

Derivative Thomae formula for singular half-periods

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A complete generalization of Thomae theorems [1] in hyperelliptic case is obtained, that is values at zero of the lowest non-vanishing derivatives of theta functions with singular characteristics of arbitrary multiplicity are expressed in terms of branch points $\{e_i\}_{i=1}^{2g+2}$ and period matrix ω .

Theorem 1. Let $\mathcal{I}_m \cup \mathcal{J}_m$ with $\mathcal{I}_m = \{i_1, \dots, i_{g+1-2m}\}$ and $\mathcal{J}_1 = \{j_1, \dots, j_{g+1+2m}\}$ be a partition of the set of indices of all $2g + 2$ branch points of hyperelliptic curve, and $[\mathcal{I}_m]$ denotes a singular characteristic of multiplicity m corresponding to $\mathcal{A}(\mathcal{I}_m) - K$. Let $\Delta(\mathcal{I}_m)$ and $\Delta(\mathcal{J}_m)$ be Vandermonde determinants built from $\{e_i \mid i \in \mathcal{I}_m\}$ and $\{e_j \mid j \in \mathcal{J}_m\}$. Then with a set $\mathcal{K} \subset \mathcal{J}_m$ of cardinality $\mathfrak{k} = 2m - 1$ or $2m$ the following relation holds

$$\frac{\partial}{\partial v_{n_1}} \cdots \frac{\partial}{\partial v_{n_m}} \theta[\mathcal{I}_m](v) \Big|_{v=0} = \epsilon \left(\frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(\mathcal{I}_m)^{1/4} \Delta(\mathcal{J}_m)^{1/4} \times \sum_{\substack{p_1, \dots, p_m \in \mathcal{K} \\ \text{all different}}} \prod_{i=1}^m \frac{\sum_{j=1}^g (-1)^{j-1} s_{j-1}(\mathcal{I}_m \cup \mathcal{K}^{(p_i)}) \omega_{jn_i}}{\prod_{k \in \mathcal{K} \setminus \{p_1, \dots, p_m\}} (e_{p_i} - e_k)}. \quad (1)$$

where $s_j(\mathcal{I})$ denotes an elementary symmetric polynomial of degree j in branch points with indices from \mathcal{I} , and $\mathcal{K}^{(p_i)} = \mathcal{K} \setminus \{p_i\}$, and ϵ satisfies $\epsilon^8 = 1$.

Theta function with characteristic $[\varepsilon]$ is defined by the formula

$$\theta[\varepsilon](v; \tau) = \exp(i\pi(\varepsilon'^t/2)\tau(\varepsilon'/2) + 2i\pi(v + \varepsilon/2)^t \varepsilon'/2) \theta(v + \varepsilon/2 + \tau \varepsilon'/2; \tau). \quad (2)$$

All half-integer characteristics are represented by partitions of $2g + 2$ indices of the form $\mathcal{I}_m \cup \mathcal{J}_m$ with $\mathcal{I}_m = \{i_1, \dots, i_{g+1-2m}\}$ and $\mathcal{J}_m = \{j_1, \dots, j_{g+1+2m}\}$, where m runs from 0 to $[(g+1)/2]$, and $[\cdot]$ means the integer part. Number m is called *multiplicity*. Infinity with index $2g + 2$ is usually omitted in the sets. The characteristic $[\mathcal{I}_m]$ corresponds to partition $\mathcal{I}_m \cup \mathcal{J}_m$ in the following way

$$\sum_{i \in \mathcal{I}_m} \mathcal{A}(e_i) - K = \varepsilon(\mathcal{I}_m)/2 + \tau \varepsilon'(\mathcal{I}_m)/2,$$

where K denotes the vector of Riemann constants. According to Riemann theorem $\theta(v + \mathcal{A}(\mathcal{I}_m) - K)$ vanishes to order m at $v = 0$, Characteristics of multiplicity 0 and 1 are called non-singular even and odd, respectively. All other characteristics are called *singular*.

Some further results are derived from Theorem 1.

Corollary 2. Let $\mathcal{I}_2 \cup \mathcal{J}_2$ with $\mathcal{I}_2 = \{i_1, \dots, i_{g-\mathfrak{k}}\}$ and $\mathcal{J}_2 = \{j_1, \dots, j_{g+1+\mathfrak{k}}\}$, where $\mathfrak{k} = 3$ or 4 , be a partition of the set of $2g + 1$ indices of finite branch points, such that singular characteristic $[\mathcal{I}_2]$, corresponding to $\mathcal{A}(\mathcal{I}_2) - K$, has multiplicity 2. Let $\Delta(\mathcal{I}_2)$ and $\Delta(\mathcal{J}_2)$ be Vandermonde determinants built from $\{e_i \mid i \in \mathcal{I}_2\}$ and $\{e_j \mid j \in \mathcal{J}_2\}$. Then

$$\frac{\partial}{\partial v_{n_1}} \frac{\partial}{\partial v_{n_2}} \theta[\mathcal{I}_2](v) \Big|_{v=0} = \epsilon \left(\frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(\mathcal{I}_2)^{1/4} \Delta(\mathcal{J}_2)^{1/4} \sum_{i,j=1}^g (\hat{S}[\{\mathcal{I}_2\}])_{i,j} \omega_{in_1} \omega_{jn_2} \quad (3)$$

with $g \times g$ matrix

$$(\hat{S}[\mathcal{I}_2])_{i,j} = (-1)^{i+j} \left(2s_{i-\mathfrak{k}+1}(\mathcal{I}_2)s_{j-\mathfrak{k}+1}(\mathcal{I}_2) - s_{i-\mathfrak{k}+2}(\mathcal{I}_2)s_{j-\mathfrak{k}}(\mathcal{I}_2) - s_{i-\mathfrak{k}}(\mathcal{I}_2)s_{j-\mathfrak{k}+2}(\mathcal{I}_2) \right), \quad (4)$$

where ϵ satisfies $\epsilon^8 = 1$, and elementary symmetric functions $s_l(\mathcal{I}_2)$ are replaced by zero when $l < 0$.

Theorem 3. For hyperelliptic curves of genera $g \geq 3$, when characteristics of multiplicity 2 exist, rank of every matrix of second derivative theta constants equals three, that is

$$\text{rank}(\partial_v^2 \theta[\mathcal{I}_2]) = 3.$$

Therefore, $\det(\partial_v^2 \theta[\mathcal{I}_2]) = 0$ in genera $g > 3$.

Conjecture 4. With a characteristic $[\mathcal{I}_\mathfrak{m}]$ of multiplicity \mathfrak{m} corresponding to a partition $\mathcal{I}_\mathfrak{m} \cup \mathcal{J}_\mathfrak{m}$ with $\mathcal{I}_\mathfrak{m} = \{i_1, \dots, i_{g-\mathfrak{k}}\}$ and $\mathcal{J}_\mathfrak{m} = \{j_1, \dots, j_{g+1+\mathfrak{k}}\}$, where $\mathfrak{k} = 2\mathfrak{m} - 1$ or $2\mathfrak{m}$, of indices of $2g + 1$ finite branch points the following holds

$$\partial_u^\mathfrak{m} \theta[\mathcal{I}_\mathfrak{m}](\omega^{-1}u)|_{u=0} = \epsilon \left(\frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(\mathcal{I}_\mathfrak{m})^{1/4} \Delta(\mathcal{J}_\mathfrak{m})^{1/4} \hat{S}[\mathcal{I}_\mathfrak{m}], \quad (5)$$

where u are non-normalized variables, and order \mathfrak{m} tensor $\hat{S}[\mathcal{I}_\mathfrak{m}]$ belongs to the \mathfrak{m} -th tensor power $\mathcal{S}_{2\mathfrak{m}-1}^{\otimes \mathfrak{m}}$ of the vector space $\mathcal{S}_{2\mathfrak{m}-1}$ spanned by $2\mathfrak{m} - 1$ vectors $\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{2\mathfrak{m}-2}$ such that $\mathbf{s}_d = (s_{j-\mathfrak{k}+d}(\mathcal{I}_\mathfrak{m}))_{j=1}^g$. The basis spanning $\hat{S}(\mathcal{I}_\mathfrak{m})$ could be found from partitions of $\mathfrak{m}(\mathfrak{m} - 1)$ of length \mathfrak{m} formed from numbers $\{0, 1, \dots, 2\mathfrak{m} - 2\}$.

As a byproduct a generalization of Bolza formulas [2] are deduced.

Proposition 5. Let $\mathcal{I}_\mathfrak{m}$ be a set of $g - \mathfrak{k}$ indices, and $\mathfrak{m} = [(\mathfrak{k} + 1)/2]$. Elementary symmetric polynomials in branch points $\{e_i \mid i \in \mathcal{I}_\mathfrak{m}\}$ of genus g hyperelliptic curve with period matrix ω are defined by

$$s_j(\mathcal{I}_\mathfrak{m}) = (-1)^j \frac{\partial_{u_{2\mathfrak{k}-4(\mathfrak{m}-1)-1}, \dots, u_{2\mathfrak{k}-5}, u_{2\mathfrak{k}+2j-1}}^\mathfrak{m} \theta[\mathcal{I}_\mathfrak{m}](\omega^{-1}u)}{\partial_{u_{2\mathfrak{k}-4(\mathfrak{m}-1)-1}, \dots, u_{2\mathfrak{k}-5}, u_{2\mathfrak{k}-1}}^\mathfrak{m} \theta[\mathcal{I}_\mathfrak{m}](\omega^{-1}u)} \Big|_{u=0}.$$

In particular,

$$e_\iota = - \frac{\partial_{u_{2(g \bmod 2)+1}, \dots, u_{2g-7}, u_{2g-1}}^{[g/2]} \theta[\{\iota\}](\omega^{-1}u)}{\partial_{u_{2(g \bmod 2)+1}, \dots, u_{2g-7}, u_{2g-3}}^{[g/2]} \theta[\{\iota\}](\omega^{-1}u)} \Big|_{u=0}.$$

Here $u = \omega v$ are non-normalized coordinates of Jacobian of the curve.

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