Linear and metric maps on trees via Markov graphs

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Abstract

The main focus of combinatorial dynamics is put on the structure of periodic points (and the corresponding orbits) of topological dynamical systems. The first result in this area is the famous Sharkovsky's theorem which completely describes the coexistence of periods of periodic points for a continuous map from the closed unit interval to itself. One feature of this theorem is that it can be proved using digraphs of a special type (the so-called periodic graphs). In this paper we use Markov graphs (which are the natural generalization of periodic graphs in case of dynamical systems on trees) as a tool to study several classes of maps on trees. The emphasis is put on linear and metric maps.

MSC 2010: 37E15, 05C20, 05C05.

Keywords: Markov graphs, Sharkovsky's theorem, maps on trees.

1 Introduction

Given a set X (finite or infinite) and a map $f : X \to X$ from X to itself the pair (X, f) is called a (combinatorial) *dynamical system*. If X is a topological space and f is a continuous map, then we obtain *topological dynamical system*. An element $x \in X$ is called *periodic point* for f if $f^n(x) = x$ for some $n \geq 1$, where f^n denotes the n-th iterate function of f. If n is the smallest number with the above property, then it is called the *period* of x. *Fixed points* are periodic points of period one. Combinatorial dynamics mainly deals with the structure of periodic points and their orbits. The first result in this area is the celebrated Sharkovsky's theorem which completely describes the coexistence of periods of periodic points for a continuous map from the closed unit interval to itself. To present this result we must consider the following linear ordering of natural numbers:

$$1 \lhd 2 \lhd 2^2 \lhd \cdots \lhd 2^n \lhd \cdots \lhd 7 \cdot 2^n \lhd 5 \cdot 2^n \lhd 3 \cdot 2^n \lhd \cdots \lhd 7 \cdot 2 \lhd 5 \cdot 2 \lhd 3 \cdot 2 \lhd \cdots \lhd 7 \lhd 5 \lhd 3.$$

The ordering $\lhd$ is called *Sharkovsky ordering* and it plays an important role in one-dimensional dynamics because of the following result.

Theorem 1.1. [10] *If the continuous map $f : [0,1] \to [0,1]$ has a periodic point of period n, then it also has a periodic point of period m for all $m \lhd n$. Moreover, for every m there exists a continuous map that has a periodic point of period m but does not have periodic points of periods n, where $m \lhd n$.*

Sharkovsky's theorem can be proved using purely combinatorial arguments which involve digraphs of a special type (see [2, 11]). Namely, let $f : [0,1] \to [0,1]$ be a continuous map and $x \in [0,1]$ be its periodic point of period $n \geq 2$. Consider the corresponding orbit $\mathrm{orb}_f(x) = \{x, f(x), \dots, f^{n-1}(x)\}$ and its natural ordering inherited from the interval, i.e. let $\mathrm{orb}_f(x) = \{x_1 < \cdots < x_n\}$. Since x is a periodic point, the restriction of f to $\mathrm{orb}_f(x)$ is a cyclic permutation of $\mathrm{orb}_f(x)$. *Periodic graph* is then defined as a directed graph with the vertex set $\{1, \dots, n-1\}$ (each $1 \leq i \leq n-1$ represents the minimal interval $[x_i, x_{i+1}]$) and with the arc set $\{(i,j) : \min\{f(x_i), f(x_{i+1})\} \leq x_j < \max\{f(x_i), f(x_{i+1})\}\}$. Since f is continuous, each cycle in the periodic graph corresponds to some periodic point of f. Moreover, if the cycle does not consists of a smaller cycle traced several times, then the period of the corresponding periodic point equals the length of a cycle.

In [1] Bernhardt used a similar approach to prove a Sharkovsky-type result for the continuous maps on finite topological trees. The corresponding digraphs are called Markov graphs and they resemble all important properties of periodic graphs.

Such a crucial role that Markov graphs play in combinatorial dynamics is a reason to study these digraphs from graph-theoretic point of view. It seems that the first results in this direction were obtained by Pavlenko [7, 8, 9]. In particular, the number of non-isomorphic periodic graphs with given number of vertices was calculated in [7]. Graph-theoretic criteria for periodic graphs and for their induced subgraphs were presented in [8] and [9], respectively.

In this paper we study several classes of maps on combinatorial trees via their Markov graphs. The emphasis is put on linear and metric maps. Roughly speaking, linear maps are those maps which preserve metric intervals between pairs of vertices and metric maps are natural generalization of homomorphisms. We obtain several "dual" criteria for linear and metric maps on trees and show that linear metric maps can be characterized as maps which minimize the number of arcs in Markov graphs. Moreover, we use linear maps to study one particular class of trees named spiders.

2 Definitions and preliminary results

To the end of this paper a *map* is just a function. If $f : X \to Y$ is a map and $A \subset X$ is some subset, then by $f|_A$ we denote the restriction of f to A. Also, the symbols $\mathrm{Im}\, f$ and $\mathrm{fix}\, f$ denote the image and the set of all fixed points of a map f, respectively.

2.1 Graphs

A *graph* G is a pair of sets (V, E), where $V = V(G)$ is the set of *vertices* and $E = E(G)$ is the set of *edges* which are unordered pairs of vertices of G. Two vertices $u, v \in V(G)$ are *adjacent* if there is an edge $uv \in E(G)$. The set $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ is called the *neighborhood* of a vertex u in a graph G. The number $d_G(u) = |N_G(u)|$ is called the *degree* of u. A graph G is called *m-regular* if $d_G(u) = m$ for all vertices $u \in V(G)$. Given a set of vertices $A \subset V(G)$ we write $G[A]$ for the subgraph of G *induced* by A, that is $G[A] = (A, E(A))$, where $E(A) = \{uv \in E(G) : u, v \in A\}$.

Two graphs are *isomorphic* if there exists a bijection between their vertex sets which preserves the adjacency in both ways. Every such a bijection is called an *isomorphism*. If two graphs G_1 and G_2 are isomorphic, then we write $G_1 \simeq G_2$.

A graph is called *connected* if for any pair of its vertices there exists a path joining them. The *distance* $d_G(u, v)$ between two vertices $u, v \in V(G)$ in a connected graph G is the number of edges in a shortest $u - v$ path. The set of vertices $[u, v]_G = \{x \in V(G) : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$ is called an *interval* between u and v. It is also convenient to write $(u, v)_X = [u, v]_X - \{u, v\}$ for any pair of vertices $u, v \in V(G)$. We put $\operatorname{diam} G = \max\{d_G(u, v) : u, v \in V(G)\}$ for the *diameter* of a connected graph G. For every set of vertices $A \subset V(G)$ in a connected graph G we set $\operatorname{diam} A = \operatorname{diam} G[A]$.

For an edge $e = uv \in E(G)$ in a connected graph G we define the next "half-space" $A_G(u, v) = \{x \in V(G) : d_G(u, x) \leq d_G(v, x)\}$.

A unique (up to isomorphism) connected 2-regular graph with $n \geq 3$ vertices is called a *cycle* and denoted by C_n.

The set of vertices $A \subset V(G)$ in a graph G is called *connected* if the induced subgraph $G[A]$ is connected. By definition the empty set is connected. The set of vertices $A \subset V(G)$ in a connected graph G is called *convex* if for any pair of vertices $u, v \in A$ we have $[u, v]_G \subset A$. Obviously, each convex set is connected. The *convex hull* $\operatorname{Conv}_G(A)$ of a given set $A \subset V(G)$ is the smallest convex set containing A. The set of vertices $A \subset V(G)$ in a connected graph G is called *Chebyshev* if for every vertex $u \in V(G)$ there exists a unique vertex $v_u \in A$ such that $d_G(u, v_u) = d_G(u, A) = \min\{d_G(u, x) : x \in A\}$. The corresponding map $\operatorname{pr}_A : V(G) \to V(G)$, where $\operatorname{pr}_A(u) = v_u$ is called a *projection* on a Chebyshev set A. Note that a constant map is a projection on a singleton subset. Also, observe that a Chebyshev set of vertices need not to be connected.

A connected graph G is called *median* if given a triple of its vertices $u, v, w \in V(G)$ the set $[u, v]_G \cap [v, w]_G \cap [u, w]_G$ is a singleton. The corresponding unique vertex is called a *median* of the triple u, v, w and denoted by $m_G(u, v, w)$.

A *tree* is a connected acyclic graph. It is easy to see that each tree is a median graph. Also, note that each connected set of vertices in a tree is Chebyshev. A vertex $u \in V(X)$ in a tree X is a *leaf* provided $d_X(u) = 1$. The set of all leaf vertices of X is denoted by $L(X)$. A *path* is a tree X with $|L(X)| \leq 2$. A path with $n \geq 1$ vertices is denoted by P_n. Similarly, a *star* is a tree X with $|L(X)| \geq |V(X)| - 1$. A tree is called *spider* if it has at most one vertex of degree at least three. If such a vertex exists, it will be called the *center* of a spider. Paths and stars are prime examples of spiders.

A *directed graph* or just a *digraph* D is a pair of sets (V, A), where $V = V(D)$ is the set of *vertices* and $A = A(D) \subset V \times V$ is the set of arcs of D. Sometimes we would write $x \to y$ for the arc (x, y). A *loop* is an arc of the form $x \to x$. The *outdegree* $d^+_D(x)$ of a vertex x in a digraph D is the number of arcs of the form $x \to y$. Similarly, the *indegree* $d^-_D(x)$ of x is the number of arcs of the form $y \to x$.

Suppose X is some set and $f : X \to X$ is a map from X to itself. A *functional graph* of the map f is a digraph with the vertex set X and the arc set $\{(x, y) : y = f(x)\}$. Thus, functional digraphs characterized as digraphs D with the property $d^+_D(x) = 1$ for every vertex $x \in V(D)$. Similarly, a digraph D is called *partial functional* if $d^+_D(x) \leq 1$ for all vertices $x \in V(D)$. Examples of functional and partial functional digraphs are provided by directed cycles and paths.

For a given digraph D its *converse* digraph D^{co} has a vertex set $V(D^{co}) = V(D)$ and there is an arc $x \to y$ in D^{co} if there is an arc $y \to x$ in D.

A digraph is called *weakly connected* if its underlying undirected graph is connected. A *weak component* of a given digraph is its maximal weakly connected subgraph.

Having some fixed linear ordering on the set of vertices $V(D) = \{x_1, \ldots, x_n\}$ of a digraph D, one can define its *adjacency matrix* which is a matrix $M_D = (a_{ij})$, where $a_{ij} = 1$ if $x_i \to x_j$ in D and $a_{ij} = 0$ otherwise. For every $k \geq 1$ the k-th power M^k_D of the adjacency matrix M_D has the following interpretation: if $M^k_D = (a^k_{ij})$, then a^k_{ij} equals the number of directed walks of length k from the vertex x_i to the vertex x_j in D.

2.2 Classes of maps between graphs

Let G_1 and G_2 be two connected graphs. A map $f : V(G_1) \to V(G_2)$ is called

1. *homomorphism* if $uv \in E(G_1)$ implies $f(u)f(v) \in E(G_2)$;
2. *metric* if $d_{G_2}(f(u), f(v)) \leq d_{G_1}(u, v)$ for all $u, v \in V(G_1)$;
3. *linear* if $f([u, v]_{G_1}) \subset [f(u), f(v)]_{G_2}$ for all $u, v \in V(G_1)$;
4. *continuous* if $[f(u), f(v)]_{G_2} \subset f([u, v]_{G_1})$ for all $u, v \in V(G_1)$;
5. *monotone* if the pre-image $f^{-1}(y)$ of every vertex $y \in V(G_2)$ is a connected set in G_1 .

For example, each injective map is monotone. Similarly, if $f : V(G_1) \to V(G_2)$ is a linear map between two connected graphs G_1 and G_2, then for all $y \in V(G_2)$ and $u, v \in f^{-1}(y)$ we have $f([u, v]_{G_1}) \subset [f(u), f(v)]_{G_2} = \{y\}$. This means that $[u, v]_{G_1} \subset f^{-1}(y)$ and therefore $f^{-1}(y)$ is a convex set. Thus, every linear map is monotone. However, not every monotone map is linear. To see this consider the path $G \simeq P_3$ with three vertices, where $V(G) = \{1,2,3\}$, $E(G) = \{12,23\}$ and its map $f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. Then f is a bijective and thus a monotone map, but $f([1,3]_G) - [f(1), f(3)]_G = \{3\} \neq \emptyset$.

Proposition 2.1. *Let G be a connected graph and $A \subset V(G)$ be a Chebyshev set. Then the projection pr_A is a monotone map.*

Proof. We prove that for every $y \in A$ and every $u \in \mathrm{pr}_A^{-1}(y)$ it holds $[u,y]_G \subset \mathrm{pr}_A^{-1}(y)$. Namely, if $x \in [u,y]_G$, then $d_G(u, \mathrm{pr}_A(x)) \leq d_G(u,x) + d_G(x, \mathrm{pr}_A(x)) \leq d_G(u,x) + d_G(x,y) = d_G(u,y) = d_G(u, \mathrm{pr}_A(u))$. Therefore, $\mathrm{pr}_A(x) = \mathrm{pr}_A(u) = y$. □

Proposition 2.2. *Let G_1, G_2 be two connected graphs and $f : V(G_1) \to V(G_2)$ be some map. Then f is a metric map if and only if $d_{G_2}(f(u), f(v)) \leq 1$ for all edges $uv \in E(G_1)$.*

Proof. The necessity of this condition is obvious. We prove the sufficiency using induction on $d_{G_1}(u,v)$. Induction basis trivially holds. Now let $d_{G_1}(u,v) = n+1$. Fix a vertex $x \in [u,v]_{G_1}$ with $ux \in E(G_1)$. Then $d_{G_1}(x,v) = n$ and thus by induction assumption $d_{G_2}(f(x), f(v)) \leq d_{G_1}(x,v) = n$. Therefore, $d_{G_2}(f(u), f(v)) \leq d_{G_2}(f(u), f(x)) + d_{G_2}(f(x), f(v)) \leq n + 1 = d_{G_1}(u,v)$. □

Corollary 2.3. *Each homomorphism between two connected graphs is a metric map.*

Corollary 2.4. *The image of a metric map between two connected graphs is always a connected set.*

Proof. Let G_1, G_2 be two connected graphs and $f : V(G_1) \to V(G_2)$ be a metric map. Assume that the image $\operatorname{Im} f$ is disconnected. Then there exists a pair of sets $A, B \subset \operatorname{Im} f$ with $\operatorname{Im} f = A \sqcup B$ and $d_{G_2}(A,B) \geq 2$. We have $V(G_1) = f^{-1}(A) \sqcup f^{-1}(B)$. Since G_1 is a connected graph, there exist two vertices $u \in f^{-1}(A)$ and $v \in f^{-1}(B)$ with $uv \in E(G_1)$. We have $d_{G_2}(f(u), f(v)) \geq d_{G_2}(A,B) \geq 2$ which contradicts Proposition 2.2. □

Further, suppose that $f : V(G_1) \to V(G_2)$ is a continuous map between two connected graphs G_1 and G_2. Then for every edge $uv \in E(G_1)$ we have $[f(u), f(v)]_{G_2} \subset f([u,v]_{G_1}) = f(\{u,v\}) = \{f(u), f(v)\}$. Therefore, $f(u) = f(v)$ or $f(u)f(v) \in E(G_2)$. From Proposition 2.2 it follows that f is a metric map. However, not every metric map is continuous. To see this consider the path $G_1 \simeq P_4$ with four vertices, where $V(G_1) = \{1,2,3,4\}$, $E(G_1) = \{12,23,34\}$, a cycle $G_2 \simeq C_4$ with four vertices, where $V(G_2) = V(G_1)$, $E(G_2) = \{12,23,34,14\}$ and the identity map $f : V(G_1) \to V(G_2)$. Then f is a metric map, but $[f(1), f(3)]_{G_2} = \{1,2,3,4\} \not\subseteq \{1,2,3\} = f([1,3]_{G_1})$.

Proposition 2.5. *Let G_1 be a connected graph, G_2 be a tree and $f : V(G_1) \to V(G_2)$ be some map. Then f is metric if and only if f is continuous.*

Proof. We must show only the necessity of the condition. Thus, let f be a metric map. We use induction on $d_{G_1}(u,v)$ to prove that $[f(u),f(v)]_{G_2} \subset f([u,v]_{G_1})$ for all pairs of vertices $u,v \in V(G_1)$. The induction basis trivially holds. Now let $d_{G_1}(u,v) = n+1$. Fix a vertex $x \in [u,v]_{G_1}$ with $ux \in E(G_1)$. By induction assumption we have $[f(x),f(v)]_{G_2} \subset f([x,v]_{G_1})$. Since $d_{G_1}(u,x) = 1$ and f is metric, it holds $d_{G_2}(f(u),f(x)) \leq 1$. But G_2 is a tree. This means that $[f(u),f(v)]_{G_2} \subset \{f(u)\} \cup [f(x),f(v)]_{G_2}$.

Finally, since $\{u\} \cup [x,v]_{G_1} = [u,v]_{G_1}$, then $[f(u),f(v)]_{G_2} \subset \{f(u)\} \cup [f(x),f(v)]_{G_2} \subset \{f(u)\} \cup f([x,v]_{G_1}) = f(\{u\} \cup [x,v]_{G_1}) = f([u,v]_{G_1})$. □

Proposition 2.6. *Let G_1, G_2 be two median graphs and $f : V(G_1) \to V(G_2)$ be some map. Then f is linear if and only if f preserves medians, i.e. $f(m_{G_1}(u,v,w)) = m_{G_2}(f(u),f(v),f(w))$ for all triplets of vertices $u,v,w \in V(G_1)$.*

Proof. Let f be a linear map. Then

$$\begin{aligned}
\{f(m_{G_1}(u,v,w))\} &= f([u,v]_{G_1} \cap [v,w]_{G_1} \cap [u,w]_{G_1}) \\
&\subset f([u,v]_{G_1}) \cap f([v,w]_{G_1}) \cap f([u,w]_{G_1}) \\
&\subset [f(u),f(v)]_{G_2} \cap [f(v),f(w)]_{G_2} \cap [f(u),f(w)]_{G_2} \\
&= \{m_{G_2}(f(u),f(v),f(w))\}
\end{aligned}$$

which yields the desired equality $f(m_{G_1}(u,v,w)) = m_{G_2}(f(u),f(v),f(w))$. Conversely, suppose that $x \in [u,v]_{G_1}$ for $u,v \in V(G_1)$. Then $m_{G_1}(u,x,v) = x$ and thus $f(x) = f(m_{G_1}(u,x,v)) = m_{G_2}(f(u),f(x),f(v))$. This implies $f(x) \in [f(u),f(v)]_{G_2}$. Therefore, $f([u,v]_{G_1}) \subset [f(u),f(v)]_{G_2}$ for any pair of vertices $u,v \in V(G_1)$. □

Proposition 2.7. *Let G_1, G_2 be two connected graphs and $f : V(G_1) \to V(G_2)$ be some map. Then for all vertices $u,v \in V(G_1)$ we have $d_{G_1}(u,v) \leq d_{G_2}(f(u),f(v))$ if and only if f is injective and its inverse $f^{-1} : f(V(G_1)) \to V(G_1)$ is a (bijective) homomorphism between the induced subgraph $G_2[f(V(G_1))]$ and G_1.*

Proof. We prove the necessity of this condition. Let $u,v \in V(G_1)$ be two distinct vertices. Then $d_{G_2}(f(u),f(v)) \geq d_{G_1}(u,v) \geq 1$ implying $f(u) \neq f(v)$. Thus, f is an injective map. Further, suppose that for a pair of vertices $u,v \in V(G_1)$ there exists an edge $f(u)f(v) \in E(G_2)$. Then clearly $u \neq v$ and therefore $1 \leq d_{G_1}(u,v) \leq d_{G_2}(f(u),f(v)) = 1$. This implies $uv \in E(G_1)$. Thus, f^{-1} is a homomorphism.

To prove the sufficiency of this condition again consider two vertices $u,v \in V(G_1)$.

Since f^{-1} is a homomorphism, f^{-1} is a metric map which yields $d_{G_1}(u,v) = d_{G_1}(f^{-1}(f(u)),f^{-1}(f(v))) \leq d_{G_2}(f(u),f(v))$. □

Corollary 2.8. *Let G be a connected graph. Then $f : V(G) \to V(G)$ is an automorphism of G if and only if $d_G(u,v) \leq d_G(f(u),f(v))$ for all $u,v \in V(G)$.*

It is easy to see that for a given connected graph G the classes of metric and linear maps of the form $f : V(G) \to V(G)$ are closed under composition of maps. However, the composition of two monotone maps may not be monotone itself. Moreover, the following proposition holds.

Proposition 2.9. *The class of monotone maps $f : V(G) \to V(G)$ for a given connected graph G with $n \geq 2$ vertices is a generating set of the semigroup of all vertex maps $V(X)^{V(X)}$.*

Proof. It is well-known that the full transformation semigroup T_n of all self-maps of an n-element set $\{1, \dots, n\}$ has a generating set consisting of the following three maps: $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 3 & \dots & n \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 1 & 3 & \dots & n \end{pmatrix}$ and $\sigma_3 = (12 \dots n)$.

Since G is connected and $n \geq 2$, G has at least one edge. Fix an edge $e = uv \in E(G)$ and a bijection $f : \{1, \dots, n\} \to V(G)$ with $f(1) = u$, $f(2) = v$. Then $f \circ \sigma_i \circ f^{-1}$ for $1 \leq i \leq 3$ is a triple of monotone maps from $V(X)$ to itself which generate all the maps from $V(X)^{V(X)}$. □

Furthermore, for a path X we can ensure that each map $\sigma : V(X) \to V(X)$ is a composition of exactly two monotone maps.

Proposition 2.10. *Let X be a path. Then for every map $\sigma : V(X) \to V(X)$ there exist two monotone maps $\sigma_i : V(X) \to V(X), i = 1, 2$ such that $\sigma = \sigma_2 \circ \sigma_1$.*

Proof. Suppose $V(X) = \{1, \dots, n\}$ and $E(X) = \{ij : 1 \leq i = j - 1 \leq n - 1\}$. Also, let $\operatorname{Im} \sigma = \{i_1 < \cdots < i_m\}$. Construct σ_1 in the following way. Let σ_1 maps the pre-image $\sigma^{-1}(i_1)$ bijectively to $\{1, \dots, |\sigma^{-1}(i_1)|\}$ and also maps the pre-image $\sigma^{-1}(i_k)$ bijectively to $\{\sum_{j=1}^{k-1} |\sigma^{-1}(i_j)| + 1, \dots, \sum_{j=1}^{k} |\sigma^{-1}(i_j)|\}$ for each $2 \leq k \leq m$. By construction, σ_1 is bijective and hence a monotone map. Similarly, let σ_2 maps the set $\{1, \dots, |\sigma^{-1}(i_1)|\}$ to i_1 and also maps $\{\sum_{j=1}^{k-1} |\sigma^{-1}(i_j)| + 1, \dots, \sum_{j=1}^{k} |\sigma^{-1}(i_j)|\}$ to i_k for each $2 \leq k \leq m$. It is easy to see that σ_2 is also monotone and $\sigma = \sigma_2 \circ \sigma_1$. □

Also, note that the composition of two projections on connected Chebyshev sets is not necessarily a projection itself. Namely, consider the graph G with $V(G) = \{1, \dots, 8\}$, $E(G) = \{12, 16, 18, 23, 25, 34, 37, 45, 48, 56, 67\}$ and two sets of vertices $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5, 6\}$. Then A_1 and A_2 are connected Chebyshev sets, but their composition $\operatorname{pr}_{A_2} \circ \operatorname{pr}_{A_1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 4 & 4 & 5 & 6 & 4 & 6 \end{pmatrix}$ is not a projection on any Chebyshev set in G.

Finally, if X is a tree and $A_i \subset V(X), i = 1, 2$ its two connected (and thus, Chebyshev) sets, then $\operatorname{pr}_{A_2} \circ \operatorname{pr}_{A_1} = \operatorname{pr}_{A_1 \cap A_2}$ if $A_1 \cap A_2 \neq \emptyset$ and $\operatorname{pr}_{A_2} \circ \operatorname{pr}_{A_1}$ is a constant map otherwise. In both cases the composition $\operatorname{pr}_{A_2} \circ \operatorname{pr}_{A_1}$ is a projection on a connected set.

2.3 Markov graphs for maps on trees

Let X be a tree and $\sigma : V(X) \to V(X)$ be some map from the vertex set of X to itself. The *Markov graph* is a digraph $\Gamma = \Gamma(X, \sigma)$ with the vertex set $V(\Gamma) = E(X)$ and the arc set $A(\Gamma) = \{(u_1v_1, u_2v_2) : u_2, v_2 \in [\sigma(u_1), \sigma(v_1)]_X\}$. Thus, vertices in Γ are the edges of X and there is an arc $u_1v_1 \to u_2v_2$ in Γ if the edge u_1v_1 "covers" u_2v_2 under σ. Note that periodic graphs are precisely Markov graphs $\Gamma(X, \sigma)$ for paths X and cyclic permutations σ.

Example 2.11. Let X be the tree with the vertex set $V(X) = \{1, \dots, 7\}$ and the edge set $E(X) = \{12, 23, 34, 45, 26, 37\}$. Also, consider the map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 3 & 6 & 2 & 4 & 2 \end{pmatrix}$ (see Figure 1). Then the corresponding Markov graph $\Gamma(X, \sigma)$ is shown in Figure 2.

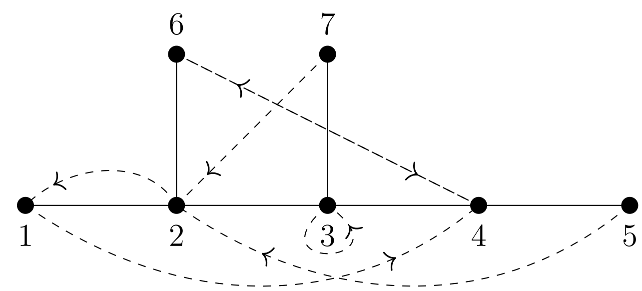

Figure 1: The pair (X, σ) from Example 2.11.

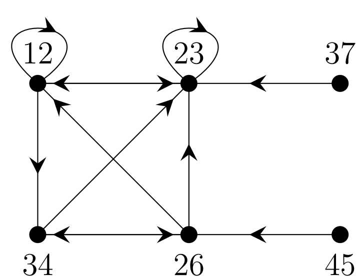

Figure 2: Markov graph $\Gamma(X, \sigma)$ for the pair (X, σ) from Example 2.11.

Proposition 2.12. [6] *Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. Put $E(\sigma) = \{e \in E(X) : d^-_\Gamma(e) \geq 1\}$. Then $E(\sigma) = E(\mathrm{Conv}_X(\mathrm{Im}\,\sigma))$. In particular, $X[E(\sigma)]$ is the connected subgraph of X.*

Now suppose that for a tree X some linear ordering of its edge set $E(X)$ is fixed. In [3] it was proved that the correspondence $\sigma \to M_{\Gamma(X,\sigma)}$ gives a homomorphism from the full transformation semigroup T_n to the semigroup $\mathrm{Mat}_{n-1}(\mathbb{F}_2)$ of $(n-1) \times (n-1)$

matrices over the two-element field. In particular, this correspondence induces an injective homomorphism from the symmetric group S_n into the general linear group $\mathrm{Gl}_{n-1}(\mathbb{F}_2)$.

Theorem 2.13. [3] *Let X be a tree and suppose that some linear ordering of the edge set $E(X)$ is fixed. Then for any pair of maps $\sigma_i : V(X) \to V(X), i = 1, 2$ it holds* $M_{\Gamma(X,\sigma_2 \circ \sigma_1)} = M_{\Gamma(X,\sigma_1)} M_{\Gamma(X,\sigma_2)} \bmod 2$.

For each tree X and its map $\sigma : V(X) \to V(X)$ one can construct the corresponding *edge labeling* $\tau_\sigma : E(X) \to V(X) \cup \{1, -1\}$ in the following way:

$$\tau_\sigma(e) = \begin{cases} u, \text{ if } \sigma(u), \sigma(v) \in A_X(u, v), \\ v, \text{ if } \sigma(u), \sigma(v) \in A_X(v, u), \\ 1, \text{ if } \sigma(u) \in A_X(u, v) \text{ and } \sigma(v) \in A_X(v, u), \\ -1, \text{ if } \sigma(u) \in A_X(v, u) \text{ and } \sigma(v) \in A_X(u, v) \end{cases}$$

for every edge $e = uv \in E(X)$. In other words, the edge $e = uv$ gets an orientation $u \to v$ provided $\tau_\sigma(e) = v$. Otherwise, the edge e is σ-positive or σ-negative depending on the sign of $\tau_\sigma(e)$.

The definition of the labeling τ_σ naturally leads to the two extremal classes of maps on trees. Namely, the map σ is called *expansive* if each vertex in the Markov graph $\Gamma(X, \sigma)$ has a loop. In other words, σ is expansive if $\operatorname{Im} \tau_\sigma \subset \{1, -1\}$. Similarly, the map σ is called *anti-expansive* if $\Gamma(X, \sigma)$ does not contain a vertex with a loop. In this case, τ_σ is an orientation of the tree X.

Proposition 2.14. [4] *Let X be a tree and $\sigma : V(X) \to V(X)$ be an anti-expansive map. Then σ has a unique fixed point.*

3 Main results

In [3] automorphisms and projections on connected sets of vertices in trees were characterized in terms of the corresponding Markov graphs. Also, from Proposition 2.2 it directly follows that a map $\sigma : V(X) \to V(X)$ on a tree X is metric if and only if the Markov graph $\Gamma(X, \sigma)$ is partial functional. To obtain a "dual" criterion for linear maps we need the following lemma.

Lemma 3.1. [3] *Let X be a tree, $\sigma : V(X) \to V(X)$ be some map and $\Gamma = \Gamma(X, \sigma)$ be its Markov graph. Then for every pair of vertices $u, v \in V(X)$ and an edge $xy \in E([\sigma(u), \sigma(v)]_X)$ there exists an edge $wz \in E([u, v]_X)$ such that $wz \to xy$ in Γ. In particular,*

$$[\sigma(u), \sigma(v)]_X \subset \bigcup_{wz \in E([u,v]_X)} [\sigma(w), \sigma(z)]_X.$$

Theorem 3.2. *Let X be a tree, $\sigma : V(X) \to V(X)$ be some map and $\Gamma = \Gamma(X, \sigma)$ be its Markov graph. Then σ is linear if and only if the converse digraph Γ^{co} is partial functional.*

Proof. First, we prove the necessity of this condition. To the contrary, suppose that σ is linear, but there exists an edge $e \in E(X)$ such that $d^-_\Gamma(e) \geq 2$. Fix a pair of edges $e_1, e_2 \in N^-_\Gamma(e)$. Then $u, v \in [\sigma(u_1), \sigma(v_1)]_X \cap [\sigma(u_2), \sigma(v_2)]_X$, where $e = uv$ and $e_i = u_i v_i$ for $i = 1, 2$. Without loss of generality, we can assume that $[v_1, v_2]_X \subset [u_1, u_2]_X$.

Consider the composition $\mathrm{pr}_e \circ \sigma$. Again, without loss of generality, suppose that $(\mathrm{pr}_e \circ \sigma)(u_1) = u$. Then $(\mathrm{pr}_e \circ \sigma)(v_1) = v$. If $(\mathrm{pr}_e \circ \sigma)(v_2) = u$, then $\sigma(v_1) \in \sigma([u_1, v_2]_X) - [\sigma(u_1), \sigma(v_2)]_X$ which contradicts to the fact that σ is a linear map. Similarly, if $(\mathrm{pr}_e \circ \sigma)(v_2) = v$, then $(\mathrm{pr}_e \circ \sigma)(u_2) = u$. This means that $\sigma(v_1) \in \sigma([u_1, u_2]_X) - [\sigma(u_1), \sigma(u_2)]_X$. A contradiction again.

Now we prove the sufficiency of this condition. Suppose that Γ^{co} is partial functional and $u, v \in V(X)$. If $d_X(u, v) \leq 1$, then clearly $\sigma([u, v]_X) \subset [\sigma(u), \sigma(v)]_X$. Thus, let $d_X(u, v) \geq 2$ and let $x \in (u, v)_X$ be some fixed vertex. We show that in this case $\sigma(x) \in [\sigma(u), \sigma(v)]_X$. Consider the vertex $y = (\mathrm{pr}_{[\sigma(u),\sigma(v)]_X} \circ \sigma)(x)$. Thus, we must prove that $\sigma(x) = y$.

First, note that $[\sigma(x), y]_X \subset [\sigma(u), \sigma(x)]_X \cap [\sigma(x), \sigma(v)]_X$. Further, from Lemma 3.1 it follows that for every edge $e \in E([\sigma(x), y]_X)$ there exists an edge $e_1 \in E([u, x]_X)$ such that $e_1 \to e$ in Γ and also there exists another edge $e_2 \in E([x, v]_X)$ with $e_2 \to e$ in Γ. Since $e_1 \neq e_2$, we obtain a contradiction with the partial functionality of Γ^{co}. Therefore, the interval $[\sigma(x), y]_X$ does not contain an edge, i.e. $\sigma(x) = y$. $\square$

Corollary 3.3. *Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. Then σ is a linear metric map if and only if each weak component in $\Gamma(X, \sigma)$ is a cycle or a path.*

Proposition 3.4. *Let X be a tree and $\sigma : V(X) \to V(X)$ be a linear map. Then σ is a metric map if and only if the image $\mathrm{Im}\,\sigma$ is a connected set.*

Proof. Corollary 2.4 asserts that we must prove only the sufficiency of this condition. Assume that there exists an edge $e = uv \in E(X)$ with $d_X(\sigma(u), \sigma(v)) \geq 2$. Fix a vertex $y \in (\sigma(u), \sigma(v))_X$. Since $\mathrm{Im}\,\sigma$ is a connected set, then $y \in \mathrm{Im}\,\sigma$ implying there is a vertex $x \in V(X)$ with $\sigma(x) = y$. If $x \in A_X(u, v)$, then $u \in [x, v]_X$ and $\sigma(u) \notin [y, \sigma(v)]_X = [\sigma(x), \sigma(v)]_X$. If $x \in A_X(v, u)$, then $v \in [u, x]_X$ and $\sigma(v) \notin [\sigma(u), y]_X = [\sigma(u), \sigma(x)]_X$. In both cases σ is not a linear map. $\square$

It is easy to see that $\sigma : V(X) \to V(X)$ is a proper coloring of a tree X if and only if $d^+_\Gamma(e) \geq 1$ for all edges $e \in E(X)$. This implies the inequality $|A(\Gamma(X, \sigma))| \geq |V(X)| - 1$ for each proper coloring σ. Similarly, σ is a constant map if and only if $|A(\Gamma(X, \sigma))| = 0$. In [3] we obtained the following bounds for the number of arcs in Markov graphs.

Proposition 3.5. [3] *Let X be a tree, $\sigma : V(X) \to V(X)$ be some map and $\Gamma = \Gamma(X, \sigma)$. Then $|\,\mathrm{Im}\,\sigma| - 1 \leq |A(\Gamma)| \leq (n - 1) \cdot \mathrm{diam}\,\mathrm{Im}\,\sigma$.*

The next theorem gives a characterization of maps which attain the lower bound from Proposition 3.5.

Theorem 3.6. *Let X be a tree, $\sigma : V(X) \to V(X)$ be some map and $\Gamma = \Gamma(X, \sigma)$. Then $|A(\Gamma)| = |\operatorname{Im}\sigma| - 1$ if and only if σ is a linear metric map.*

Proof. First, we prove the sufficiency of this condition using induction on $|V(X)|$. Induction basis trivially holds. Suppose that $|V(X)| \geq 2$. Fix a leaf vertex $u \in L(X)$ and the corresponding edge $e = uu_0$. Also, put $X' = X - \{u\}$ and $\sigma' = \operatorname{pr}_{V(X)-\{u\}} \circ \sigma$. Obviously, X' is a tree and σ' is a linear metric map on X'. By induction assumption $|A(\Gamma')| = |\operatorname{Im}\sigma'| - 1$, where $\Gamma' = \Gamma(X', \sigma')$. Further, we consider the following four cases.

Case 1: $d^+_\Gamma(e) = d^-_\Gamma(e) = 0$.

If $u \in \operatorname{Im}\sigma$, then from the equality $d^-_\Gamma(e) = 0$ it follows that σ is a constant map and therefore we are done. Otherwise, let $u \notin \operatorname{Im}\sigma$. Since $d^+_\Gamma(e) = 0$ implies $\sigma(u) = \sigma(u_0)$, then $\operatorname{Im}\sigma = \operatorname{Im}\sigma'$. Also, in this case we have $A(\Gamma) = A(\Gamma')$ and the desired is proved.

Case 2: $d^+_\Gamma(e) = 0$ and $d^-_\Gamma(e) = 1$.

The equality $d^-_\Gamma(e) = 1$ implies $u \in \operatorname{Im}\sigma$. Again, from the equality $d^+_\Gamma(e) = 0$ it follows that $\sigma(u) = \sigma(u_0)$. In other words, in this case $|\operatorname{Im}\sigma| = |\operatorname{Im}\sigma'| + 1$. Let $e' = xy \in E(X)$ be an edge with $N^+_\Gamma(e') = \{e\}$. Then $N^+_{\Gamma'}(e') = \emptyset$. Thus, in this case $|A(\Gamma)| = |A(\Gamma')| + 1$ implying $|A(\Gamma)| = |\operatorname{Im}\sigma'| - 1 + 1 = |\operatorname{Im}\sigma'| = |\operatorname{Im}\sigma| - 1$.

Case 3: $d^+_\Gamma(e) = 1$ and $d^-_\Gamma(e) = 0$.

Since every linear map is monotone and $\sigma(u) \neq \sigma(u_0)$, then $\sigma^{-1}(\sigma(u)) = \{u\}$. This means that $\sigma(u) \in \operatorname{Im}\sigma$, but $\sigma(u) \notin \operatorname{Im}\sigma'$. Meanwhile, if $u \in \operatorname{Im}\sigma$, then from the equality $d^-_\Gamma(e) = 0$ it follows that σ is a constant map and we are done. Otherwise, $u \notin \operatorname{Im}\sigma$ and thus $|\operatorname{Im}\sigma| = |\operatorname{Im}\sigma'| + 1$. Therefore, in this case $|A(\Gamma)| = |A(\Gamma')| + 1$ which yields $|A(\Gamma)| = |\operatorname{Im}\sigma| - 1$.

Case 4: $d^+_\Gamma(e) = d^-_\Gamma(e) = 1$.

In this case we have $|A(\Gamma)| = |A(\Gamma')| + 2$. Also, clearly $u, \sigma(u) \in \operatorname{Im}\sigma - \operatorname{Im}\sigma'$. This means that $|\operatorname{Im}\sigma| = |\operatorname{Im}\sigma'| + 2$ and we are done.

Now we prove the necessity of this condition. Put $E' = E(\operatorname{Conv}_X(\operatorname{Im}\sigma))$. By Proposition 2.12, we have

$$\begin{aligned} |\operatorname{Im}\sigma| - 1 = |A(\Gamma)| = \sum_{e \in E(X)} d^+_\Gamma(e) \geq |\{e \in E(X) : d^-_\Gamma(e) \geq 1\}| = \\ = |E'| \geq |\operatorname{Im}\sigma| - 1 \end{aligned}$$

This means that $d^-_\Gamma(e) \leq 1$ for all $e \in E(X)$. Hence, the map σ is linear on X (see Theorem 3.2). Moreover, the equality $|E'| = |\operatorname{Im}\sigma| - 1$ asserts that $\operatorname{Im}\sigma$ is a connected set. By Proposition 3.4 the map σ is metric on X. □

Given a tree X and a pair of maps $\sigma_i : V(X) \to V(X), i = 1, 2$ we write $\sigma_1 \leq_m \sigma_2$ if $\Gamma(X, \sigma_1) \subset \Gamma(X, \sigma_2)$. In other words, $\sigma_1 \leq_m \sigma_2$ if for all edges $uv \in E(X)$ we have $\sigma_1(u) = \sigma_1(v)$ and $\sigma_2(u) = \sigma_2(v)$, or $[\sigma_1(u), \sigma_1(v)]_X \subset [\sigma_2(u), \sigma_2(v)]_X$. Relation $\leq_m$ establishes a preordering of the set $V(X)^{V(X)}$ which is called *Markov preordering.* Indeed, for every two different constant maps σ_1 and σ_2 we have $\sigma_1 \leq_m \sigma_2$ and $\sigma_2 \leq_m \sigma_1$ but $\sigma_1 \neq \sigma_2$. Thus, generally speaking, $\leq_m$ is not an antisymmetric relation. However, in [3] it was proved that Markov preordering is a partial ordering of the set $\{\sigma \in V(X)^{V(X)} : |\operatorname{Im}\sigma| \geq 3\}$. For the results about maximal elements in Markov preordering see [4].

Theorem 3.7. *Let X be a tree with $|V(X)| \geq 3$ and $f : V(X) \to V(X)$ be some map. Then*

1. *the map f is linear if and only if for every pair of maps $\sigma_i : V(X) \to V(X), i = 1, 2$ with $\sigma_1 \leq_m \sigma_2$ we have $f \circ \sigma_1 \leq_m f \circ \sigma_2$;*

2. *the map f is metric if and only if for every pair of maps $\sigma_i : V(X) \to V(X), i = 1, 2$ with $\sigma_1 \leq_m \sigma_2$ we have $\sigma_1 \circ f \leq_m \sigma_2 \circ f$;*

Proof. We prove the first claim. Let f be a linear map and $uv \in E(X)$ be some edge in X. If $\sigma_1|_{\{u,v\}}$ and $\sigma_2|_{\{u,v\}}$ is a pair of constant maps, then the compositions $f \circ \sigma_1|_{\{u,v\}}$ and $f \circ \sigma_2|_{\{u,v\}}$ are also constant. Therefore, in this case $N^+_{\Gamma(X, f\circ\sigma_1)}(uv) = N^+_{\Gamma(X, f\circ\sigma_2)}(uv) = \emptyset$.

Otherwise, we have $\sigma_1(u), \sigma_1(v) \in [\sigma_2(u), \sigma_2(v)]_X$. This implies $f(\sigma_1(u)), f(\sigma_1(v)) \in f([\sigma_2(u), \sigma_2(v)]_X) \subset [f(\sigma_2(u)), f(\sigma_2(v))]_X$ as f is linear. But an interval in a tree is a convex set. Thus, $[f(\sigma_1(u)), f(\sigma_1(v))]_X \subset [f(\sigma_2(u)), f(\sigma_2(v))]_X$ which means that $f \circ \sigma_1 \leq_m f \circ \sigma_2$.

Now suppose that f is a non-linear map. Then there exists a pair of vertices $u, v \in V(X)$ such that $f([u, v]_X) - [f(u), f(v)]_X \neq \emptyset$. In other words, there exists a vertex $w \in [u, v]_X$ with $f(w) \notin [f(u), f(v)]_X$. Without loss of generality, we can assume that u and w are adjacent in X. Consider two new maps $\sigma_1 = \operatorname{pr}_{\{u,w\}}$ and $\sigma_2 = \sigma' \circ \operatorname{pr}_{\{u,w\}}$, where $\sigma' : \{u, w\} \to V(X)$, $\sigma'(u) = u$, $\sigma'(w) = v$. It is easy to see that $\sigma_1 \leq_m \sigma_2$. On the other hand, $f \circ \sigma_1|_{\{u,w\}}$ (as well as $f \circ \sigma_2|_{\{u,w\}}$) is a non-constant map and $f(\sigma_1(w)) = f(w) \in [f(\sigma_1(u)), f(\sigma_1(w))]_X = [f(u), f(w)]_X$, but $f(w) \notin [f(u), f(v)]_X = [f(\sigma_2(u)), f(\sigma_2(w))]_X$. In other words, $f \circ \sigma_1 \not\leq_m f \circ \sigma_2$.

Now we prove the second claim. Let f be a metric map. Then $f(u) = f(v)$ or $f(u)f(v) \in E(X)$ for every edge $uv \in E(X)$. If $f(u) = f(v)$, then $\sigma_1 \circ f|_{\{u,v\}}$ and $\sigma_2 \circ f|_{\{u,v\}}$ are two constant maps. Therefore, in this case $N^+_{\Gamma(X,\sigma_1\circ f)}(uv) = N^+_{\Gamma(X,\sigma_2\circ f)}(uv) = \emptyset$.

Let $f(u)f(v) \in E(X)$. If $\sigma_1|_{\{f(u),f(v)\}}$ and $\sigma_2|_{\{f(u),f(v)\}}$ are two constant maps, then $\sigma_1 \circ f|_{\{u,v\}}$ and $\sigma_2 \circ f|_{\{u,v\}}$ are also constant maps. Thus, $\sigma_1 \circ f \leq_m \sigma_2 \circ f$. Otherwise, $[\sigma_1(f(u)), \sigma_1(f(v))]_X \subset [\sigma_2(f(u)), \sigma_2(f(v))]_X$ which also means that $\sigma_1 \circ f \leq_m \sigma_2 \circ f$.

Suppose that f is a non-metric map. Then there exists an edge $uv \in E(X)$ with $d_X(f(u), f(v)) \geq 2$. Fix two vertices $w, t \in [f(u), f(v)]_X$ such that $f(u)$ is adjacent with w and w is adjacent with t (it could be $t = f(v)$, but $w \in (f(u), f(v))_X$ since

$d_X(f(u), f(v)) \geq 2$). Consider two maps $\sigma_1 = \mathrm{pr}_{\{f(u),w\}}$ and $\sigma_2 = \sigma' \circ \mathrm{pr}_{\{f(u),w,t\}}$, where $\sigma' : \{f(u), w, t\} \to V(X)$, $\sigma'(f(u)) = \sigma'(t) = f(u)$, $\sigma'(w) = w$. It is easy to see that $\sigma_1 \leq_m \sigma_2$. Also, from the inequality $d_X(f(u), f(v)) \geq 2$ and definitions of $\sigma_i, i = 1, 2$ it follows that $\sigma_1 \circ f|_{\{u,v\}}$ is non-constant. Moreover, $\sigma_1(f(u)) = f(u)$ and $\sigma_1(f(v)) = w$, but $\sigma_2(f(u)) = \sigma_2(f(v)) = f(u)$. In other words, $[\sigma_1(f(u)), \sigma_1(f(v))]_X \nsubseteq [\sigma_2(f(u)), \sigma_2(f(v))]_X = \{f(u)\}$ which is equivalent to $\sigma_1 \circ f \not\leq_m \sigma_2 \circ f$. □

Corollary 3.8. *Let X be a tree and $\sigma_i : V(X) \to V(X), i = 1, 2$ be a pair of maps. The following conditions are equivalent:*

1. $\sigma_1 \leq_m \sigma_2$*;*
2. $\mathrm{pr}_A \circ \sigma_1 \leq_m \mathrm{pr}_A \circ \sigma_2$ *for every connected set $A \subset V(X)$;*
3. $\sigma_1 \circ \mathrm{pr}_A \leq_m \sigma_2 \circ \mathrm{pr}_A$ *for every connected set $A \subset V(X)$.*

Proof. These are follow from the fact that each projection on a connected set of vertices in a tree is a linear metric map and Theorem 3.7. □

It is easy to see that for a tree X the existence of a map $\sigma : V(X) \to V(X)$ with a Markov graph $\Gamma(X, \sigma)$ which has a vertex of maximum degree (equal to $|E(X)|$) implies that X is a path. Similarly, let X be a tree and $\sigma : V(X) \to V(X)$ be a map with $\Gamma(X, \sigma)$ being a nontrivial symmetric sum of two digraphs. In [5] it is proved that in this case X is a spider of degree at most three.

Theorem 3.9. *Let X be a tree and $\sigma : V(X) \to V(X)$ be a map such that its Markov graph $\Gamma(X, \sigma)$ is a path. Then X is a spider.*

Proof. We divide the proof into the following claims.

Claim 1: For every $k \geq 1$ the Markov graph $\Gamma(X, \sigma^k)$ has no loops.

Fix some linear ordering of the edge set $E(X) = \{e_1, \ldots, e_{n-1}\}$. From Theorem 2.13 it follows that $M_{\Gamma(X,\sigma^k)} = M^k_{\Gamma(X,\sigma)} \bmod 2$. Therefore, the existence of a loop in $\Gamma(X, \sigma^k)$ is equivalent to the existence of $1 \leq i \leq n-1$ for which $(M^k_{\Gamma(X,\sigma)})_{ii} = 1 \bmod 2$. This implies $(M^k_{\Gamma(X,\sigma)})_{ii} \geq 1$ which means that there exists a closed walk of length k passing through the edge e_i in $\Gamma(X, \sigma)$. Therefore, $\Gamma(X, \sigma)$ contains a cycle which is impossible since $\Gamma(X, \sigma)$ is a path.

Claim 2: Each periodic point of σ is a fixed point.

To the contrary, suppose that $u \in V(X)$ is a periodic point of σ with period $m \geq 2$. Then $|\operatorname{fix} \sigma^m| \geq m \geq 2$ and thus from Proposition 2.14 it follows that $\Gamma(X, \sigma^m)$ has a loop. A contradiction with Claim 1.

Claim 3: Each vertex $u \in V(X) - \operatorname{fix} \sigma$ with $d_X(u) \geq 3$ is a periodic point of σ.

Fix three vertices $x_1, x_2, x_3 \in N_X(u)$. Since $\sigma(u) \neq u$, it holds $\sigma(x_i) \neq \sigma(u)$ for every $1 \leq i \leq 3$. Thus, $\Gamma(X, \sigma)$ has the following three arcs: $ux_i \to \sigma(u)\sigma(x_i), 1 \leq i \leq 3$.

Since $\Gamma(X,\sigma)$ is a path, without loss of generality, we can assume that there are two directed walks: one from $\sigma(u)\sigma(x_1)$ to ux_2 and another one from $\sigma(u)\sigma(x_2)$ to ux_3 in $\Gamma(X,\sigma)$. In other words, there exist two numbers $k, m \geq 1$ such that $\{\sigma^{k+1}(u), \sigma^{k+1}(x_1)\} = \{u, x_2\}$ and $\{\sigma^{m+1}(u), \sigma^{m+1}(x_2)\} = \{u, x_3\}$. If $\sigma^{k+1}(u) = u$ (similarly, $\sigma^{m+1}(u) = u$), then u is a periodic point of σ. Since $\sigma(u) \neq u$, u has a period of at least two. Therefore, suppose $\sigma^{k+1}(u) = x_2$, $\sigma^{k+1}(x_1) = u$ and $\sigma^{m+1}(u) = x_3$, $\sigma^{m+1}(x_2) = u$. This implies $\sigma^{k+m+2}(u) = \sigma^{m+1}(x_2) = u$. Thus, again u is a periodic point for σ with period greater than one.

Combining Claim 2 and Claim 3, we can conclude that each vertex in X of degree at least three is a fixed point of σ. However, the map σ is anti-expansive and therefore using Proposition 2.14 we obtain that $|\operatorname{fix} \sigma| = 1$ which completes the proof. $\square$

Note that not every spider X admits a map σ with $\Gamma(X,\sigma)$ being a path. For example, let X be a tree with $V(X) = \{1, \ldots, 6\}$ and $E(X) = \{12, 23, 34, 45, 26\}$. Then X is a spider with the vertex 2 being its center. However, X does not admit a map σ with a path $\Gamma(X,\sigma)$. Nevertheless, paths and stars X admit maps σ with $\Gamma(X,\sigma)$ being a path. Moreover, in [3] it was proved that if $\Gamma(X,\sigma)$ is a cycle, then X is a star. Finally, note that for every proper coloring $\sigma : V(X) \to \{u, v\}$, where $uv \in E(X)$ the Markov graph $\Gamma(X,\sigma)$ is weakly connected and σ is a metric map.

Proposition 3.10. *For every spider X there exists a linear map $\sigma : V(X) \to V(X)$ such that $\Gamma(X,\sigma)$ is weakly connected.*

Proof. At first, let $X \simeq P_n$ be a path with $n \geq 1$ vertices. Thus, let $V(X) = \{u_1, \ldots, u_n\}$ and $E(X) = \{u_i u_j : 1 \leq i = j - 1 \leq n - 1\}$. Consider the following map

$$\sigma(x) = \begin{cases} u_{i+1}, & \text{if } x = u_i \text{ for } 1 \leq i \leq n-1, \\ u_n, & \text{if } x = u_n \end{cases}$$

for all $x \in V(X)$. Then $\Gamma(X,\sigma)$ is a path.

Now let X be a nontrivial spider and $u \in V(X)$ be its center. Put $L(X) = \{v_1, \ldots, v_n\}$. Therefore, $V(X) = \bigcup_{1 \leq i \leq n} [u, v_i]_X$ and for each pair $1 \leq i \leq n$, $1 \leq j \leq d_X(u, v_i)$ there exists a unique vertex $x_{i,j} \in [u, v_i]_X$ such that $d_X(u, x_{i,j}) = j$.

Without loss of generality, we can assume that $d_X(u, v_1) \geq \cdots \geq d_X(u, v_n)$. Put

$$\sigma(x) = \begin{cases} u, & \text{if } x = u, \\ x_{i+1, d_X(u, v_{i+1})}, & \text{if } x = x_{i,j},\ 1 \leq j \leq d_X(u, v_i) \text{ and } i \neq n, \\ x_{1, d_X(u, v_1)}, & \text{if } x = x_{n,j} \text{ and } 1 \leq j \leq d_X(u, v_n) \end{cases}$$

for all $x \in V(X)$. Then the map σ is linear and $\Gamma(X,\sigma)$ is weakly connected. $\square$

Consider the tree X with $V(X) = \{1, \ldots, 6\}$, $E(X) = \{12, 23, 34, 35, 26\}$ and the map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 2 & 6 & 1 & 2 \end{pmatrix}$. It is easy to see that σ is linear and the Markov graph $\Gamma(X,\sigma)$ is weakly connected. However, the tree X is not a spider as it has two vertices of degree three.

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