) and \(g^{-1}(b_i)\) are unions of cylindrical sets for all \(1 \leq i \leq q\). Hence, from the definitions of the measure \(\mu\) and the metric \(d_\mu\) we obtain

\[
d_\mu(f, g) = \mu(\Sigma_f \triangle \Sigma_g) = \frac{1}{m_n} \sum_{i=1}^{m_n} (f(C_{v_i}) \oplus g(C_{v_i})),
\]

where

\[
f(C_{v_i}) \oplus g(C_{v_i}) = \begin{cases} 
1, & \text{if } f(C_{v_i}) \neq g(C_{v_i}) \\
0, & \text{if } f(C_{v_i}) = g(C_{v_i})
\end{cases}.
\]
Since
\[ \frac{1}{m_n} \sum_{i=1}^{m_n} f(C_{v_i}) \oplus g(C_{v_i}) = d_{H_m}(f(C_{v_1}), \ldots, f(C_{v_{m_n}}), g(C_{v_1}), \ldots, g(C_{v_{m_n}})), \]
using (8) we have
\[ d_{\mu}(f, g) = d_{H_m}(f(C_{v_1}), \ldots, f(C_{v_{m_n}}), g(C_{v_1}), \ldots, g(C_{v_{m_n}})). \]
Hence the mapping \( \varphi_n \) is an isometry between \((\text{Fun}(n), d_{\mu})\) and \(H_m(q)\).

Every cylindrical set corresponding to a vertex of the \(n\)-th level splits into the union of \(s_n+1\) cylindrical subsets corresponding to vertices of the \((n+1)\)th level in \(T_{\tau}\). Every \(n\)-determined function is \((n+1)\)-determined. Thus we can define an injection \(\chi_n : H_m(q) \hookrightarrow H_{m+1}(q)\) by the rule:
\[ \chi_n(x_1, \ldots, x_{m_n}) = (x_1, \ldots, x_{1}, \ldots, x_{m_n}, \ldots, x_{m_n}). \]
Then for any positive integer \(n\) the diagram
\[
\begin{array}{ccc}
\text{Fun}(n) & \xrightarrow{\psi_n} & \text{Fun}(n+1) \\
\downarrow{\varphi_n} & & \downarrow{\varphi_{n+1}} \\
\hat{H}_m(q) & \xrightarrow{\chi_n} & \hat{H}_{m+1}(q)
\end{array}
\]
is commutative. Therefore, the spaces
\[ \bigcup_{n=1}^{\infty} \text{Fun}(n) = C(\partial T_{\tau}, B) \quad \text{and} \quad \lim_{\to}(\hat{H}_{m_i}(q), \psi_{s_i}) = \mathcal{H}(\tau, q) \]
are isometric. The proof of the theorem is complete. \(\square\)

Define an equivalence \(\sim\) on the set of all measurable functions \(\text{Measurable}(\partial T_{\tau}, B)\). Let \(f \sim g\) iff for every \(Y \subseteq B\) the sets \(f^{-1}(Y)\) and \(g^{-1}(Y)\) coincide up to measure zero sets.

The following statement is the answer to the question (A).

**Corollary 1.** Let \(B = \{b_1, \ldots, b_q\}\) be an alphabet, \(q > 2\), \(\tau = (m_1, m_2, \ldots)\) be an increasing divisible sequence with the sequence of its ratios \((s_1, s_2, \ldots)\). Assume that \(T_{\tau}\) is a spherically homogeneous rooted tree of the spherical index \([s_1; s_2; \ldots]\) and \(u\) is a supernatural number with \(\text{char}(\tau) = u\). Then the completion of the \(u\)-periodic Hamming space \(\mathcal{H}(u, q)\) on the alphabet \(B\)
is isometric to the space of all measurable functions $\text{Measurable}(\partial T_\tau, B)$ (well-defined up to measure zero sets) with the metric $d_\mu$ defined by (7).

As the space $(\partial T_\tau, \mu)$ is a standard probability space and independent of the choice of a divisible sequence $\tau$, the space of all measurable functions $\text{Measurable}(\partial T_\tau, B)$ (well-defined up to measure zero sets) with metric $d_\mu$ independent of the choice of a divisible sequence too. Thus we get an answer to the question (B) formulated in [1].

**Corollary 2.** For every infinite strictly increasing divisible sequence $\tau = (m_1, m_2, \ldots)$ the completion of the space $\mathcal{H}(\tau, q)$ is isometric to the completion of the space $\mathcal{H}(2^\infty, q)$ or completion of the space $\mathcal{H}(I, q)$ of all periodic sequences over the alphabet $B$.

**References**


**Contact Information**

B. Oliynyk
Department of Mechanics and Mathematics,
Kyiv Taras Shevchenko University,
Volodymyrska 64, Kyiv, 01601, Ukraine
E-Mail: bogdana.oliynyk@gmail.com

Received by the editors: 11.04.2013
and in final form 18.04.2013.