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Isometry groups of non standard metric products

Research Article

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Abstract: We consider isometry groups of a fairly general class of non standard products of metric spaces. We present sufficient conditions under which the isometry group of a non standard product of metric spaces splits as a permutation group into direct or wreath product of isometry groups of some metric spaces.

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1. Introduction

Let (X_i, d_i), $i = 1, \ldots, n$, be metric spaces. To define a metric on their cartesian product $X = \prod_{i=1}^n X_i$ one can use, for instance, one of the following equalities:

$$d\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big) = d_1(x_1,y_1)+\cdots+d_n(x_n,y_n), \quad \widetilde{d}\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big) = \sqrt{d_1^2(x_1,y_1)+\cdots+d_n^2(x_n,y_n)}\,.$$

There are different generalizations of these constructions. In the case $n = 2$ they include μ-products [1], f-products [5], warped products [3], etc. Following A. Bernig, T. Foertsch, V. Schroeder [2] we consider *non standard metric products* or Φ-*products* of metric spaces. Let us recall the precise definition. A function $\Phi\colon [0,\infty)^n \to [0,\infty)$ is called *admissible* if it satisfies the following conditions:

(A) $\Phi(p_1, p_2, \ldots, p_n) = 0$ iff $p_1 = p_2 = \ldots = p_n = 0$;

(B) $\Phi(q_1, \ldots, q_n) \le \Phi(r_1, \ldots, r_n) + \Phi(p_1, \ldots, p_n)$ for any $q_i, r_i, p_i \in [0,\infty)$ such that $q_i \le r_i + p_i$, $1 \le i \le n$.

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Then the function

$$d_\Phi\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big) = \Phi\big(d_1(x_1,y_1),\dots,d_n(x_n,y_n)\big)$$

is a metric on X [2].

Definition 1.1.

The metric space (X, d_Φ) is called the Φ-product of metric spaces $X_1,\dots,X_n$.

Wreath products of metric spaces [6] arise as a special case of Φ-products of metric spaces. The aim of this article is to describe the isometry group of the Φ-product of $X_1,\dots,X_n$, under certain conditions for Φ. We will show the relation

$$(\operatorname{Isom} X, X) \geq (\operatorname{Isom} X_1, X_1)\times\cdots\times(\operatorname{Isom} X_n, X_n). \tag{1}$$

For $n = 2$ we describe a family of functions Φ for which the relation (1) is an equality. More generally, we show that for certain Φ-products of two metric spaces, its isometry group splits as a permutation group into the direct product of isometry groups of naturally defined subspaces. We also present sufficient conditions on Φ under which the isometry group of the Φ-product of $X_1,\dots,X_n$ is isomorphic as a permutation group to the wreath product of the isometry groups of spaces $X_1,\dots,X_n$.

2. Preliminaries

We will need the following.

Proposition 2.1.

Let $\Phi\colon [0,\infty)^n \to [0,\infty)$ be an admissible function. Then

$$\Phi(q_1,\dots,q_{i-1},0,q_{i+1},\dots,q_n) \leq \Phi(q_1,\dots,q_{i-1},q_i,q_{i+1},\dots,q_n),$$

for all $q_i \in [0,\infty)$, $1 \leq i \leq n$.

Proof. If we replace q_j, r_j, p_j by $q_j, q_j, 0$, $1 \leq j \leq n$, $j \neq i$, and q_i, r_i, p_i by $0, q_i, 0$ respectively in condition (B), then we obtain

$$\Phi(q_1,\dots,q_{i-1},0,q_{i+1},\dots,q_n) \leq \Phi(q_1,\dots,q_{i-1},q_i,q_{i+1},\dots,q_n) + \Phi(0,\dots,0) = \Phi(q_1,\dots,q_{i-1},q_i,q_{i+1},\dots,q_n). \qquad \square$$

Let q be a positive real number. It is easy to see that the function

$$\widetilde{\Phi}(p_1,p_2,\dots,p_n) = \begin{cases} 0 & \text{if } p_1 = p_2 = \ldots = p_n = 0, \\ q & \text{otherwise,} \end{cases}$$

is admissible. Therefore, for arbitrary metric spaces $X_1,\dots,X_n$ one can consider their $\widetilde{\Phi}$-product. The isometry group of $(X, d_{\widetilde{\Phi}})$ is isomorphic as a permutation group to the symmetric group $S_{|X|}$. This is the largest possible isometry group of Φ-products of $X_1,\dots,X_n$.

In general, we obtain the following statement describing a candidate for the smallest possible isometry group of a Φ-product.

Proposition 2.2.

Let X be a Φ-product of metric spaces $X_1,\dots,X_n$, $n\ge 2$. Then the transformation group $(\operatorname{Isom} X, X)$ contains a subgroup isomorphic to the direct product of transformation groups

$$(\operatorname{Isom} X_1, X_1)\times\cdots\times(\operatorname{Isom} X_n, X_n).$$

Thus, the direct product $\prod_{i=1}^n(\operatorname{Isom} X_i, X_i)$ is contained in the isometry group of the Φ-product of $X_1,\dots,X_n$ for any admissible Φ. In the next section, we consider conditions on Φ so that the isometry group of the space (X,d_Φ) is the smallest possible.

3. Direct product of transformation groups and Φ-products of metric spaces

In this section, we consider Φ-products $(X_1\times X_2, d_\Phi)$ of two metric spaces $(X_1,d_1),(X_2,d_2)$. We will use the following notation. For each $a_1\in X_1$, $a_2\in X_2$ let

$$X^2_{a_1}=\{(a_1,x_2): x_2\in X_2\},\qquad X^1_{a_2}=\{(x_1,a_2): x_1\in X_1\}$$

be subspaces of $(X_1\times X_2,d_\Phi)$. The points of $X^2_{a_1}$ are in natural one-to-one correspondence with the points of X_2, while the points of spaces $X^1_{a_2}$ are in natural one-to-one correspondence with the points of X_1. With these identifications, let the group $\operatorname{Isom} X^2_{a_1}$ act on X_2 and $\operatorname{Isom} X^1_{a_2}$ act on X_1.

Lemma 3.1.

(i) *For arbitrary $a_1,b_1\in X_1$ the spaces $X^2_{a_1}$ and $X^2_{b_1}$ are isometric.*

(ii) *For arbitrary $a_2,b_2\in X_2$ the spaces $X^1_{a_2}$ and $X^1_{b_2}$ are isometric.*

Proof. (i) Let $g\colon X^2_{a_1}\to X^2_{b_1}$ be a one-to-one correspondence between $X^2_{a_1}$ and $X^2_{b_1}$ given by the equality $g((a_1,x_2))=(b_1,x_2)$ for all $x_2\in X_2$. For different y_2 and z_2 from X_2 the equalities

$$d_\Phi\big(g((a_1,y_2)),g((a_1,z_2))\big)=d_\Phi((b_1,y_2),(b_1,z_2))=\Phi\big(d_1(b_1,b_1),d_2(y_2,z_2)\big)=\Phi(0,d_2(y_2,z_2))$$

hold. As $d_\Phi((a_1,y_2),(a_1,z_2))=\Phi(0,d_2(y_2,z_2))$,

$$d_\Phi(g((a_1,y_2)),g((a_1,z_2)))=d_\Phi((a_1,y_2),(a_1,z_2)).$$

Therefore, g is an isometry between $X^2_{a_1}$ and $X^2_{b_1}$.

(ii) The proof of this statement is similar to the proof of (i). □

Note that groups $(\operatorname{Isom} X^2_{a_1},X_2)$ and $(\operatorname{Isom} X_2,X_2)$ (respectively $(\operatorname{Isom} X^1_{a_2},X_1)$ and $(\operatorname{Isom} X_1,X_1)$) are not necessarily isomorphic. Moreover, the spaces $X^2_{a_1}$ and X_2 (resp. $X^1_{a_2}$ and X_1) are not necessarily isometric, see Example 3.5 below.

Denote by C_i the set of values of the metric d_i, $i=1,2$. Assume that

$$\inf_{q_1\in C_1,\,q_1\neq 0}\Phi(q_1,0)>\sup_{q_2\in C_2}\Phi(0,q_2),\qquad \inf_{q_2\in C_2,\,q_2\neq 0}\Phi(0,q_2)>\frac{1}{2}\sup_{q_1\in C_1}\Phi(q_1,0). \tag{2}$$

Note that the estimates

$$\sup_{q_1\in C_1}\Phi(q_1,0)<\infty,\qquad \sup_{q_2\in C_2}\Phi(0,q_2)<\infty\qquad\text{and}\qquad \inf_{q_1\in C_1,\,q_1\neq 0}\Phi(q_1,0)>0,\qquad \inf_{q_2\in C_2,\,q_2\neq 0}\Phi(0,q_2)>0$$

follow from the inequalities (2).

Theorem 3.2.
Let $\Phi\colon [0,\infty)^2 \to [0,\infty)$ *be an admissible function, and let* (X_1,d_1), (X_2,d_2) *be metric spaces. Assume that* Φ *satisfies* (2) *and the following condition holds:*

$$\Phi(q_1,q_2) = \Phi(q_1,0) + \Phi(0,q_2). \tag{3}$$

Then

$$(\operatorname{Isom} X, X) \simeq (\operatorname{Isom} X^1_{a_2}, X_1) \times (\operatorname{Isom} X^2_{a_1}, X_2)$$

for any $(a_1,a_2) \in X_1 \times X_2$.

Proof. We shall show that each $(g_1,g_2) \in \operatorname{Isom} X^1_{a_2} \times \operatorname{Isom} X^2_{a_1}$ is an isometry of (X,d_Φ). The element (g_1,g_2) acts on $X_1 \times X_2$ coordinate-wise. Therefore, it suffices to show that this transformation preserves the metric d_Φ:

$$d_\Phi\big((x_1,x_2)^{(g_1,g_2)}, (y_1,y_2)^{(g_1,g_2)}\big) = d_\Phi\big((x_1^{g_1},x_2^{g_2}),(y_1^{g_1},y_2^{g_2})\big) = \Phi\big(d_1(x_1^{g_1},y_1^{g_1}), d_2(x_2^{g_2},y_2^{g_2})\big).$$

From (3) we have

$$\begin{aligned}\Phi\big(d_1(x_1^{g_1},y_1^{g_1}), d_2(x_2^{g_2},y_2^{g_2})\big) &= \Phi\big(d_1(x_1^{g_1},y_1^{g_1}),0\big) + \Phi\big(0, d_2(x_2^{g_2},y_2^{g_2})\big)\\ &= \Phi\big(d_1(x_1^{g_1},y_1^{g_1}), d_2(a_2,a_2)\big) + \Phi\big(d_1(a_1,a_1), d_2(x_2^{g_2},y_2^{g_2})\big)\\ &= d_\Phi\big((x_1^{g_1},a_2),(y_1^{g_1},a_2)\big) + d_\Phi\big((a_1,x_2^{g_2}),(a_1,y_2^{g_2})\big).\end{aligned}$$

As $g_1 \in \operatorname{Isom} X^1_{a_2}$, $g_2 \in \operatorname{Isom} X^2_{a_1}$, the following equalities hold:

$$d_\Phi\big((x_1^{g_1},a_2),(y_1^{g_1},a_2)\big) = d_\Phi((x_1,a_2),(y_1,a_2)), \qquad d_\Phi\big((a_1,x_2^{g_2}),(a_1,y_2^{g_2})\big) = d_\Phi((a_1,x_2),(a_1,y_2)).$$

Therefore,

$$\begin{aligned}d_\Phi\big((x_1,x_2)^{(g_1,g_2)},(y_1,y_2)^{(g_1,g_2)}\big) &= d_\Phi((x_1,a_2),(y_1,a_2)) + d_\Phi((a_1,x_2),(a_1,y_2))\\ &= \Phi(d_1(x_1,y_1),0) + \Phi(0,d_2(x_2,y_2)) = d_\Phi((x_1,x_2),(y_1,y_2)).\end{aligned}$$

Hence,

$$(\operatorname{Isom} X, X) \geq (\operatorname{Isom} X^1_{a_2}, X_1) \times (\operatorname{Isom} X^2_{a_1}, X_2).$$

Let $\varphi \in \operatorname{Isom} X$. We show that there exist $g_1 \in \operatorname{Isom} X^1_{a_2}$ and $g_2 \in \operatorname{Isom} X^2_{a_1}$, such that φ acts on $X_1 \times X_2$ as $(g_1,g_2) \in \operatorname{Isom} X^1_{a_2} \times \operatorname{Isom} X^2_{a_1}$ does. Let (y_1,y_2) be a point from $X_1 \times X_2$, u_1 a point from X_1. Then

$$d_\Phi((y_1,y_2),(u_1,y_2)) = \Phi(d_1(y_1,u_1), d_2(y_2,y_2)) = \Phi(d_1(y_1,u_1),0) \leq \sup_{q_1 \in C_1} \Phi(q_1,0).$$

Denote by (z_1,z_2) the value $\varphi(y_1,y_2)$ and by (w_1,w_2) the value $\varphi(u_1,y_2)$. Using (3) we get

$$d_\Phi(\varphi(y_1,y_2),\varphi(u_1,y_2)) = d_\Phi((z_1,z_2),(w_1,w_2)) = \Phi(d_1(z_1,w_1),d_2(z_2,w_2)) = \Phi(d_1(z_1,w_1),0) + \Phi(0,d_2(z_2,w_2)).$$

Assume that $z_2 \neq w_2$. Then using (2) we obtain

$$\begin{aligned}\Phi(d_1(z_1,w_1),0) + \Phi(0,d_2(z_2,w_2)) &\geq \inf_{q_1 \in C_1,\, q_1 \neq 0} \Phi(q_1,0) + \inf_{q_2 \in C_2,\, q_2 \neq 0} \Phi(0,q_2) > \frac{1}{2}\sup_{q_1 \in C_1} \Phi(q_1,0) + \frac{1}{2}\sup_{q_1 \in C_1} \Phi(q_1,0)\\ &= \sup_{q_1 \in C_1} \Phi(q_1,0).\end{aligned}$$

We have

$$d_\Phi(\varphi(y_1,y_2),\varphi(u_1,y_2)) > \sup_{q_1\in C_1} \Phi(q_1,0).$$

But φ is an isometry of the space X. Hence $z_2 = w_2$, i.e. $\varphi(u_1,y_2) = (w_1,z_2)$. Then the mapping φ acts as an isometry between subspaces of the form $X^1_{a_2}$, $a_2 \in X_2$.

Denote by g_1 the restriction of φ on $X^1_{y_2}$. Then $z_1 = g_1(y_1)$. We shall show that φ acts on each subspace of the form $X^1_{a_2}$, $a_2 \in X_2$, as g_1. Fix arbitrary $(b_1,b_2) \in X_1\times X_2$. Assume that $\varphi(b_1,b_2) = (h_1,h_2)$. We shall show that $h_1 = g_1(b_1)$. Indeed, in the opposite case from (2) it follows

$$d_\Phi((b_1,y_2),(b_1,b_2)) = \Phi(0,d_2(y_2,b_2)) < \inf_{q_1\in C_1,\,q_1\neq 0} \Phi(q_1,0)$$

and

$$\begin{aligned} d_\Phi(\varphi(b_1,y_2),\varphi(b_1,b_2)) &= d_\Phi((b_1^{g_1},z_2),(h_1,h_2)) = \Phi\big(d_1(b_1^{g_1},h_1),d_2(z_2,h_2)\big) = \Phi(d_1(b_1^{g_1},h_1),0)+\Phi(0,d_2(z_2,h_2)) \\ &\geq \inf_{q_1\in C_1,\,q_1\neq 0} \Phi(q_1,0). \end{aligned}$$

Therefore, the isometry φ acts on each subspace $X^1_{a_2}$, $a_2 \in X_2$, as g_1.

Let now u_2 be a point from X_2. Then

$$d_\Phi((y_1,y_2),(y_1,u_2)) = \Phi(0,d_2(y_2,u_2)) \leq \sup_{q_2\in C_2} \Phi(0,q_2) < \inf_{q_1\in C_1,\,q_1\neq 0} \Phi(q_1,0). \tag{4}$$

Suppose that $\varphi(y_1,u_2) = (v_1,v_2)$. Using (3) we obtain

$$d_\Phi(\varphi(y_1,y_2),\varphi(y_1,u_2)) = d_\Phi((z_1,z_2),(v_1,v_2)) = \Phi(d_1(z_1,v_1),d_2(z_2,v_2)) = \Phi(d_1(z_1,v_1),0)+\Phi(0,d_2(z_2,v_2)).$$

Assume that $z_1 \neq v_1$. We have

$$d_\Phi(\varphi(y_1,y_2),\varphi(y_1,u_2)) \geq \inf_{q_1\in C_1,\,q_1\neq 0} \Phi(q_1,0). \tag{5}$$

Combining (4) and (5), we obtain $z_1 = v_1$. Therefore, φ acts as an isometry between subspaces of the form $X^2_{a_1}$, $a_1 \in X_1$.

Denote by g_2 the restriction of φ on $X^2_{y_1}$. Then $z_2 = g_2(y_2)$. Assume now that $h_2 \neq g_2(b_2)$. Using (2) we get

$$d_\Phi((y_1,b_2),(b_1,b_2)) = \Phi(d_1(y_1,b_1),0) \leq \sup_{q_1\in C_1,\,q_1\neq 0} \Phi(q_1,0), \tag{6}$$

$$\begin{aligned} d_\Phi(\varphi(y_1,b_2),\varphi(b_1,b_2)) &= d_\Phi((z_1,b_2^{g_2}),(h_1,h_2)) = \Phi\big(d_1(z_1,h_1),d_2(b_2^{g_2},h_2)\big) = \Phi(d_1(z_1,h_1),0)+\Phi(0,d_2(b_2^{g_2},h_2)) \\ &\geq \inf_{q_1\in C_1,\,q_1\neq 0} \Phi(q_1,0) + \sup_{q_2\in C_2} \Phi(0,q_2) > \frac{1}{2}\sup_{q_1\in C_1} \Phi(q_1,0) + \frac{1}{2}\sup_{q_1\in C_1} \Phi(q_1,0) = \sup_{q_1\in C_1} \Phi(q_1,0). \end{aligned} \tag{7}$$

Combining (6) and (7), we obtain $h_2 = g_2(b_2)$. Therefore, the isometry φ acts on each subspace of the form $X^2_{a_1}$, $a_1 \in X_1$, as g_2. From Lemma 3.1 it follows, that $g_1 \in \operatorname{Isom} X^1_{a_2}$, $g_2 \in \operatorname{Isom} X^2_{a_1}$ for any $(a_1,a_2) \in X_1\times X_2$. Finally, for arbitrary $(x_1,x_2) \in X_1\times X_2$ we have

$$\varphi(x_1,x_2) = (x_1,x_2)^{(g_1,g_2)} = (x_1^{g_1},x_2^{g_2}). \qquad \square$$

Corollary 3.3.

Let $\Phi\colon [0,\infty)^2 \to [0,\infty)$ be an admissible function, and let (X_1,d_1), (X_2,d_2) be metric spaces. Assume that Φ satisfies (2) and the following condition holds:

$$\Phi(q_1,q_2) = \Phi(q_1,0)+\Phi(0,q_2).$$

If $\operatorname{Isom} X^1_{a_2} = \operatorname{Isom} X_1$, $\operatorname{Isom} X^2_{a_1} = \operatorname{Isom} X_2$ for some $(a_1,a_2) \in X_1\times X_2$, then

$$(\operatorname{Isom} X, X) \simeq (\operatorname{Isom} X_1, X_1)\times(\operatorname{Isom} X_2, X_2).$$

Example 3.4.
Let (X_1, d_1) and (X_2, d_2) be uniformly discrete metric spaces of finite diameters D_1, D_2 respectively. And let r_1, r_2 be positive numbers, such that for arbitrary points $x_i, y_i \in X_i$, $x_i \neq y_2$, the inequalities $d_i(x_i, y_i) \geq r_i$ hold, $i = 1, 2$. Denote $\Phi_1(q_1, q_2) = q_1 + q_2$. Then the function $\Phi_1(q_1, q_2)$ is admissible. If the inequalities

$$r_1 > D_2 \geq r_2 > \frac{1}{2} D_1 \qquad \text{or} \qquad r_2 > D_1 \geq r_1 > \frac{1}{2} D_2$$

hold, then the inequalities (2) hold as well. Therefore

$$\operatorname{Isom}(X_1 \times X_2, d_{\Phi_1}) \simeq \operatorname{Isom} X_1 \times \operatorname{Isom} X_2.$$

Example 3.5.
Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let

$$\Phi_2(q_1, q_2) = \begin{cases} 0 & \text{if} \quad q_1 = q_2 = 0, \\ 4 & \text{if} \quad q_1 \neq 0,\ q_2 = 0, \\ 3 & \text{if} \quad q_1 = 0,\ q_2 \neq 0, \\ 7 & \text{in other cases.} \end{cases}$$

Then

$$\operatorname{Isom}(X_1 \times X_2, d_{\Phi_2}) \simeq S_{|X_1|} \times S_{|X_2|}.$$

4. Wreath products of groups and Φ-products

In this section we consider Φ-products $(X_1 \times \cdots \times X_n, d_\Phi)$ of n metric spaces $(X_1, d_1), \ldots, (X_n, d_n)$, $n \geq 2$. Let us recall the definition of the wreath product of transformation groups. Let $(G_1, X_1), \ldots, (G_n, X_n)$ be a sequence of transformation groups. Following [7], the transformation group $\left(G, \prod_{i=1}^n X_i\right)$ is called the *wreath products of groups* $(G_1, X_1), \ldots, (G_n, X_n)$ if for all elements $u \in G$ the following conditions hold:

1) if $(x_1, \ldots, x_n)^u = (y_1, \ldots, y_n)$, then for all i, $1 \leq i \leq n$, the value of y_i depends only on x_1, …, x_i;
2) for fixed $x_1, \ldots, x_{i-1}$ the mapping $g_i(x_1, \ldots, x_{i-1})$ defined by the equality

$$g_i(x_1, \ldots, x_{i-1})(x_i) = y_i, \qquad x_i \in X_i,$$

   is a permutation on the set X_i which belongs to G_i. Denote the wreath products of groups $(G_1, X_1), \ldots, (G_n, X_n)$ by $\wr_{i=1}^n (G_i, X_i)$.

It follows from this definition that each element $u \in G$ can be represented as a so-called table $u = [g_1, g_2(x_1), \ldots, g_n(x_1, \ldots, x_{n-1})]$, where $g_1 \in G_1$, $g_i(x_1, \ldots, x_{i-1}) \in G_i^{X_1 \times \cdots \times X_{i-1}}$, $2 \leq i \leq n$. An element $u \in G$ acts on $(x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$ by the rule

$$(x_1, \ldots, x_n)^u = \left(x_1^{g_1}, x_2^{g_2(m_1)}, \ldots, x_n^{g_n(x_1, \ldots, x_{n-1})}\right).$$

We can consider the space δT of paths in a rooted level homogeneous tree T as some Φ-product of discrete metric spaces. Indeed, let T be a finite n-levels rooted tree with root v_0. Recall that a rooted tree T is called *level homogeneous* [4] if it is homogenous on every level. Such a tree is uniquely determined by its *level index*, i.e. by a finite sequence of cardinal numbers $[k_0; k_1; k_2; \ldots; k_n]$, where k_i is the number of edges joining a vertex of the i-th level with vertices of the $(i+1)$-st level. *A rooted path* is a finite sequence of vertices $(v_0, v_1, \ldots, v_n)$ such that $\{v_i, v_{i+1}\} \in E(T)$ for every i, $0 \leq i \leq n-1$. The metric space δT is defined to be the set of all rooted paths of T equipped with a natural ultrametric

$$\rho(\gamma_1, \gamma_2) = \frac{1}{m+1},$$

where m is the length of the maximal common part of rooted paths γ_1 and γ_2.

Let now $(X_1, d_1), \ldots, (X_n, d_n)$ be discrete spaces, i.e., for different points $u, v \in X_i$, $d_i(u, v) = 1$, $1 \le i \le n$. And let $|X_i| = k_i$, $1 \le i \le n$. We can introduce the function $\Phi_3\colon [0,\infty)^n \to [0,\infty)$ putting

$$\Phi_3(q_1,\ldots,q_n) = \begin{cases} q_1 & \text{if} \quad q_1 \neq 0, \\ \frac{1}{2} q_2 & \text{if} \quad q_1 = 0 \ \text{ and } \ q_2 \neq 0, \\ \ldots & \\ \frac{1}{n} q_n & \text{if} \quad q_1 = \ldots = q_{n-1} = 0 \ \text{ and } \ q_n \neq 0, \\ 0 & \text{if} \quad q_1 = \ldots = q_n = 0. \end{cases}$$

It is clear that Φ_3 is admissible. Therefore, one can consider the Φ_3-product of the spaces $X_1, \ldots, X_n$.

It is easy to see that the space δT of paths in the rooted level homogeneous tree T and the Φ_3-product of discrete metric spaces $X_1, \ldots, X_n$ are isometric. It is well known that the isometry group of the space δT is isomorphic as a permutation group to the wreath product of symmetric groups S_{k_i}, $i = 1, \ldots, n$. Therefore, the isometry group of the space $(X_1 \times \cdots \times X_n, d_{\Phi_3})$ is isomorphic as a permutation group to the wreath product of isometry groups of discrete spaces X_i, $i = 1, \ldots, n$. In this section we generalize this result by extending the class of metric spaces and introducing restrictions on the function Φ.

Let now (X_i, d_i), $i = 1, \ldots, n$, be arbitrary metric spaces. And let as before C_i be the set of values of the metric d_i, $1 \le i \le n$. Assume that there exist functions $f_i\colon [0,\infty) \to [0,\infty)$, $1 \le i \le n$, such that

$$\Phi(q_1,\ldots,q_n) = \begin{cases} f_1(q_1) & \text{if} \quad q_1 \neq 0, \\ f_2(q_2) & \text{if} \quad q_1 = 0 \ \text{ and } \ q_2 \neq 0, \\ \ldots & \\ f_n(q_n) & \text{if} \quad q_1 = \ldots = q_{n-1} = 0 \ \text{ and } \ q_n \neq 0, \\ 0 & \text{if} \quad q_1 = \ldots = q_n = 0, \end{cases} \tag{8}$$

for arbitrary $q_i \ge 0$, $1 \le i \le n$. For each $i, 1 \le i \le n$, denote by $\widehat{X}_i$ the space $(X_i, \widehat{d}_i)$, where for $u, v \in X_i$,

$$\widehat{d}_i(u,v) = \begin{cases} f_i(d_i(u,v)) & \text{if} \quad u \neq v, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that for all i, $1 \le i \le n-1$,

$$\inf_{q_i \in C_i,\, q_i \neq 0} f_i(q_i) > \sup_{q_{i+1} \in C_{i+1}} f_{i+1}(q_{i+1}). \tag{9}$$

Theorem 4.1.

Let $\Phi\colon [0,\infty)^n \to [0,\infty)$ be an admissible function such that conditions (8) and (9) hold. Then the isometry group of the Φ-product X of metric spaces $X_1, X_2, \ldots, X_n$ is isomorphic as a permutation group to the wreath product of isometry groups of spaces $\widehat{X}_i$, $i = 1, \ldots, n$,

$$(\operatorname{Isom}(X, d_\Phi), X) \simeq \wr_{i=1}^n (\operatorname{Isom}\widehat{X}_i, X_i).$$

Proof. Consider arbitrary

$$\varphi = [g_1, g_2(x_1), \ldots, g_n(x_1, \ldots, x_{n-1})] \in \wr_{i=1}^n \operatorname{Isom}\widehat{X}_i.$$

We shall show that φ is an isometry of (X, d_Φ). By the definition of the wreath product of permutation groups [7] the element φ acts on $\prod_{i=1}^n X_i$. Therefore, it is sufficient to show that φ preserves the metric d_Φ. Indeed,

$$\begin{aligned} d_\Phi\big(\varphi(x_1, x_2, \ldots, x_n), \varphi(y_1, y_2, \ldots, y_n)\big) &= d_\Phi\Big(\big(x_1^{g_1}, x_2^{g_2(x_1)}, \ldots, x_n^{g_n(x_1,\ldots,x_{n-1})}\big), \big(y_1^{g_1}, y_2^{g_2(y_1)}, \ldots, y_n^{g_n(y_1,\ldots,y_{n-1})}\big)\Big) \\ &= \Phi\Big(d_1(x_1^{g_1}, y_1^{g_1}), d_2\big(x_2^{g_2(x_1)}, y_2^{g_2(y_1)}\big), \ldots, d_n\big(x_n^{g_n(x_1,\ldots,x_{n-1})}, y_n^{g_n(y_1,\ldots,y_{n-1})}\big)\Big). \end{aligned}$$

Using (8), we have

$$d_\Phi\big(\varphi(x_1,x_2,\dots,x_n),\varphi(y_1,y_2,\dots,y_n)\big)=\begin{cases} f_1(d_1(x_1^{g_1},y_1^{g_1})) & \text{if } x_1^{g_1}\neq y_1^{g_1},\\ f_2\big(d_2\big(x_2^{g_2(x_1)},y_2^{g_2(y_1)}\big)\big) & \text{if } x_1^{g_1}=y_1^{g_1},\ x_2^{g_2(x_1)}\neq y_2^{g_2(y_1)},\\ \dots & \\ 0 & \text{if } x_1^{g_1}=y_1^{g_1},\ \dots,\ x_n^{g_n(x_1,\dots,x_{n-1})}=y_n^{g_n(y_1,\dots,y_{n-1})}.\end{cases} \tag{10}$$

But $g_1\in \operatorname{Isom}\widehat{X}_1$. Then $x_1^{g_1}=y_1^{g_1}$ iff $x_1=y_1$. Hence, $x_1^{g_1}=y_1^{g_1}$ iff $g_2(x_1)=g_2(y_1)$, and so on. With similar reasoning, using (10), we get

$$d_\Phi\big(\varphi(x_1,\dots,x_n),\varphi(y_1,\dots,y_n)\big)=\begin{cases} f_1(d_1(x_1^{g_1},y_1^{g_1})) & \text{if } x_1\neq y_1,\\ f_2\big(d_2\big(x_2^{g_2(x_1)},y_2^{g_2(x_1)}\big)\big) & \text{if } x_1=y_1 \text{ and } x_2\neq y_2,\\ \dots & \\ 0 & \text{if } x_1=y_1,\ \dots,\ x_n=y_n.\end{cases}$$

As g_1 is an isometry of $\widehat{X}_1$, $f_1(d_1(x_1^{g_1},y_1^{g_1}))=f_1(d_1(x_1,y_1))$. Since $g_i(x_1,\dots,x_{i-1})\in\operatorname{Isom}\widehat{X}_i$ the following equalities hold:

$$f_i\big(d_i\big(x_i^{g_i(x_1,\dots,x_{i-1})},y_i^{g_i(x_1,\dots,x_{i-1})}\big)\big)=f_i(d_i(x_i,y_i)),\qquad 2\le i\le n.$$

Therefore,

$$\begin{aligned} d_\Phi\big(\varphi(x_1,\dots,x_n),\varphi(y_1,\dots,y_n)\big) &= \begin{cases} f_1(d_1(x_1,y_1)) & \text{if } x_1\neq y_1,\\ f_2(d_2(x_2,y_2)) & \text{if } x_1=y_1 \text{ and } x_2\neq y_2,\\ \dots & \\ f_n(d_n(x_n,y_n)) & \text{if } x_1=y_1,\ \dots,\ x_{n-1}=y_{n-1},\ x_n\neq y_n,\\ 0 & \text{if } x_1=y_1,\ \dots,\ x_n=y_n,\end{cases}\\ &= d_\Phi\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big).\end{aligned}$$

Let now φ be an isometry of (X,d_Φ). Fix a point $(x_1,\dots,x_n)$ from X and assume that $\varphi(x_1,\dots,x_n)=(z_1,\dots,z_n)$. Let $\varphi(y_1,y_2,\dots,y_n)=(w_1,w_2,\dots,w_n)$ for some $y_i\in X_i$, $1\le i\le n$, $(y_1,\dots,y_n)\neq(x_1,\dots,x_n)$. We have

$$d_\Phi\big((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)\big)=\Phi\big(d_1(x_1,y_1),d_2(x_2,y_2),\dots,d_n(x_n,y_n)\big)=f_j(d_j(x_j,y_j)), \tag{11}$$

where j is the smallest number such that $y_1=x_1,\ \dots,\ y_{j-1}=x_{j-1},\ y_j\neq x_j$. Using (8), we obtain

$$\begin{aligned} d_\Phi\big(\varphi(x_1,x_2,\dots,x_n),\varphi(x_1,y_2,\dots,y_n)\big) &= d_\Phi\big((z_1,z_2,\dots,z_n),(w_1,w_2,\dots,w_n)\big)\\ &=\Phi\big(d_1(z_1,w_1),d_2(z_2,w_2),\dots,d_n(z_n,w_n)\big)=f_k(d_k(x_k,y_k)),\end{aligned} \tag{12}$$

where k is the smallest number such that $z_1=w_1,\ \dots,\ z_{k-1}=w_{k-1},\ z_k\neq w_k$. Combining (11), (12) and (9), we get $j=k$. This means that for all i, $1\le i\le n$, the value y_i depends only on $x_1,\dots,x_i$ and for fixed $x_1,\dots,x_{i-1}$ the mapping φ acts on X_i as some isometry $g_i(x_1,\dots,x_{i-1})$. Therefore, there exists a table

$$\big[g_1,g_2(x_1),\dots,g_n(x_1,\dots,x_{n-1})\big]$$

such that $g_1\in\operatorname{Isom}\widehat{X}_1$, $g_i(x_1,\dots,x_{i-1})\in(\operatorname{Isom}\widehat{X}_i)^{X_1\times\dots\times X_{i-1}}$. And the table $[g_1,g_2(x_1),\dots,g_n(x_1,\dots,x_{n-1})]$ acts on X as φ does. This completes the proof. □

Corollary 4.2.

Let $\Phi\colon [0,\infty)^n \to [0,\infty)$ be an admissible function such that conditions (8) and (9) hold. If

$$\operatorname{Isom}(X_i, f_i(d_i)) = \operatorname{Isom}(X_i, d_i)$$

for all i, $1 \le i \le n$, then

$$(\operatorname{Isom} X, X) \simeq \wr_{i=1}^{n} (\operatorname{Isom} X_i, X_i).$$

Example 4.3.

Let (X_1, d_1) and (X_2, d_2) be metric spaces of finite diameters D_1, D_2. Assume that there exists a positive number r such that for arbitrary points $x_1, x_2 \in X_1$, $x_1 \neq x_2$, the inequality $d_1(x_1, x_2) \ge r$ holds. Let $\Phi_3(q_1, q_2) = \max(q_1, q_2)$. If $r > D_2$, then $\operatorname{Isom}(X_1 \times X_2, d_{\Phi_3}) \simeq (\operatorname{Isom} X_1) \wr (\operatorname{Isom} X_2)$.

Example 4.4.

Let $X_i = \mathbb{Z}$ and d_i be the Euclidean distance, $1 \le i \le n$. It is easy to see that the function

$$\Phi_5(q_1, \dots, q_n) = \begin{cases} n + 1 - \dfrac{1}{q_1 + 1} & \text{if} \quad q_1 \neq 0, \\ n - \dfrac{1}{q_2 + 1} & \text{if} \quad q_1 = 0 \ \text{ and } \ q_2 \neq 0, \\ \dots & \\ 2 - \dfrac{1}{q_n + 1} & \text{if} \quad q_1 = \ldots = q_{n-1} = 0 \ \text{ and } \ q_n \neq 0, \\ 0 & \text{if} \quad q_1 = \ldots = q_n = 0, \end{cases}$$

is admissible and satisfies (8) and (9). Therefore, one can consider the Φ_5-product $(\mathbb{Z} \times \dots \times \mathbb{Z}, d_{\Phi_5})$ of X_i, $1 \le i \le n$. The set of values of the metric d_{Φ_5} is bounded, while each d_i takes arbitrary large values.

It follows from Theorem 4.1 that the isometry group of $(\mathbb{Z} \times \dots \times \mathbb{Z}, d_{\Phi_5})$ is isomorphic as a permutation group to the wreath product of isometry groups of $(X_i, \widehat{d}_i)$, $i = 1, \dots, n$, where for arbitrary $u, v \in X_i$,

$$\widehat{d}_i(u, v) = \begin{cases} n + 2 - i - \dfrac{1}{d_i(u, v) + 1} & \text{if} \quad u \neq v, \\ 0 & \text{in other cases.} \end{cases}$$

Recall, metric spaces (Y, d_1) and (Y, d_2) are called *isomorphic* [8] if there exists a *scale*, that is a strictly increasing continuous function $s\colon \mathbb{R}^+ \to \mathbb{R}^+$, $s(0) = 0$, such that $d_1 = s(d_2)$. It is easy to observe that if metric spaces (Y, d_1) and (Y, d_2) are isomorphic then their isometry groups $\operatorname{Isom}(Y, d_1)$ and $\operatorname{Isom}(Y, d_2)$ are equal.

For each i, $1 \le i \le n$, the spaces $(\mathbb{Z}, \widehat{d}_i)$ and $(\mathbb{Z}, d_i)$ are isomorphic. Indeed, if we consider a scale $s : \mathbb{R}^+ \to \mathbb{R}^+$ given by the equality

$$s(t) = \begin{cases} n + 2 - i - \dfrac{1}{t + 1} & \text{if} \quad t \ge 1, \\ \left(n + \dfrac{3}{2} - i\right) t & \text{if} \quad 0 \le t \le 1, \end{cases}$$

then $\widehat{d}_i = s(d_i)$ on $\mathbb{Z}$. Therefore, the isometry group of the space $(\mathbb{Z}, \widehat{d}_i)$ is isomorphic to the infinite dihedral group D_∞. Hence, the isometry group of $(\mathbb{Z} \times \dots \times \mathbb{Z}, d_{\Phi_5})$ is isomorphic as a permutation group to the wreath product of n infinite dihedral groups D_∞:

$$\big(\operatorname{Isom}(\mathbb{Z} \times \dots \times \mathbb{Z}, d_{\Phi_5}), \mathbb{Z} \times \dots \times \mathbb{Z}\big) \simeq \wr_{i=1}^{n} (D_\infty, \mathbb{Z}).$$

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