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On Separation of Variables for Integrable Equations of Soliton Type

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Abstract

We propose a general scheme for separation of variables in the integrable Hamiltonian systems on orbits of the loop algebra $\mathfrak{sl}(2,\mathbb{C})\times\mathcal{P}(\lambda,\lambda^{-1})$. In particular, we illustrate the scheme by application to modified Korteweg—de Vries (MKdV), sin(sinh)-Gordon, nonlinear Schrödinger, and Heisenberg magnetic equations.

Introduction

Let us make a brief review of the problem.

After the fundamental paper [23], B. Dubrovin (see [8]) proposed a separation of variables for finite gap KdV system. B. Dubrovin shows that the poles of an appropriately normalized Baker—Akhiezer function for the auxiliary linear spectral problem are the separation variables. The new variables evolve on a hyperelliptic Riemannian surface $\mathcal{R}$ of genus g. The genus coincides with the number of degrees of freedom of the finite gap phase space.

In the papers [14,18,20,25] a separation of variables is realized for sin-Gordon equation, nonlinear Schrödinger equation, and the classic Thirring model. The case of sin-Gordon equation appears to be completely similar to the KdV system. However, the cases of nonlinear Schrödinger equation and Thirring model have a distinction: *the number of degrees of freedom is greater by one than the genus of the corresponding spectral curve.* Here the papers [14, 18, 25] suggest the separation of variables on a reduced phase space. Later, the complex Liouville torus of nonlinear Schrödinger equation was proven to be the generalized Jacobian of a singular Riemannian surface (see [24]).

The ideas of the early papers on the integration of finite gap systems were generalized by E. Sklyanin [27, 28] and partly extended to the quantum integrable models [29, 30].

At the beginning of the 90s a new technique of separation of variables appeared that effectively uses bi-hamiltonian, or multi-hamiltonian, properties of integrable systems, see [3, 5–7] and [10–12, 22]. The main result in this direction is the diagonalization of recursive Nijenhuis operator. In the papers [11, 22] the method is applied to KdV and Boussinesq hierarchies, and classical finite-dimensional systems.

The papers [9, 31, 32] investigate the connection between the problem of separation of variables and the parametrization of compact tori by symmetric products of Riemannian surfaces. According to [31, 32], *if a change of variables reduces Liouville 1-form to a sum of meromorphic differentials on the corresponding Riemannian surface, then we say that the new variables are the separation variables.*

We propose a method of separation of variables for integrable Hamiltonian systems that is connected with the orbit structure of affine Lie algebras. The fact that finite gap phase space of an integrable soliton hierarchy has an orbital structure was established in [15,16].[1] The Hamiltonian systems in question obey the equations of Lax type, and hence the separation variables are points on the corresponding *spectral curve*. Note that such systems are multi-Hamiltonian, which connects our results with the results of [10–12, 22].

In Sections 1 and 3 we reproduce the key results from [15, 16] about finite gap phase spaces for integrable equations as orbits of loop algebra. We illustrate our scheme by the examples of modified Korteweg-de Vries (MKdV) system, sin(sinh)-Gordon equation, nonlinear Schrödinger equation, and Heisenberg magnetic chain.

This paper is organized as follows. Sections 1 and 2 are devoted to MKdV system and sin(sinh)-Gordon equation. In Section 1 we construct adjoint Poisson spaces and define the orbits regarded as phase spaces for MKdV system and sin(sinh)-Gordon equation. The construction is discussed in more detail in [4]. In Section 2 we describe the scheme for separation of variables and illustrate it by application to MKdV system and sin(sinh)-Gordon equation. We show that the separation of variables is achieved on both orbits simultaneously. In Sections 3 we construct adjoint Poisson spaces and define the orbits regarded as phase spaces for nonlinear Schrödinger equation and Heisenberg magnetic chain. In Sections 4 and 5 we similarly consider separation of variables for nonlinear Schrödinger equation and Heisenberg magnetic chain, accordingly.

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1 Phase spaces for MKdV system and sin-Gordon equation as orbits in $\mathfrak{sl}(2,\mathbb{C})\otimes\mathcal{P}(\lambda,\lambda^{-1})$

First, let us recall some constructions from [15, 16]. Take the algebra $\mathfrak{sl}(2,\mathbb{C})$ with the basis

$$H=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix},\qquad X=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\qquad Y=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

[1]The results of [15, 16] are partially covered by [13]. However the authors of [13] took no notice of the remarkable *duality* between pairs of soliton equations: MKdV and sin-Gordon equations, KdV and Liouville equations, nonlinear Schrödinger and Heisenberg magnetic equations, etc. The duality is evident if one uses the orbital approach. The pairs of dual equations have common Liouville torus and separation variables. Sometimes, there exists a gauge equivalence between the equations of a pair, and the equivalence extends to the total infinite phase space [26].

Suppose $\mathcal{P}(\lambda,\lambda^{-1})$ is the algebra of Laurent polynomials in λ. Denote by $\widetilde{\mathfrak{g}}$ the algebra $\mathfrak{sl}(2,\mathbb{C})\otimes\mathcal{P}(\lambda,\lambda^{-1})$. Then

$$H^{2m}=\lambda^m H,\qquad X^{2m+1}=\lambda^m X,\qquad Y^{2m+1}=\lambda^{m+1}Y \tag{1.1}$$

is a *basis* in $\widetilde{\mathfrak{g}}$.

Consider the operator

$$d=2\lambda\frac{d}{d\lambda}+\mathrm{ad}_H;$$

we call it the *operator of principal grading.* It is easy to prove that the basis elements (1.1) are the eigenvectors of d. We call the eigenvalues of d the *degrees.* The superscripts in the lefthand sides of (1.1) indicate the corresponding principal degrees of the basis elements. By $\mathfrak{g}_l$, $l\in\mathbb{Z}$, denote an eigenspace of principal degree l. It is evident that

$$\mathfrak{g}_{2m}=\mathrm{span}_{\mathbb{C}}\{H^{2m}\},\qquad \mathfrak{g}_{2m+1}=\mathrm{span}_{\mathbb{C}}\{X^{2m+1},Y^{2m+1}\}.$$

Decompose $\widetilde{\mathfrak{g}}$ into two subalgebras

$$\widetilde{\mathfrak{g}}_+=\sum_{l\geqslant 0}\mathfrak{g}_l,\qquad \widetilde{\mathfrak{g}}_-=\sum_{l<0}\mathfrak{g}_l,\qquad \widetilde{\mathfrak{g}}=\widetilde{\mathfrak{g}}_++\widetilde{\mathfrak{g}}_-.$$

Further, consider the ad-*invariant bilinear forms*

$$\langle A(\lambda),B(\lambda)\rangle_k=\mathrm{res}\,\lambda^{-k-1}\,\mathrm{Tr}\,A(\lambda)B(\lambda),\quad A(\lambda),\ B(\lambda)\in\widetilde{\mathfrak{g}},\quad k\in\mathbb{Z}. \tag{1.2}$$

We use the forms to define the spaces dual to $\widetilde{\mathfrak{g}}_+$ and $\widetilde{\mathfrak{g}}_-$.

Example 1. Let $k=-1$. We have

$$(\widetilde{\mathfrak{g}}_-)^*=\widetilde{\mathfrak{g}}_++\mathfrak{g}_{-1},\qquad (\widetilde{\mathfrak{g}}_+)^*=\sum_{l\leqslant -2}\mathfrak{g}_l, \tag{1.3}$$

where $(\widetilde{\mathfrak{g}}_-)^*$ and $(\widetilde{\mathfrak{g}}_+)^*$ contain only the nonzero functionals on $\widetilde{\mathfrak{g}}_\pm$.

Example 2. Let $k=N\geqslant 0$. Then

$$(\widetilde{\mathfrak{g}}_-)^*=\sum_{l\geqslant 2N+1}\mathfrak{g}_l,\qquad (\widetilde{\mathfrak{g}}_+)^*=\sum_{l\leqslant 2N}\mathfrak{g}_l.$$

Fix $N\geqslant 0$. Consider $M^{N+1}\subset\widetilde{\mathfrak{g}}$, where an element $\widehat{\mu}(\lambda)\in M^{N+1}$ has the form

$$\widehat{\mu}(\lambda)=\begin{pmatrix}\alpha(\lambda)&\beta(\lambda)\\ \gamma(\lambda)&-\alpha(\lambda)\end{pmatrix}$$

with

$$\alpha(\lambda)=\sum_{m=0}^{N}\lambda^m\alpha_{2m},\qquad \beta(\lambda)=\sum_{m=0}^{N+1}\lambda^{m-1}\beta_{2m-1},\qquad \gamma(\lambda)=\sum_{m=0}^{N+1}\lambda^m\gamma_{2m-1}.$$

We call M^{N+1} the *N-gap sector of* $\widetilde{\mathfrak{g}}$, or shortly the *finite gap sector.*

Because the factor-algebra $\widetilde{\mathfrak{g}}_-/\sum_{l\leqslant -2N-4}\mathfrak{g}_l$ acts effectively on M^{N+1}, the coadjoint action of $\widetilde{\mathfrak{g}}_-$ with respect to the form $\langle\ ,\ \rangle_{-1}$ is well defined on M^{N+1}. The same is true for the coadjoint action of $\widetilde{\mathfrak{g}}_+$ with respect to the form $\langle\ ,\ \rangle_N$, indeed, the factor-algebra $\widetilde{\mathfrak{g}}_+/\sum_{l\geqslant 2N+2}\mathfrak{g}_l$ acts effectively on M^{N+1}.

Let $C(M^{N+1})$ be the space of smooth functions on M^{N+1}. For all f_1, $f_2\in C(M^{N+1})$ define the *first Lie-Poisson bracket* by the formula

$$\{f_1,f_2\}_1=\sum_{m,n=0}^{N}\sum_{a,b=1}^{3}P_{ab}^{mn}(-1)\frac{\partial f_1}{\partial\mu_m^a}\frac{\partial f_2}{\partial\mu_n^b}, \tag{1.4}$$

where

$$P_{ab}^{mn}(-1)=\langle\widehat{\mu}(\lambda),[Z_a^{-m-1},Z_b^{-n-1}]\rangle_{-1},$$

$$Z_1^m=H^m,\qquad Z_2^m=Y^m,\qquad Z_3^m=X^m,$$
$$\mu_m^1=\alpha_m,\qquad \mu_m^2=\beta_m,\qquad \mu_m^3=\gamma_m.$$

With the same notation, define the *second Lie-Poisson bracket* by the formula

$$\{f_1,f_2\}_2=\sum_{m,n=0}^{N}\sum_{a,b=1}^{3}P_{ab}^{mn}(N)\frac{\partial f_1}{\partial\mu_m^a}\frac{\partial f_2}{\partial\mu_n^b}, \tag{1.5}$$

where

$$P_{ab}^{mn}(N)=\langle\widehat{\mu}(\lambda),[Z_a^{-m+N},Z_b^{-n+N}]\rangle_N.$$

One can see that the functions β_{2N+1} and γ_{2N+1} *annihilate the bracket (1.4)*

$$\{\beta_{2N+1},f\}_1=0,\qquad \{\gamma_{2N+1},f\}_1=0\qquad \text{for all}\quad f\in C(M^{N+1}).$$

Thus, we can assume without loss of generality that

$$\beta_{2N+1}=\gamma_{2N+1}=const \tag{1.6}$$

and restrict the bracket (1.4) to the subspace $M_{con}^{N+1}\subset M^{N+1}$ with the constraints (1.6), clearly, $\dim M_{con}^{N+1}=3(N+1)$. The first Lie-Poisson bracket is nondegenerate on M_{con}^{N+1}. We use the set γ_{2m-1}, β_{2m-1}, α_{2m}, $m=0,1,\ldots,N$, as *coordinate functions* in M_{con}^{N+1}. We call the fixed coordinates β_{2N+1}, γ_{2N+1} the *external parameters.*

We see that, on one hand, $M_{con}^{N+1}\subset(\widetilde{\mathfrak{g}}_-)^*$ with respect to $\langle\ ,\ \rangle_{-1}$, see Example 1, and, on the other hand, $M_{con}^{N+1}\subset(\widetilde{\mathfrak{g}}_+)^*$ with respect to $\langle\ ,\ \rangle_N$, see Example 2.

In addition to the brackets (1.4) and (1.5), one can define N intermediate brackets with the Poisson tensors

$$P_{ab}^{mn}(k)=\langle\widehat{\mu}(\lambda),[Z_a^{-m+k},Z_b^{-n+k}]\rangle_k,\qquad k=0,\ldots,N-1. \tag{1.7}$$

Now, consider the ad*-invariant function

$$I(\lambda)=-\det\widetilde{\mu}(\lambda)=h_{-1}\lambda^{-1}+h_0+\cdots+h_{2N+1}\lambda^{2N+1}.$$

Then we have

$$h_\nu = \sum_{m+n=\nu} \left(\alpha_{2m}\alpha_{2n} + \gamma_{2m-1}\beta_{2n-1}\right), \qquad \nu = -1, 0, \ldots, 2N+1. \tag{1.8}$$

The Kostant-Adler scheme [1] implies the following assertions.

Proposition 1. *All functions h_ν, $\nu = -1, 0, \ldots, 2N+1$ determined by* (1.8) *mutually commute with respect to the brackets* (1.4), (1.5), *and the intermediate brackets with the Poisson tensors* (1.7).

Proposition 2. *The functions h_ν, $\nu = N, \ldots, 2N$, are functionally independent and annihilate the bracket* (1.4).

Consider the algebraic variety $\mathcal{O}_1^N \subset M_{con}^{N+1}$ defined by the set of equations $h_\nu = c_\nu$, $\nu = N, \ldots, 2N$, where c_ν are arbitrary fixed complex numbers. $\mathcal{O}_1^N$ is an orbit of coadjoint action of the subalgebra $\widetilde{\mathfrak{g}}_-$ and $\dim \mathcal{O}_1^N = 2(N+1)$.

Proposition 3. *The functions h_ν, $\nu = -1, \ldots, N-1$, are functionally independent and annihilate the bracket* (1.5).

Consider the algebraic variety $\mathcal{O}_2^N \subset M_{con}^{N+1}$ defined by the set of equations $h_\nu = c_\nu$, $\nu = -1, \ldots, N-1$, where c_ν are arbitrary fixed complex numbers. $\mathcal{O}_2^N$ is an orbit of coadjoint action of the subalgebra $\widetilde{\mathfrak{g}}_+$ and $\dim \mathcal{O}_2^N = 2(N+1)$.

It is obvious that the orbits $\mathcal{O}_1^N$ and $\mathcal{O}_2^N$ are the *symplectic leaves* with respect to the first and the second Lie-Poisson brackets, accordingly.

Further, the functions h_{-1}, h_0, $\ldots$, h_{N-1}, regarded as Hamiltonians with respect to the first Lie-Poisson bracket, generate non-trivial flows on M_{con}^{N+1}

$$\frac{\partial \mu_m^a}{\partial \tau_\nu} = \{\mu_m^a, h_\nu\}_1, \quad \nu = -1, 0, \ldots, N-1. \tag{1.9}$$

The equations (1.9) can be written with the help of the second Lie-Poisson bracket and the functions h_N, $\ldots$, h_{2N} regarded as Hamiltonians. Namely (see [16]), one has

$$\{\mu_m^a, h_\nu\}_1 = -\{\mu_m^a, h_{\nu+N+1}\}_2.$$

Proposition 4. *The system* (1.9) *reduced to the orbit $\mathcal{O}_1^N$ is equivalent to the finite gap complex MKdV hierarchy.*

The system (1.9) *reduced to the orbit $\mathcal{O}_2^N$ is equivalent to finite gap sin(sinh)-Gordon equation.*

Below we give the outline of the proof which may be found in full detail in [4].

First, rewrite (1.9) in matrix form

$$\frac{\partial \widehat{\mu}(\lambda)}{\partial \tau_\nu} = [\nabla_2 h_{\nu+N+1}, \widehat{\mu}(\lambda)] = [\widehat{\mu}(\lambda), \nabla_1 h_\nu], \tag{1.10}$$

where

$$\nabla_1 h = \sum_{m=0}^{N} \left(\frac{\partial h}{\partial \alpha_{2m}} H^{-2m-2} + \frac{\partial h}{\partial \beta_{2m-1}} Y^{-2m-1} + \frac{\partial h}{\partial \gamma_{2m-1}} X^{-2m-1} \right),$$

$$\nabla_2 h = \sum_{m=0}^{N} \left(\frac{\partial h}{\partial \alpha_{2m}} H^{-2m+2N} + \frac{\partial h}{\partial \beta_{2m-1}} Y^{-2m+2N+1} + \frac{\partial h}{\partial \gamma_{2m-1}} X^{-2m+2N+1} \right).$$

The hamiltonian flows along τ_ν and $\tau_{\nu'}$ commute, which implies the compatibility condition in the form of *zero curvature equations*. In particular, assigning $\tau_{N-1} = x$, $\tau_{N-2} = t$, we obtain

$$\frac{\partial \nabla_2 h_{2N}}{\partial t} - \frac{\partial \nabla_2 h_{2N-1}}{\partial x} + [\nabla_2 h_{2N}, \nabla_2 h_{2N-1}] = 0,$$

where

$$\nabla_2 h_{2N} = \begin{pmatrix} \alpha_{2N} & \beta_{2N+1} \\ \lambda\gamma_{2N+1} & -\alpha_{2N} \end{pmatrix},$$

$$\nabla_2 h_{2N-1} = \begin{pmatrix} \alpha_{2N-2} + \lambda\alpha_{2N} & \beta_{2N-1} + \lambda\beta_{2N+1} \\ \lambda\gamma_{2N-1} + \lambda^2\gamma_{2N+1} & -(\alpha_{2N-2} + \lambda\alpha_2 N) \end{pmatrix}.$$

Recall that β_{2N+1} and γ_{2N+1} are the fixed external parameters.

The reduction of (1.9) onto the orbit $\mathcal{O}_1^N$ gives the equation

$$\frac{\partial \alpha_{2N}}{\partial t} = \frac{\partial \alpha_{2N-2}}{\partial x}, \tag{1.11}$$

equivalent to MKdV equation with respect to the function $\alpha_{2N}(x,t) = u(x,t)$. Indeed, reducing the equation (1.9) as $\nu = N-1$ to the orbit $\mathcal{O}_1^N$ we obtain

$$\beta_{2N-1} = \frac{1}{2\beta_{2N+1}} \left(c_{2N} - \frac{\partial \alpha_{2N}}{\partial x} - \alpha_{2N}^2 \right), \tag{1.12a}$$

$$\gamma_{2N-1} = \frac{1}{2\beta_{2N+1}} \left(c_{2N} + \frac{\partial \alpha_{2N}}{\partial x} - \alpha_{2N}^2 \right), \tag{1.12b}$$

$$\alpha_{2N-2} = \frac{1}{4\beta_{2N+1}} \left(\frac{\partial^2 \alpha_{2N}}{\partial x^2} - 2\alpha_{2N}^3 + 2c_{2N}\alpha_{2N} \right). \tag{1.12c}$$

It is readily seen that combining (1.11) and (1.12c) we get the complex MKdV equation. Two real subalgebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1) \cong \mathfrak{sl}(2,\mathbb{R})$ of $\mathfrak{sl}(2,\mathbb{C})$ give rise to two real MKdV equations (the so-called ±MKdV).

In the same time, the reduction of (1.9) onto the orbit $\mathcal{O}_2^N$ leads to sin(sinh)-Gordon equation. Let $\mathcal{O}_2^N \cup \mathfrak{g}_{-1}$ be the *base* for the orbit $\mathcal{O}_2^N$. The 1-parameter subgroup $G_0 = \exp \mathfrak{g}_0$ parametrizes the base in a natural way

$$\gamma_{-1} = \sqrt{h_{-1}}\, e^{u}, \qquad \beta_{-1} = \sqrt{h_{-1}}\, e^{-u}.$$

Then the equations (1.9) imply

$$\alpha_{2N} = \frac{1}{2} \frac{\partial}{\partial x} u, \tag{1.13}$$

where $x \equiv \tau_{N-1}$ as above; the corresponding flow is called *stationary*. The Hamiltonian h_N gives rise to an *evolutionary* flow. In the case of the subalgebra $\mathfrak{sl}(2,\mathbb{R})$ we have

$$\frac{\partial \alpha_{2N}}{\partial t} = 2\beta_{2N+1}\sqrt{h_{-1}} \sinh u. \tag{1.14}$$

Combining (1.13) and (1.14) we obtain sinh-Gordon equation.

In the case of the subalgebra $\mathfrak{su}(2)$ we have to assign $\alpha_{2m} = i a_{2m}$, $a_{2m} \in \mathbb{R}$, and $\gamma_{2m-1} = -\beta_{2m-1}^*$, therefore $\beta_{2N+1} = \gamma_{2N+1} = ib$, $\gamma_{-1} = -ire^{iu}$, $\beta_{-1} = -ire^{-iu}$. Then we come to sin-Gordon equation

$$\frac{\partial^2 u}{\partial t \partial x} = 4rb \sin u.$$

2 Separation of variables for MKdV and sin(sinh)-Gordon equation

Definition 1. Suppose we have the variables (λ_k, w_k), $k = 1, \ldots, N+1$, such that

(i) they are quasi-canonically conjugate, that is

$$\{\lambda_k, w_l\}_1 = f(\lambda_k)\delta_{kl}, \qquad \{\lambda_k, \lambda_l\}_1 = \{w_k, w_l\}_1 = 0,$$

where $f(\lambda)$ is an arbitrary smooth function;

(ii) they reduce Liouville 1-form[2] to a sum of meromorphic differentials on the corresponding Riemannian surface.

We call (λ_k, w_k), $k = 1, \ldots, N+1$, *separation variables.*

Consider the orbit $\mathcal{O}_1^N$, $\dim \mathcal{O}_1^N = 2(N+1)$. One can parameterize the orbit using any subset of $2(N+1)$ variables from $\{\alpha_{2m}, \beta_{2m-1}, \gamma_{2m-1}\}$, $m = 0, 1, \ldots, N$. The most natural way to obtain the parameterization is to eliminate one of the subsets $\{\beta_{2m-1}\}$ or $\{\gamma_{2m-1}\}$. The reason for this is the nilpotency of the basis elements that correspond to the subsets.

Note that the correspondence between the elimination variables, which we chose to parameterize the orbit, and the nilpotent elements of the basis of the algebra is a crucial feature of our scheme and applies to all examples.

We chose to parameterize the orbit $\mathcal{O}_1^N$ by the variables $\{\gamma_{2m-1}, \alpha_{2m}\}$, $m = 0, 1, \ldots, N$, that is we eliminate the set $\{\beta_{2m-1}\}$. From the orbit equations we find

$$\beta_{2m-1} = \sum_{j=0}^{N+1} (\Gamma^+)^{-1}_{mj}(c_{N+j} - A_{N+j}),\ m = 0, \ldots N+1, \quad c_{2N+1} = \beta_{2N+1}\gamma_{2N+1}, \tag{2.1}$$

where

$$\Gamma^+ = \begin{bmatrix} \gamma_{2N+1} & \gamma_{2N-1} & \cdots & \gamma_1 & \gamma_{-1} \\ 0 & \gamma_{2N+1} & \cdots & \gamma_3 & \gamma_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{2N+1} & \gamma_{2N-1} \\ 0 & 0 & \cdots & 0 & \gamma_{2N+1} \end{bmatrix} \quad \text{and} \quad A_\nu = \sum_{\substack{m+n=\nu,\\ 0\leqslant m,n\leqslant N}} \alpha_{2m}\alpha_{2n}.$$

Now, using the parameterization (2.1), we find expressions for the Hamiltonians $h_{-1}, h_0, \ldots, h_{N-1}$

$$h_{n-1} = \sum_{m,j=0}^{N+1} \Gamma^-_{nm}(\Gamma^+)^{-1}_{mj}(c_{N+j} - A_{N+j}) + A_{n-1}, \quad n = 0, \ldots N, \tag{2.2}$$

where

$$\Gamma^- = \begin{bmatrix} \gamma_{-1} & 0 & \cdots & 0 & 0 \\ \gamma_1 & \gamma_{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_{2N-1} & \gamma_{2N-3} & \cdots & \gamma_{-1} & 0 \end{bmatrix}.$$

[2]We call Ω Liouville 1-form if $d\Omega = \omega$, where ω is a symplectic 2-form.

Note that the expressions (2.2) are linear in c_ν, $\nu = N, \ldots, 2N+1$.

Clearly, one can obtain an analogous parametrization of the orbit $\mathcal{O}_2^N$ by the set $\{\alpha_{2m}, \gamma_{2m-1}\}$, $m = 0, \ldots, N$.

To proceed we need to define the *characteristic polynomial*

$$Q(\varkappa, \lambda) = \det\big(\mu(\lambda) - \varkappa \cdot I\big),$$

where I denotes 2×2 identity matrix. By the substitution $\varkappa = w\lambda^{-1}$ the equation $Q(\varkappa, \lambda) = 0$ becomes transformed into the standard equation of a hyperelliptic curve of genus $N+1$

$$P(w, \lambda) = \lambda^2 Q(w\lambda^{-1}, \lambda) = w^2 - \lambda(h_{-1} + h_0\lambda + \cdots + h_{2N+1}\lambda^{2N+2}) = 0. \tag{2.3}$$

Recall that on the orbit $\mathcal{O}_1^N$ we have $h_\nu = c_\nu$, $\nu = N, \ldots, 2N$. Denote by (w_k, λ_k) a root of $P(w, \lambda)$ on the orbit, that is

$$w_k^2 = \lambda_k(h_{-1} + h_0\lambda_k + \cdots h_{N-1}\lambda_k^N + c_N\lambda_k^{N+1} + c_{N+1}\lambda_k^{N+2} + \cdots + c_{2N+1}\lambda_k^{2N+2}). \tag{2.4}$$

We proceed to show that the set $\{(w_k, \lambda_k)\}$, $k = 1, \ldots, N+1$, defines another parametrization of the orbit $\mathcal{O}_1^N$. We have to find the explicit relation between the sets $\{(w_1, \lambda_1), \ldots, (w_{N+1}, \lambda_{N+1})\}$ and $\{\alpha_0, \alpha_2, \ldots, \alpha_{2N}, \gamma_{-1}, \gamma_1, \ldots, \gamma_{2N-1}\}$.

Solving (2.4) for the Hamiltonians $h_{-1}, h_0, \ldots, h_{N-1}$ one gets

$$\begin{aligned}
&h_{-1} = \frac{1}{W}[W_1\left(\tfrac{w^2}{\lambda}\right) - c_N W_1(\lambda^{N+1}) - \cdots - c_{2N+1}W_1(\lambda^{2N+2})]\\
&h_0 = \frac{1}{W}[W_2\left(\tfrac{w^2}{\lambda}\right) - c_N W_2(\lambda^{N+1}) - \cdots - c_{2N+1}W_2(\lambda^{2N+2})]\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&h_{N-1} = \frac{1}{W}[W_{N+1}\left(\tfrac{w^2}{\lambda}\right) - c_N W_{N+1}(\lambda^{N+1}) - \cdots - c_{2N+1}W_{N+1}(\lambda^{2N+2})].
\end{aligned} \tag{2.5}$$

where $W = \prod(\lambda_i - \lambda_j)$ is Vandermonde determinant of $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$. By $W_i(f(\lambda, w))$ we denote the determinant of Vandermonde matrix with the i-th column replaced by $\big(f(\lambda_1, w_1), \ldots, f(\lambda_{N+1}, w_{N+1})\big)^t$.

On the orbit the formulas (2.2) and (2.5) define the same set of functions. We see that both (2.2) and (2.5) are linear in c_ν, $\nu = N, \ldots, 2N+1$. As $\{c_\nu\}$ is the set of *independent* parameters one can equate the corresponding terms. Namely, we get

$$\frac{\gamma_{2m-1}}{\gamma_{2N+1}} = -\frac{W_{m+1}(\lambda^{N+1})}{W}, \quad m = 0, \ldots N.$$

This implies that the set $\{\lambda_k\}$ is, in fact, the set of roots of the polynomial $\gamma(\lambda)$

$$\gamma(\lambda_k) = 0,$$

while the variables $\{w_k\}$ satisfy the equalities

$$w_k^2 = \lambda_k^2\left(\alpha^2(\lambda_k) - \gamma(\lambda_k)\beta(\lambda_k)\right) = \lambda_k^2\alpha^2(\lambda_k), \qquad k = 1, \ldots, N+1.$$

Theorem 1. *Suppose the orbit $\mathcal{O}_1^N$ has the coordinates $(\alpha_{2m}, \gamma_{2m-1})$, $m = 0, 1, \ldots, N$, as above. Then the new coordinates (λ_k, w_k), $k = 1, \ldots, N+1$, defined by the formulas*

$$\gamma(\lambda_k) = 0, \qquad w_k = \varepsilon\lambda_k\alpha(\lambda_k), \qquad \text{where} \quad \varepsilon^2 = 1, \tag{2.6}$$

have the following properties:

(1) *a pair (w_k, λ_k) is a root of the characteristic polynomial* (2.3)*.*

(2) *a pair (λ_k, w_k) is quasi-canonically conjugate with respect to the first Lie-Poisson bracket* (1.4)*:*

$$\{\lambda_k, \lambda_l\}_1 = 0, \qquad \{\lambda_k, w_l\}_1 = \varepsilon\lambda_k\delta_{kl}, \qquad \{w_k, w_l\}_1 = 0; \tag{2.7}$$

(3) *the corresponding Liouville 1-form is*

$$\Omega_{-1} = \sum_k \varepsilon\lambda_k^{-1} w_k \, d\lambda_k.$$

Proof. (1) The assertion is a direct consequence of (2.3) and (2.6).

(2) It is evident that

$$\{\lambda_k, \lambda_l\}_1 = 0.$$

Indeed, since λ_k, $k = 1, \ldots, N+1$, depend only on γ_{2m-1}, $m = 0, \ldots, N$, and γ_{2m-1} mutually commute, λ_k also mutually commute.

Let us calculate the bracket of λ_k and w_l

$$\{\lambda_k, w_l\}_1 = \sum_{m,n} \left(\frac{\partial\lambda_k}{\partial\gamma_{2m-1}} \frac{\partial w_l}{\partial\alpha_{2n}} - \frac{\partial\lambda_k}{\partial\alpha_{2n}} \frac{\partial w_l}{\partial\gamma_{2m-1}} \right) \{\gamma_{2m-1}, \alpha_{2n}\}_1.$$

From (2.6) we have

$$\frac{\partial\lambda_k}{\partial\alpha_{2n}} = 0, \qquad \frac{\partial\lambda_k}{\partial\gamma_{2m-1}} = -\frac{\lambda_k^m}{\gamma'(\lambda_k)}, \qquad \frac{\partial w_l}{\partial\alpha_{2n}} = \varepsilon\lambda_l^{n+1}. \tag{2.8}$$

Further $\{\gamma_{2m-1}, \alpha_{2n}\}_1 = -\gamma_{2(m+n)+1}$ when $m + n < N$ and $\{\gamma_{2m-1}, \alpha_{2n}\}_1 = 0$ when $m + n \geqslant N$. Thus, we obtain

$$\{\lambda_k, w_l\}_1 = \frac{\sum\limits_{m+n<N} \varepsilon\lambda_k^m\lambda_l^{n+1}\gamma_{2(m+n)+1}}{\gamma'(\lambda_k)} = \frac{\varepsilon\lambda_l}{\gamma'(\lambda_k)} \frac{\gamma(\lambda_l) - \gamma(\lambda_k)}{\lambda_l - \lambda_k}.$$

As $k \neq l$ it is evident that $\{\lambda_k, w_l\}_1 = 0$ while $\gamma(\lambda_l) = \gamma(\lambda_k) = 0$. As $k = l$ we get

$$\{\lambda_k, w_k\}_1 = \lim_{\lambda_l \to \lambda_k} \frac{\varepsilon\lambda_l}{\gamma'(\lambda_k)} \frac{\gamma(\lambda_l) - \gamma(\lambda_k)}{\lambda_l - \lambda_k} = \varepsilon\lambda_k.$$

Thus,

$$\{\lambda_k, w_l\}_1 = \varepsilon\lambda_k\delta_{kl}.$$

Let us calculate the bracket of w_k and w_l

$$\{w_k, w_l\}_1 = \sum_{m,n} \left(\frac{\partial w_k}{\partial \gamma_{2m-1}} \frac{\partial w_l}{\partial \alpha_{2n}} - \frac{\partial w_k}{\partial \alpha_{2n}} \frac{\partial w_l}{\partial \gamma_{2m-1}} \right) \{\gamma_{2m-1}, \alpha_{2n}\}_1.$$

From (2.6) follows that

$$\frac{\partial w_k}{\partial \gamma_{2m-1}} = \varepsilon \left[\alpha(\lambda_k) + \lambda_k \alpha'(\lambda_k)\right] \frac{\partial \lambda_k}{\partial \gamma_{2m-1}},$$

then, using (2.8), we obtain

$$\{w_k, w_l\}_1 = \left(\frac{\lambda_l \left[\alpha(\lambda_k) + \lambda_k \alpha'(\lambda_k)\right]}{\gamma'(\lambda_k)} - \frac{\lambda_k \left[\alpha(\lambda_l) + \lambda_l \alpha'(\lambda_l)\right]}{\gamma'(\lambda_l)} \right) \frac{\gamma(\lambda_l) - \gamma(\lambda_k)}{\lambda_l - \lambda_k},$$

hence

$$\{w_k, w_l\}_1 = 0.$$

(3) From (2.7) it follows that Liouville 1-form on the orbit $\mathcal{O}_1^N$ is

$$\Omega_{-1} = \sum_k \varepsilon \lambda_k^{-1} w_k \, d\lambda_k.$$

The reduction to Liouville torus is done by fixing the values of Hamiltonians h_{-1}, h_0, …, h_{N-1}. On the torus w_k is the algebraic function of λ_k due to(2.3). After the reduction the form Ω_{-1} becomes a sum of meromorphic differentials on the Riemann surface $P(w, \lambda) = 0$. ■

The next theorem is proven similarly.

Theorem 2. *Suppose the orbit $\mathcal{O}_2^N$ has the coordinates $(\alpha_{2m}, \gamma_{2m-1})$, $m = 0, 1, \ldots, N$. Then the new coordinates (λ_k, w_k), $k = 1, \ldots, N+1$, defined by the formulas*

$$\gamma(\lambda_k) = 0, \qquad w_k = \varepsilon \lambda_k \alpha(\lambda_k), \qquad \textit{where} \qquad \varepsilon^2 = 1,$$

have the following properties:

(1) *a pair (w_k, λ_k) is a root of the characteristic polynomial* (2.3)*;*

(2) *a pair (λ_k, w_k) is quasi-canonically conjugate with respect to the second Lie-Poisson bracket* (1.5)*:*

$$\{\lambda_k, \lambda_l\}_2 = 0, \qquad \{\lambda_k, w_l\}_2 = -\varepsilon \lambda_k^{N+2} \delta_{kl}, \qquad \{w_k, w_l\}_2 = 0; \tag{2.9}$$

(3) *the corresponding Liouville 1-form is*

$$\Omega_N = -\sum_k \varepsilon \lambda_k^{-(N+2)} w_k \, d\lambda_k.$$

Let us summarize our scheme of obtaining the separation variables. First, we parameterize the orbit by eliminating a subset of group coordinates corresponding to nilpotent basis elements. Next, we restrict the curve $P(w,\lambda)=0$ onto the orbit, where $P(w,\lambda)$ is the characteristic polynomial

$$P(w,\lambda) = \det\big(\mu(\lambda) - w \cdot I\big),$$

I is identity matrix. We use the set $\{\lambda_k, w_k\}$, $k = 1,\dots,N+1$, where $P(w_k,\lambda_k) = 0$, to define another parametrization of the orbit. Then, we equate expressions for Hamiltonians in the coordinates of two parameterizations of the orbit in order to obtain the link between the two sets of orbit coordinates. Finally, the set $\{\lambda_k, w_k\}$ is the set of separation variables.

Further, in Sections 4 and 5 we apply the scheme to nonlinear Schrödinger equation and Heisenberg magnetic chain.

3 Phase spaces for nonlinear Schrödinger equation and Heisenberg magnetic chain as orbits in $\mathfrak{sl}(2,\mathbb{C}) \otimes \mathcal{P}(z,z^{-1})$

Here we use the construction from Section 1 with *homogeneous grading*. That is,

$$X^l = z^l X, \qquad Y^l = z^l Y, \qquad H^l = z^l H \tag{3.1}$$

be the *basis* in $\widetilde{\mathfrak{g}} \simeq \mathfrak{sl}(2,\mathbb{C}) \bigotimes \mathcal{P}(z,z^{-1})$.

Note the well-known fact that the Lie algebra from Sections 1–2 can be realized as the subalgebra of $\mathfrak{sl}(2,\mathbb{C}) \otimes \mathcal{P}(z,z^{-1})$ invariant with respect to an automorphism of order 2, see [17], [19], [21].

By $\mathfrak{g}_l$, $l \in \mathbb{Z}$, denote an eigenspace of homogeneous degree l. It is evident that

$$\mathfrak{g}_l = \mathrm{span}_{\mathbb{C}}\{X^l,\, Y^l,\, H^l\}.$$

Decompose $\widetilde{\mathfrak{g}}$ into two subalgebras

$$\widetilde{\mathfrak{g}}_+ = \sum_{l\geqslant 0} \mathfrak{g}_l, \qquad \widetilde{\mathfrak{g}}_- = \sum_{l<0} \mathfrak{g}_l, \qquad \widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-.$$

Use the same ad-*invariant bilinear forms* (1.2) to define the spaces dual to $\widetilde{\mathfrak{g}}_+$ and $\widetilde{\mathfrak{g}}_-$.

Fix $N \geqslant 0$. Consider $M^{N+1} \subset \widetilde{\mathfrak{g}}$, where an element $\widehat{\mu}(z) \in M^{N+1}$ has the form

$$\widehat{\mu}(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & -\alpha(z) \end{pmatrix}$$

with

$$\alpha(\lambda) = \sum_{m=0}^{N+1} z^m \alpha_m, \qquad \beta(z) = \sum_{m=0}^{N+1} z^m \beta_m, \qquad \gamma(z) = \sum_{m=0}^{N+1} z^m \gamma_m.$$

As above, we call M^{N+1} the *N-gap sector of* $\widetilde{\mathfrak{g}}$, or shortly the *finite gap sector.*

For all $f_1, f_2 \in C(M^{N+1})$ define two Lie-Poisson brackets

$$\{f_1, f_2\}_1 = \sum_{m,n=0}^{N+1} \sum_{a,b=1}^{3} P_{ab}^{mn}(-1) \frac{\partial f_1}{\partial \mu_m^a} \frac{\partial f_2}{\partial \mu_n^b} \tag{3.2}$$

and

$$\{f_1, f_2\}_2 = \sum_{m,n=0}^{N+1} \sum_{a,b=1}^{3} P_{ab}^{mn}(N+1) \frac{\partial f_1}{\partial \mu_m^a} \frac{\partial f_2}{\partial \mu_n^b}, \tag{3.3}$$

where

$$P_{ab}^{mn}(-1) = \langle \widehat{\mu}(z), [Z_a^{-m-1}, Z_b^{-n-1}] \rangle_{-1},$$
$$P_{ab}^{mn}(N+1) = \langle \widehat{\mu}(z), [Z_a^{-m+N+1}, Z_b^{-n+N+1}] \rangle_{N+1},$$

$$Z_1^m = H^m, \qquad Z_2^m = Y^m, \qquad Z_3^m = X^m,$$
$$\mu_m^1 = \alpha_m, \qquad \mu_m^2 = \beta_m, \qquad \mu_m^3 = \gamma_m.$$

One can see that $M^{N+1} \subset (\widetilde{\mathfrak{g}}_-)^*$ with respect to $\langle\ ,\ \rangle_{-1}$ and, in the same time, $M^{N+1} \subset (\widetilde{\mathfrak{g}}_+)^*$ with respect to $\langle\ ,\ \rangle_{N+1}$.

Next, introduce the ad*-invariant function

$$I(z) = -\det \widetilde{\mu}(z) = h_0 + h_1 z + \cdots + h_{2N+2} z^{2N+2},$$

where

$$h_\nu = \sum_{m+n=\nu} (\alpha_m \alpha_n + \gamma_m \beta_n), \qquad \nu = 0, 1, \ldots, 2N+2. \tag{3.4}$$

One can easily prove that the functions α_{N+1}, β_{N+1}, γ_{N+1} annihilate the bracket (3.2).

In order to obtain nonlinear Schrödinger equation we have to assign

$$\beta_{N+1} = \gamma_{N+1} = 0, \qquad \alpha_{N+1} = const \neq 0. \tag{3.5}$$

After the restriction of the bracket (3.2) to the subspace $M_{con}^{N+1} \subset M^{N+1}$ with the constrains (3.5) we get $\dim M_{con}^{N+1} = 3(N+1)$. The bracket (3.2) is nondegenerate on M_{con}^{N+1}. We use the set $\gamma_m, \beta_m, \alpha_m$, $m = 0, 1, \ldots N$ as *coordinate functions* in M_{con}^{N+1}. We call the fixed coordinate α_{N+1} the *external parameter*.

On the other hand, the functions α_{N+1}, β_{N+1}, γ_{N+1} commute with all Hamiltonians h_ν, $\nu = N+2, N+3, \ldots, 2N+2$, with respect to the bracket (3.3) and give rise to nontrivial flows on M^{N+1}, therefore are Hamiltonians. That is, the bracket (3.3) is considered on M^{N+1}, $\dim M^{N+1} = 3(N+2)$, and the set γ_m, β_m, α_m, $m = 0, 1, \ldots N+1$ serve as *coordinate functions* in M^{N+1}.

The following assertions are immediately derived from the Kostant-Adler scheme [1].

Proposition 5. *All functions h_ν, $\nu = 0, 1, \ldots, 2N+2$ determined by* (3.4) *mutually commute with respect to the brackets* (3.2) *and* (3.3).

Proposition 6. *The functions h_ν, $\nu = N+1, \dots, 2N+1$ are functionally independent and annihilate the bracket* (3.2)*.*

Consider the algebraic variety $\mathcal{O}_1^N \subset M_{con}^{N+1}$ defined by the set of equation $h_\nu = c_\nu$, $\nu = N+1, \dots, 2N+1$, where c_ν are arbitrary fixed complex numbers. $\mathcal{O}_1^N$ is an orbit of coadjoint action of the subalgebra $\widetilde{\mathfrak{g}}_-$ and $\dim \mathcal{O}_1^N = 2(N+1)$*.*

Proposition 7. *The functions h_ν, $\nu = 0, \dots, N+1$ are functionally independent and annihilate the bracket* (3.3)*.*

Consider the algebraic variety $\mathcal{O}_2^{N+1} \subset M^{N+1}$ defined by the set of equation $h_\nu = c_\nu$, $\nu = 0, \dots, N+1$, where c_ν are arbitrary fixed complex numbers. $\mathcal{O}_2^{N+1}$ is an orbit of coadjoint action of the subalgebra $\widetilde{\mathfrak{g}}_+$ and $\dim \mathcal{O}_2^{N+1} = 2(N+2)$*.*

Further, the functions $h_0, h_1, \dots, h_N$ regarded as Hamiltonians with respect to the bracket (3.2) give rise to nontrivial flows on M_{con}^{N+1}

$$\frac{\partial \mu_m^a}{\partial \tau_\nu} = \{\mu_m^a, h_\nu\}_1, \qquad \nu = 0, 1, \dots N. \tag{3.6}$$

The functions $h_{N+2}, \dots, h_{2N+2}$ regarded as Hamiltonians with respect to the bracket (3.3) give rise to nontrivial flows on M^{N+1}

$$\frac{\partial \mu_m^a}{\partial \tau_\nu} = -\{\mu_m^a, h_{\nu+N+2}\}_2, \qquad \nu = 0, 1, \dots N. \tag{3.7}$$

Proposition 8. *The system* (3.6) *reduced to the orbit $\mathcal{O}_1^N$ is equivalent to finite gap nonlinear Schrödinger equation.*

The system (3.7) *reduced to the orbit $\mathcal{O}_2^{N+1}$ is equivalent to finite gap Heisenberg magnetic chain.*

Here we give the outline of the proof.

Consider the orbit $\mathcal{O}_1^N$. Rewrite (3.6) in matrix form. In particular, assigning $\tau_N = x$, $\tau_{N-1} = t$, we obtain

$$\frac{\partial \widehat{\mu}(z)}{\partial x} = [\widehat{\mu}(z), \nabla_1 h_N] = [\nabla_2 h_{2N+1}, \widehat{\mu}(z)], \tag{3.8a}$$

$$\frac{\partial \widehat{\mu}(z)}{\partial \tau} = [\widehat{\mu}(z), \nabla_1 h_{N-1}] = [\nabla_2 h_{2N}, \widehat{\mu}(z)], \tag{3.8b}$$

where

$$\nabla_2 h_{2N+1} = \begin{pmatrix} z\alpha_{2N+1} + \alpha_N & \beta_N \\ \gamma_N & -(z\alpha_{2N+1} + \alpha_N) \end{pmatrix},$$

$$\nabla_2 h_{2N} = \begin{pmatrix} z^2\alpha_{2N+1} + z\alpha_N + \alpha_{N-1} & z\beta_N + \beta_{N-1} \\ z\gamma_N + \gamma_{N-1} & -(z^2\alpha_{N+1} + z\alpha_N + \alpha_{N-1}) \end{pmatrix}.$$

We use the real subalgebra $\mathfrak{su}(2)$ of $\mathfrak{sl}(2,\mathbb{C})$, that is assign $\alpha_m = ia_m$, $\gamma_m = \mp\beta_m^*$, then the compatibility condition for (3.8) gives

$$2ia_{N+1}\frac{\partial \beta_N}{\partial t} = -\frac{\partial^2 \beta_N}{\partial x^2} - 2\beta_N|\beta_N|^2 - 2\beta_N h_{2N},$$

which coincides with nonlinear Schrödinger equation with respect to the function $\beta_N(x,t) = \psi(x,t)$ as $a_{N+1} = \frac{1}{2}$, $h_N = a_N = 0$,

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + 2\varepsilon\psi|\psi|^2, \qquad \varepsilon^2 = 1.$$

Consider the orbit $\mathcal{O}_2^{N+1}$. Note, that $\dim\mathcal{O}_2^{N+1} = 2(N+2)$ and the set h_{N+2}, h_{N+3}, $\ldots$, h_{2N+2} is insufficient to provide Liouville integrability. Since the functions α_{N+1}, β_{N+1}, and γ_{N+1} are in involution with the set $h_{N+2}, h_{N+3}, \ldots, h_{2N+2}$ with respect to the bracket (3.3) one can take any of them as an extra Hamiltonian. Here we chose α_{N+1}.

Rewrite (3.7) in matrix form. In particular, by assigning $\tau_{N+2} = x$ and $\tau_{N+3} = t$ we obtain

$$\frac{\partial\widehat{\mu}(z)}{\partial x} = [\nabla_2 h_{N+2}, \widehat{\mu}(z)] = [\widehat{\mu}(z), \nabla_1 h_0], \tag{3.9a}$$

$$\frac{\partial\widehat{\mu}(z)}{\partial t} = [\nabla_2 h_{N+3}, \widehat{\mu}(z)] = [\widehat{\mu}(z), \nabla_1 h_1], \tag{3.9b}$$

where

$$\nabla_1 h_0 = z^{-1}\begin{pmatrix}\alpha_0 & \beta_0\\ \gamma_0 & -\alpha_0\end{pmatrix},$$

$$\nabla_1 h_1 = z^{-1}\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & -\alpha_1\end{pmatrix} + z^{-2}\begin{pmatrix}\alpha_0 & \beta_0\\ \gamma_1 & -\alpha_1\end{pmatrix}.$$

Let us replace the coordinates α_m, β_m, γ_m, $m = 0, 1, \ldots, N+1$, according to the formulas

$$\alpha_m = i\mu_m^3, \qquad \beta_m = \mu_m^1 - i\mu_m^2, \qquad \gamma_m = -\mu_m^1 - i\mu_m^2.$$

Then

$$\{\mu_m^i, \mu_n^j\}_2 = \varepsilon_{ijk}\mu_{m+n-N-1}^k.$$

Introduce the vector notation $\boldsymbol{\mu}_m = (\mu_m^1, \mu_m^2, \mu_m^3)^t$, $m = 0, 1, \ldots, N+1$. Now the orbit $\mathcal{O}_2^{N+1}$ is determined by the equations

$$\begin{aligned}
&(\boldsymbol{\mu}_0, \boldsymbol{\mu}_0) = -c_0,\\
&2(\boldsymbol{\mu}_0, \boldsymbol{\mu}_1) = -c_1,\\
&\ldots\ldots\ldots\ldots\ldots\\
&\sum_{m+n=N+1}(\boldsymbol{\mu}_m, \boldsymbol{\mu}_n) = -c_{N+1},
\end{aligned}$$

where $(\cdot,\cdot)$ denotes the dot product. The equations (3.9) are written in the form

$$\frac{\partial\boldsymbol{\mu}_m}{\partial x} = 2[\boldsymbol{\mu}_0, \boldsymbol{\mu}_{m+1}], \tag{3.10a}$$

$$\frac{\partial\boldsymbol{\mu}_m}{\partial t} = 2[\boldsymbol{\mu}_1, \boldsymbol{\mu}_{m+1}] + 2[\boldsymbol{\mu}_0, \boldsymbol{\mu}_{m+2}], \tag{3.10b}$$

where $[\cdot,\cdot]$ denotes the cross product.

By reduction of (3.10) to the orbit $\mathcal{O}_2^{N+1}$ we obtain

$$\boldsymbol{\mu}_1 = \frac{1}{2c_0}\left[\boldsymbol{\mu}_0, \frac{\partial \boldsymbol{\mu}_0}{\partial x}\right] + \frac{c_1}{2c_0}\boldsymbol{\mu}_0.$$

Taking into account the compatibility condition

$$\frac{\partial \boldsymbol{\mu}_0}{\partial t} = \frac{\partial \boldsymbol{\mu}_1}{\partial x},$$

we get

$$\frac{\partial \boldsymbol{\mu}_0}{\partial t} = \frac{1}{2c_0}\left[\boldsymbol{\mu}_0, \frac{\partial^2 \boldsymbol{\mu}_0}{\partial x^2}\right] + \frac{c_1}{2c_0}\frac{\partial \boldsymbol{\mu}_0}{\partial x}. \tag{3.11}$$

When $c_1 = 0$, the equation (3.11) becomes the well-known *classic Heisenberg magnetic equation*, also called *isotropic Landau-Livshits equation*.

4 Separation of variables for nonlinear Schrödinger equation

Consider the orbit $\mathcal{O}_1^N$, $\dim \mathcal{O}_1^N = 2(N+1)$. The most natural way to parameterize the orbit, which we already noted in Section 2, is to eliminate the subset $\{\beta_m\}$ (or $\{\gamma_m\}$), $m = 0, 1, \ldots, N+1$. Then, roots of $\gamma(\lambda)$ (or $\beta(\lambda)$) give a half of separation variables. This way is applied in [18]. However, in the case of nonlinear Schrödinger equation the finite gap phase space is determined by the constrains (3.5). That is, the polynomials $\beta(\lambda)$ and $\gamma(\lambda)$ have the order N, therefore the set of roots is insufficient to parameterize the orbit $\mathcal{O}_1^N$.

In order to solve this problem we change coordinates. We use coordinates as in [24]. Let

$$T = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \qquad S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \tag{4.1}$$

be the basis in $\mathfrak{sl}(2,\mathbb{C})$. It is easily shown that $[T,S] = S$, $[T,R] = -R$, $[S,R] = 2T$. Then

$$T^m = z^m T, \qquad R^m = z^m R, \qquad S^m = z^m S.$$

is a *basis* in $\widetilde{\mathfrak{g}} \simeq \mathfrak{sl}(2,\mathbb{C}) \otimes \mathcal{P}(z, z^{-1})$. An element $\widehat{\mu}(z) \in M^{N+1}$ has the form

$$\widehat{\mu}(z) = \begin{pmatrix} \frac{1}{2}[r(z)+s(z)] & \frac{1}{2}[r(z)-s(z)-2t(z)] \\ \frac{1}{2}[s(z)-r(z)-2t(z)] & -\frac{1}{2}[r(z)+s(z)] \end{pmatrix},$$

where

$$t(z) = \sum_{m=0}^{N+1} z^m t_m, \qquad r(z) = \sum_{m=0}^{N+1} z^m r_m, \qquad s(z) = \sum_{m=0}^{N+1} z^m s_m.$$

Note that t_m, r_m, s_m, $m = 0, 1, \ldots, N+1$ are defined by the formulas:

$$\begin{aligned} t_m &= \langle \widehat{\mu}(z), T^{-m-1} \rangle_{-1} = \langle \widehat{\mu}(z), T^{-m+N+1} \rangle_{N+1}, \\ r_m &= \langle \widehat{\mu}(z), R^{-m-1} \rangle_{-1} = \langle \widehat{\mu}(z), R^{-m+N+1} \rangle_{N+1}, \\ s_m &= \langle \widehat{\mu}(z), S^{-m-1} \rangle_{-1} = \langle \widehat{\mu}(z), S^{-m+N+1} \rangle_{N+1}, \qquad m = 0, 1, \ldots, N+1. \end{aligned} \tag{4.2}$$

With the new coordinates M^{N+1}_{con} is defined by constrains

$$t_{N+1} = 0, \qquad r_{N+1} = s_{N+1} = \sqrt{c_{2N+2}} \neq 0.$$

One can see that the subsets $\{r_m\}$ and $\{s_m\}$, $m = 0, 1, \ldots, N+1$ have nilpotent corresponding basis elements. For the reason we chose one of these subsets, namely $\{r_m\}$ here, to parameterize the orbit $\mathcal{O}^N_1$. From the orbit equations we find

$$r_m = \sum_{j=0}^{N+1} (S^+)^{-1}_{mj} (c_{j+N+1} - B_{j+N+1}), \; m = 0, \ldots, N+1, \quad c_{2N+2} = r_{N+1} s_{N+1}, \tag{4.3}$$

where

$$S^+ = \begin{bmatrix} s_{N+1} & s_N & \ldots & s_1 & s_0 \\ 0 & s_{N+1} & \ldots & s_2 & s_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & s_{N+1} & s_N \\ 0 & 0 & \ldots & 0 & s_{N+1} \end{bmatrix} \qquad \text{and} \qquad B_\nu = \sum_{\substack{m+n=\nu, \\ 0 \leqslant m,n \leqslant N}} t_m t_n.$$

Using the parameterization (4.3), we find the expressions for the Hamiltonians $h_0, h_1, \ldots, h_N$

$$h_n = \sum_{mj=0}^{N+1} S^-_{nm} (S^+)^{-1}_{mj} (c_{j+N+1} - B_{j+N+1}) + B_n, \qquad n = 0, \ldots N, \tag{4.4}$$

where

$$S^- = \begin{bmatrix} s_0 & 0 & \ldots & 0 & 0 \\ s_1 & s_0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_N & s_{N-1} & \ldots & s_0 & 0 \end{bmatrix}.$$

To proceed we define the *characteristic polynomial*

$$P(w, z) = \det\big(\mu(z) - w \cdot I\big). \tag{4.5}$$

The equation $P(w, z) = 0$ has a form of the standard equation of a hyperelliptic curve of genus $N+1$

$$P(w, z) = w^2 - (h_0 + h_1 z + \cdots + h_{2N+2} z^{2N+2}) = 0. \tag{4.6}$$

On the orbit $\mathcal{O}^N_1$ we have $h_\nu = c_\nu$, $\nu = N+1, \ldots, 2N+1$. Denote by (w_k, z_k) a root of $P(w, z)$ on the orbit, that is

$$w_k^2 = h_0 + h_1 z_k + \cdots h_N z_k^N + c_{N+1} z_k^{N+1} + \cdots c_{2N+2} z_k^{2N+2}. \tag{4.7}$$

We proceed to show that the set $\{(w_k, z_k)\}$, $k = 0, 1, \ldots, N+1$ defines another parameterization of the orbit $\mathcal{O}_1^N$. We have to find the explicit relation between the sets $\{(w_1, z_1), \ldots, (w_{N+1}, z_{N+1})\}$ and $\{t_0, t_1, \ldots, t_N, s_0, s_1, \ldots, s_N\}$.

Solving (4.7) for Hamiltonians $h_0, h_1, \ldots, h_N$ one gets

$$\begin{aligned} h_0 &= \frac{1}{W}[W_1\left(w^2\right) - c_{N+1}W_1(z^{N+1}) - \cdots - c_{2N+2}W_1(z^{2N+2})] \\ h_1 &= \frac{1}{W}[W_2\left(w^2\right) - c_{N+1}W_2(z^{N+1}) - \cdots - c_{2N+2}W_2(z^{2N+2})] \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ h_N &= \frac{1}{W}[W_{N+1}\left(w^2\right) - c_{N+1}W_{N+1}(z^{N+1}) - \cdots - c_{2N+2}W_{N+1}(z^{2N+2})], \end{aligned} \tag{4.8}$$

where W and $W_i(f(z,w))$ denote the same as in (2.5).

On the orbit $\mathcal{O}_1^N$ the formulas (4.4) and (4.8) define the same set of functions. We see that both (4.4) and (4.8) are linear in c_ν, $\nu = N+1, \ldots, 2N+2$. As $\{c_\nu\}$ is the set of *independent* parameters one can equate the corresponding terms. Namely, we obtain

$$\frac{s_m}{s_{N+1}} = \frac{W_{m+1}(z^{N+1})}{W}, \qquad m = 0, 1, \ldots, N.$$

This implies that the set $\{z_k\}$ is the set of roots of the polynomial $s(z)$

$$s(z_k) = 0,$$

while the variables $\{w_k\}$ satisfy the equalities

$$w_k^2 = t^2(z_k) + s(z_k)r(z_k) = t^2(z_k), \qquad k = 1, \ldots, N+1.$$

Theorem 3. *Suppose the orbit $\mathcal{O}_1^N$ has the coordinates (t_m, s_m), $m = 0, 1, \ldots, N$, as above. Then the new coordinates (z_k, w_k), $k = 1, \ldots, N+1$, defined by the formulas*

$$s(z_k) = 0, \qquad w_k = \varepsilon t(z_k), \qquad \textit{where} \quad \varepsilon^2 = 1, \tag{4.9}$$

have the following properties:

(1) *a pair (w_k, z_k) is a root of the characteristic polynomial* (4.6)*.*

(2) *a pair (z_k, w_k) is canonically conjugate with respect to the Lie-Poisson bracket* (3.2)*:*

$$\{z_k, z_l\}_1 = 0, \qquad \{z_k, w_l\}_1 = \varepsilon\delta_{kl}, \qquad \{w_k, w_l\}_1 = 0; \tag{4.10}$$

(3) *the corresponding Liouville 1-form is*

$$\Omega_{-1} = \sum_k \varepsilon w_k \, dz_k.$$

Proof. (1) The assertion is a direct consequence of (4.6) and (4.9).

(2) It is evident that

$$\{z_k, z_l\}_1 = 0,$$

since z_k, $k=1,\dots,N+1$ depend only on s_m, $m=0,1,\dots,N$ and s_m mutually commute.

Let us calculate the brackets $\{z_k,w_l\}_1$ and $\{w_k,w_l\}_1$. From (4.9) we have

$$\frac{\partial z_k}{\partial t_n}=0,\qquad \frac{\partial z_k}{\partial s_m}=-\frac{z_k^m}{s'(z_k)},\qquad \frac{\partial w_l}{\partial t_n}=\varepsilon z_l^n,\qquad \frac{\partial w_l}{\partial s_m}=\varepsilon t'(z_l)\frac{\partial z_l}{\partial s_m}.$$

Further $\{s_m,t_n\}_1=-s_{m+n}$ when $m+n\leqslant N$ and $\{s_m,t_n\}_1=0$ when $m+n>N$. Thus, we obtain

$$\{z_k,w_l\}_1=\frac{\varepsilon}{s'(z_k)}\frac{s(z_k)-s(z_l)}{z_k-z_l},\qquad \{w_k,w_l\}_1=\left(\frac{t'(z_k)}{s'(z_k)}-\frac{t'(z_l)}{s'(z_l)}\right)\frac{s(z_k)-s(z_l)}{z_k-z_l}.$$

Thus,

$$\{z_k,w_l\}_1=\varepsilon\delta_{kl},\qquad \{w_k,w_l\}_1=0.$$

(3) From (4.10) it follows that Liouville 1-form on the orbit $\mathcal{O}_1^N$ is

$$\Omega_{-1}=\sum_k \varepsilon w_k\,dz_k.$$

The reduction to Liouville torus is done by fixing the values of Hamiltonians h_0, h_1, …, h_N. On the torus w_k is the algebraic function of z_k due to (4.6). After the reduction the form Ω_{-1} becomes a sum of meromorphic differentials on the Riemann surface $P(w,z)=0$. ∎

Given a set of pairs (z_k,w_k), $k=1,\dots,N+1$, one can find the set (t_m,s_m), $m=0,\dots,N$, such that the equations (4.9) are satisfied. Thus, we can define a homomorphism

$$\mathbb{C}^{2N+2}\to\mathcal{O}_1^N \tag{4.11}$$

that takes the set of pairs (z_k,w_k), $k=1,\dots,N+1$, to a point of the orbit $\mathcal{O}_1^N$. Since the orbit $\mathcal{O}_1^N$ is topologically trivial, this implies that the map (4.11) is global.

After the reduction (4.11) turns into the map that takes the $(N+1)$th symmetric power of Riemann surface to Loiuville torus:

$$\mathrm{Sym}\{\mathcal{R}\times\mathcal{R}\times\cdots\times\mathcal{R}\}\mapsto T^{N+1}.$$

5 Separation of variables for Heisenberg magnetic chain

Consider the orbit $\mathcal{O}_2^{N+2}$, $\dim\mathcal{O}_2^{N+2}=2(N+2)$. We chose to parameterize the orbit $\mathcal{O}_2^{N+2}$ by the variables $\{\gamma_m,\alpha_m\}$, $m=0,1,\dots,N+1$, that is we eliminate the set $\{\beta_m\}$, which corresponding basis elements are nilpotent.

From the orbit equation we find

$$\beta_m=\sum_{j=0}^{N+1}(\widetilde{\Gamma}^-)^{-1}_{mj}(c_j-\widetilde{A}_j),\qquad m=0,\dots N+1, \tag{5.1}$$

where

$$\widetilde{\Gamma}^- = \begin{bmatrix} \gamma_0 & 0 & \dots & 0 & 0 \\ \gamma_1 & \gamma_0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_N & \gamma_{N-1} & \dots & \gamma_0 & 0 \\ \gamma_{N+1} & \gamma_N & \dots & \gamma_1 & \gamma_0 \end{bmatrix} \qquad \text{and} \qquad \widetilde{A}_\nu = \sum_{\substack{m+n=\nu,\\ 0\leqslant m,n\leqslant N+1}} \alpha_m \alpha_n.$$

Now, using the parameterization (5.1), we find expressions for the Hamiltonians h_{N+2}, $h_{N+3}, \dots, h_{2N+2}$

$$h_{n+N+1} = \sum_{m,j=0}^{N+1} \widetilde{\Gamma}^+_{nm} (\widetilde{\Gamma}^-)^{-1}_{mj} (c_j - \widetilde{A}_j) + \widetilde{A}_{n+N+1}, \quad n = 1, \dots N+1, \tag{5.2}$$

where

$$\widetilde{\Gamma}^+ = \begin{bmatrix} 0 & \gamma_{N+1} & \dots & \gamma_2 & \gamma_1 \\ 0 & 0 & \dots & \gamma_3 & \gamma_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \gamma_{N+1} \end{bmatrix}.$$

Note that the expressions (5.2) are linear in c_ν, $\nu = N, \dots, 2N+1$.

To proceed we use the same characteristic polynomial (4.5)

On the orbit $\mathcal{O}_2^{N+1}$ we have h_ν, $\nu = 0, 1, \dots, N+1$. Denote by (w_k, z_k) a root of $P(w,z)$ on the orbit, that is

$$w_k^2 = c_0 + c_1 z_k + \cdots c_{N+1} z_k^{N+1} + h_{N+2} z_k^{N+2} + \cdots h_{2N+2} z_k^{2N+2}. \tag{5.3}$$

The set $\{(w_k, z_k)\}$, $k = 0, 1, \dots, N+1$ is insufficient to parameterize $\mathcal{O}_2^{N+1}$. Let us fix α_{N+1} and γ_{N+1} regarded as Hamiltonians and consider below the reduced orbit $\mathcal{O}_{2red}^{N+1}$. If we find an explicit relation between the sets $\{(w_1, z_1), \dots, (w_{N+1}, z_{N+1})\}$ and $\{\alpha_0, \alpha_1, \dots, \alpha_N, \gamma_0, \gamma_1, \dots, \gamma_N\}$ we show that the set $\{(w_k, z_k)\}$, $k = 0, 1, \dots, N+1$ defines another parameterization of the orbit $\mathcal{O}_{2red}^{N+1}$.

Theorem 4. *Suppose the orbit $\mathcal{O}_{2red}^{N+1}$ has the coordinates (α_m, γ_m), $m = 0, 1, \dots, N$, as above. Then the new coordinates (z_k, w_k), $k = 1, \dots, N+1$, defined by the formulas*

$$\gamma(z_k) = 0, \qquad w_k = \varepsilon \alpha(z_k), \qquad \textit{where} \quad \varepsilon^2 = 1, \tag{5.4}$$

have the following properties:

(1) *a pair (w_k, z_k) is a root of the characteristic polynomial (4.6).*

(2) *a pair (z_k, w_k) is quasi-canonically conjugate with respect to the Lie-Poisson bracket (3.3):*

$$\{z_k, z_l\}_2 = 0, \qquad \{z_k, w_l\}_2 = -\varepsilon z_k^{N+2} \delta_{kl}, \qquad \{w_k, w_l\}_2 = 0; \tag{5.5}$$

(3) *the corresponding Liouville 1-form is*

$$\Omega_{N+1} = -\sum_k \varepsilon z_k^{-(N+2)} w_k\, dz_k.$$

Proof. (1) The assertion is a direct consequence of (4.6) and (5.4).

(2) It is evident that

$$\{z_k, z_l\}_2 = 0,$$

since z_k, $k = 1, \ldots, N+1$ depend only on γ_m, $m = 0, 1, \ldots, N$ and γ_m mutually commute.

Let us calculate the brackets $\{z_k, w_l\}_2$ and $\{w_k, w_l\}_2$. From (5.4) we have

$$\frac{\partial z_k}{\partial \alpha_n} = 0, \qquad \frac{\partial z_k}{\partial \gamma_m} = -\frac{z_k^m}{\gamma'(z_k)}, \qquad \frac{\partial w_l}{\partial \alpha_n} = \varepsilon z_l^n, \qquad \frac{\partial w_l}{\partial \gamma_m} = \varepsilon \alpha'(z_l) \frac{\partial z_l}{\partial \gamma_m}.$$

Further $\{\gamma_m, \alpha_n\}_2 = -\gamma_{m+n-N-1}$ when $m + n \leqslant N + 1$ and $\{\gamma_m, \alpha_n\}_2 = 0$ when $m + n > N + 1$. Thus, we obtain

$$\{z_k, w_l\}_2 = \frac{1}{\gamma'(z_k)} \frac{z_k^{N+2}\gamma(z_l) - z_l^{N+2}\gamma(z_k)}{z_k - z_l},$$
$$\{w_k, w_l\}_2 = \left(\frac{1}{\gamma'(z_k)} - \frac{1}{\gamma'(z_l)}\right) \frac{z_k^{N+2}\gamma(z_l) - z_l^{N+2}\gamma(z_k)}{z_k - z_l},$$

whence

$$\{z_k, w_l\}_2 = -z_k^{N+2}\delta_{kl}, \qquad \{w_k, w_l\}_2 = 0.$$

(3) From (5.4) it follows that Liouville 1-form on the orbit $\mathcal{O}_{2red}^{N+1}$ is

$$\Omega_{N+1} = -\sum_k \varepsilon z_k^{-(N+2)} w_k\, dz_k.$$

The reduction to Liouville torus is done by fixing the values of Hamiltonians h_0, h_1, …, h_N. On the torus w_k is the algebraic function of z_k due to (4.6). After the reduction the form Ω_{-1} becomes a sum of meromorphic differentials on the Riemann surface $P(w, z) = 0$. ■

Remark 1. When $\gamma_{N+1} = 0$, $\alpha_{N+1} \neq 0$ one can replace $\alpha_m, \beta_m, \gamma_m$ by t_m, s_m, r_m, $m = 0, 1, \ldots, N$, according to (4.2). In this case the new coordinates (z_k, w_k), $k = 1, \ldots, N+1$, defined by the formulas

$$s(z_k) = 0, \qquad w_k = \varepsilon t(z_k), \qquad \text{where} \quad \varepsilon^2 = 1. \tag{5.6}$$

and have the same properties as in Theorem 4.

The results of Theorem 3 and 4 can be summarized as follows. Liouville tori for the nonlinear Schrödinger equation and Heisenberg magnetic chain have the same number of parameterizing variables z_k and each variable belongs to the hyperelliptic curve (4.6) of genus $g = N + 1$. In other words, the common Liouville torus is the Jacobi variety of the curve (4.6).

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