ECCENTRIC DIGRAPHS OF UNIQUE POINT ECCENTRIC GRAPHS

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Let G be a simple finite connected undirected graph and $u \in V(G)$ be its vertex. A vertex $v \in V(G)$ is called an *eccentric vertex for* u if $d_G(u,v) = e_G(u)$, where $e_G(u) = \max\{d_G(u,x) : x \in V(G)\}$ denotes the *eccentricity* of a vertex u. One way to capture the local metric structure of a connected graph G is to consider the so-called *eccentric digraph* $Ecc(G)$, which is a digraph with $V(Ecc(G)) = V(G)$ and there is an arc $u \to v$ if v is an eccentric vertex for u.

A graph G is called *unique eccentric point* graph [2] if every its vertex $u \in V(G)$ has a unique eccentric vertex. In other words, G is a uep-graph if it has a *functional* (the out-degree of every vertex equals one) eccentric digraph $Ecc(G)$. Self-centered uep-graphs were characterized in [2] and their structure was extensively studied in [1]. Note that a uep-graph is self-centered if and only if its eccentric digraph is a disjoint union of 2-cycles.

A pair of vertices $u, v \in V(G)$ is called *diametral* if $d_G(u,v) = diam(G)$. It is clear that any diametral pair of vertices in G forms a cycle of length two in $Ecc(G)$. It turns out that eccentric digraphs of uep-graphs can not have cycles of other lengths.

Proposition 1. *The eccentric digraph of a nontrivial uep-graph has cycles only of length two.*

Question: does any cycle of length two in a uep-graph corresponds to some diametral pair (this is not true for general graphs)?

A *block* of a graph is its maximal biconnected subgraph. A graph is called a *block graph* if it is isomorphic to the intersection graph of the collection of all blocks in some graph. For example, each tree is a block graph.

For a pair of natural numbers $m, k \in \mathbb{Z}_+$ define the digraph $D_{m,k}$ as follows: $V(D_{m,k}) = \{u, v, x_1, \ldots, x_m, y_1, \ldots, y_k\}$, $E(D_{m,k}) = \{(u,v),(v,u)\} \cup \{(x_i, u), (y_j, v) : 1 \le i \le m, 1 \le j \le k\}$. For example, $D_{0,0}$ is just a directed 2-cycle.

In general, the problem of characterizing eccentric digraphs of uep-graphs up to isomorphism seems to be very hard. However, in the class of block graphs we have the following result.

Theorem 1. *Let G be a uep-graph which is also a block graph. Then $Ecc(G)$ is isomorphic to $D_{m,k}$ for $m = k = 0$ or $m = k = 1$ or $m, k \geq 2$. Conversely, for every such $D_{m,k}$ there exists a uep-graph which is a block graph (even a tree) with $Ecc(G)$ being isomorphic to $D_{m,k}$.*

Trees which are uep-graphs were characterized in [2]. It turns out, that we can extend the same characterization to connected block graphs.

Theorem 2. *A connected block graph is a uep-graph if and only if it has exactly two central and two peripheral vertices.*

References

[1] Göbel F., Veldman V.L. *Even graphs* // J. Graph Theory — 1986. — V. 10. — P. 225–239.

[2] Parthasarathy K.R., Nandakumar R. *Unique eccentric point graphs* // Discrete Math. — 1983. — V. 46. — P. 69–74.

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